Kuramoto model of synchronization: equilibrium and non equilibrium aspects

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Plan

- Kuramoto and Sakaguchi models, the role of noise
- Inertia and the connection with statistical mechanics
- Equilibrium vs. non equilibrium
- Detailed balance
- Complete phase diagram: First and second order phase transition
- An intermezzo on long-range interactions and the HMF model
- Hysteresis and bistability
- Linear stability analysis
- α-HMF model and zero mode dominance
- α-Kuramoto
- Summary

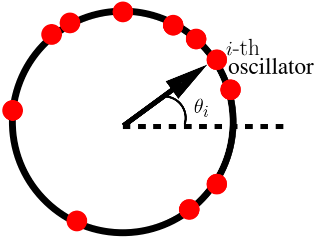
The Kuramoto model

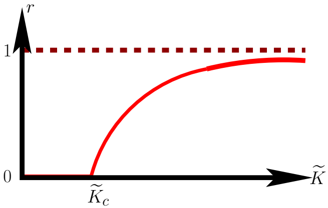
A framework to study spontaneous synchronization:
N globally coupled oscillators with distributed natural frequencies

$$\frac{d\theta_i}{dt} = \omega_i + \frac{\tilde{K}}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

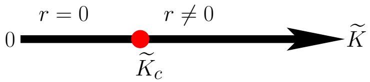
$\tilde{K}$ coupling constant, ω_i quenched random variables with distribution $g(\omega)$
Order parameter, fraction of phase-locked oscillators: $r = |1/N \sum_j \exp(i\theta_j)|$

High $\tilde{K}$: Synchronized phase , $r > 0$
Low $\tilde{K}$: Incoherent phase, $r = 0$.
For unimodal $g(\omega)$ continuous transition on tuning $\tilde{K}$.





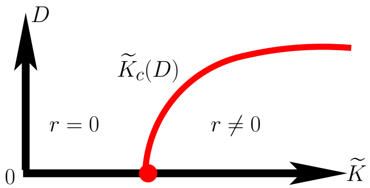
The Sakaguchi model



Stochastic fluctuations of the ω_i in time

$$\frac{d\theta_i}{dt} = \omega_i + \frac{\tilde{K}}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \eta_i(t)$$

$$< \eta_i(t) >= 0\,,\; < \eta_i(t)\eta_j(t') >= 2D\delta_{ij}\delta(t-t')$$



2nd order dynamics

Two dynamical variables: θ_i (Phase); v_i (Angular velocity)

$$\frac{d\theta_i}{dt} = v_i$$

$$m\frac{dv_i}{dt} = -\gamma v_i + \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \eta_i(t)$$

where m is the inertia and γ the friction constant.
Motivation:

- An adaptive frequency can explain the slower approach to synchronization observed in a particular firefly (the Pteropyx mallacae) Ermentrout (1991)
- Phase dynamics in electric power distribution networks in the mean-field limit Filatrella, Nielsen and Pedersen (2008), Rohden, Sorge, Timme and Witthaut (2012), Olmi and Torcini (2014)

Previous studies

- No noise: Simulations for a Lorentzian $g(\omega)$ show a first-order synchronization transition Tanaka, Lichtenberg and Oishi (1997)
- Analysis in the continuum limit, based on a suitable Fokker-Planck equation in the limit $N \to \infty$ for a Lorentzian $g(\omega)$: either larger inertia or larger ω spread makes the system harder to synchronize Acebron and Spigler (1998); Acebron, Bonilla and Spigler (2000)
- HOWEVER, THE COMPLETE PHASE DIAGRAM HAS NOT BEEN ADDRESSED

Rescaling

One can analyze the model in the reduced parameter space (T, σ, m)

$$\frac{d\theta_i}{dt} = v_i$$

$$\frac{dv_i}{dt} = F_i + \eta_i(t) = -\frac{1}{\sqrt{m}} v_i + \sigma\omega_i + \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \eta_i(t)$$

where now:

- $g(\omega)$ has zero average and unit width
- $< \eta_i(t)\eta_j(t') >= \frac{2T}{\sqrt{m}}\delta_{ij}\delta(t - t')$

Two steps

- Subtracting average motion: $\theta_i \Rightarrow \theta_i + < \omega > t$, $v_i \Rightarrow v_i + < \omega > t$, $\omega_i \Rightarrow \omega_i + < \omega >$
- Rescaling: $t' = t\sqrt{K/m}$, $v_i' = v_i\sqrt{m/K}$, $1/\sqrt{m'} = \gamma/\sqrt{Km}$, $\sigma' = \gamma\sigma/K$, $T' = T/K$.

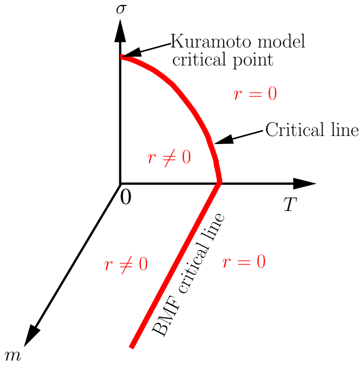
Equilibrium and nonequilibrium

The role of σ

- $\sigma = 0$, $\Rightarrow$ no external drive $\Rightarrow$ detailed balance $\Rightarrow$ equilibrium stationary state
- $\sigma > 0 \Rightarrow$ non equilibrium stationary state

Continuous transition lines

- $m = T = 0$, $\sigma > 0$, Kuramoto
- $m = 0$, $T > 0$, $\sigma > 0$, Sakaguchi, continuous transition, critical line
- $\sigma = 0$ Hamiltonian system + heat-bath (Brownian Mean Field model), continuous transition, critical line Chavanis (2013)



Detailed balance

Fokker-Planck equation for the N-body distribution

$$\frac{\partial f_N(\mathbf{x})}{\partial t} = -\sum_{i=1}^{2N} \frac{\partial [A_i(\mathbf{x}) f_N(\mathbf{x})]}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{2N} \frac{\partial^2 [B_{i,j}(\mathbf{x}) f_N(\mathbf{x})]}{\partial x_i \partial x_j}$$

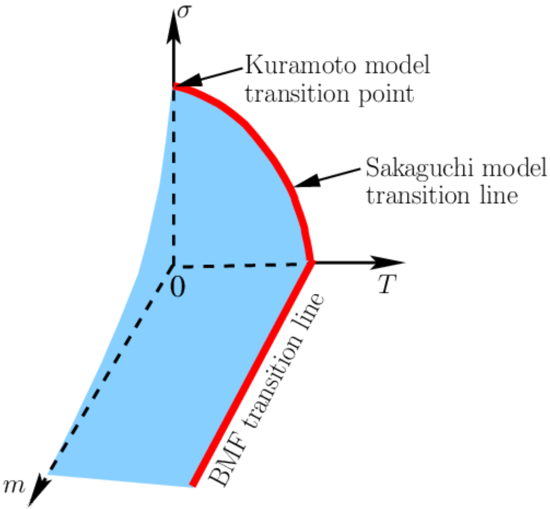
$$\mathbf{x} = (\theta_1, \ldots, \theta_N; v_1, \ldots, v_N) \ \mathbf{A}(\mathbf{x}) = (v_1, \ldots, v_N; F_1, \ldots, F_N) \ B_{i,j} = \delta_{i,j} 2T$$

Detailed balance conditions (Risken)

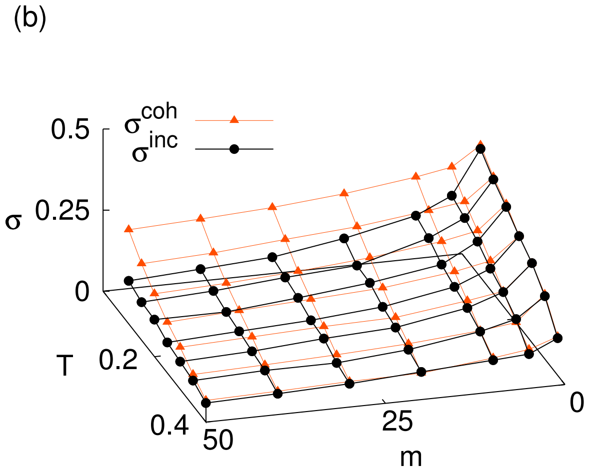
$$\epsilon_i \epsilon_j B_{i,j}(\epsilon \mathbf{x}) = B_{i,j}(\mathbf{x}) \ , \ \epsilon_i A_i(\epsilon \mathbf{x}) f_N^s(\mathbf{x}) = -A_i(\mathbf{x}) f_N^s(\mathbf{x}) + \sum_{j=1}^{2N} \frac{\partial [B_{i,j}(\mathbf{x}) f_N^s(\mathbf{x})]}{\partial x_j}$$

where $\epsilon_i = \pm 1$ is the parity with respect to time reversal and f_N^s is a stationary solution of the Fokker-Planck equation.
These conditions can be satisfied only when $\sigma = 0$ and, as a consequence $f_N^s \propto \exp(-H/T)$

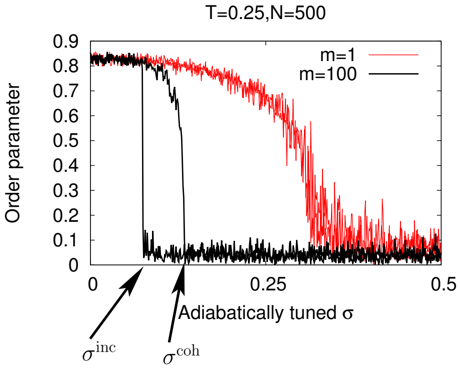
Phase diagram-I



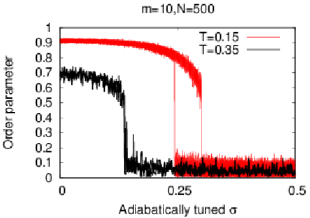
Phase diagram-II



Hysteresis

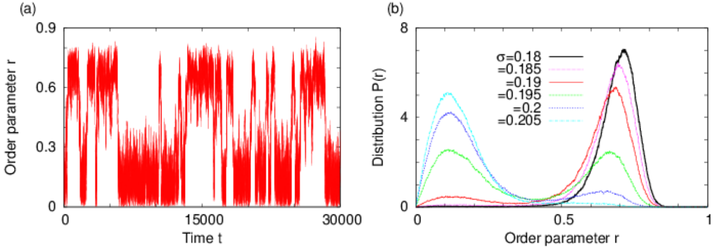


T=0.25,N=500
Order parameter
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.25
0.5
m=1
m=100
Adiabatically tuned σ
σ^{inc}
σ^{coh}



m=10,N=500
Order parameter
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.25
0.5
T=0.15
T=0.35
Adiabatically tuned σ

Bistability



For $m = 20$, $T = 0.25$, $N = 100$, and a Gaussian $g(\omega)$ with zero mean and unit width, (left) shows, at $\sigma = 0.195$, r vs. time in the stationary state, while (right) shows the distribution $P(r)$ at several σ's around 0.195.

$N \to \infty$ continuum limit

Single-particle distibution $f(\theta, v, \omega, t)$: Fraction of oscillators at time t and for each ω which have phase θ and angular velocity v (Periodic in θ and normalized).
Evolution by Kramers equation

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial v}\Big(\frac{v}{\sqrt{m}} - \sigma\omega - r\sin(\psi - \theta)\Big)f + \frac{T}{\sqrt{m}}\frac{\partial^2 f}{\partial v^2},$$

with self-consistent order parameter

$$r\exp(i\psi) = \iiint d\theta dv d\omega g(\omega)\exp(i\theta)f(\theta, v, \omega, t)$$

Homogeneous ($r = 0$) solution

$$f^{\mathrm{inc}} = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{(v - \sigma\omega\sqrt{m})^2}{2T}\right)$$

Linear stability results

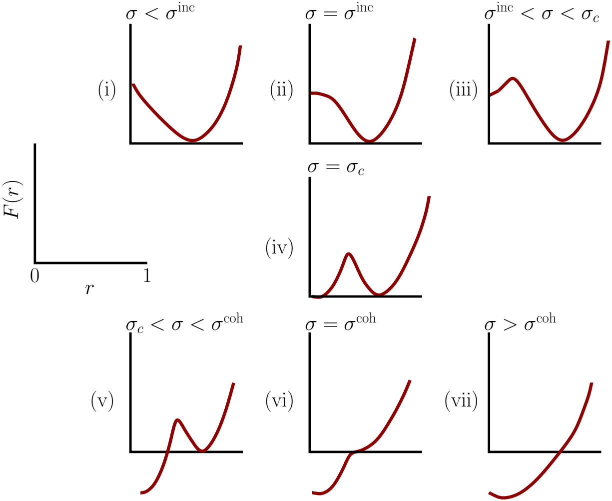
Stability analysis gives σ^{inc}:
$f(\theta, v, \omega, t) = f^{\text{inc}}(\theta, v, \omega) + e^{\lambda t}\delta f(\theta, v, \omega)$

$$\frac{2T}{e^{mT}} = \sum_{p=0}^{\infty} \frac{(-mT)^p(1+\frac{p}{mT})}{p!} \int_{-\infty}^{+\infty} \frac{g(\omega)d\omega}{1+\frac{p}{mT}+i\frac{\sigma\omega}{T}+\frac{\lambda}{T\sqrt{m}}}.$$

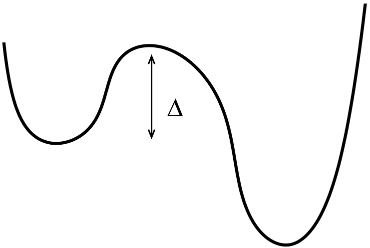
`Acebron, Bonilla and Spigler (2000)`

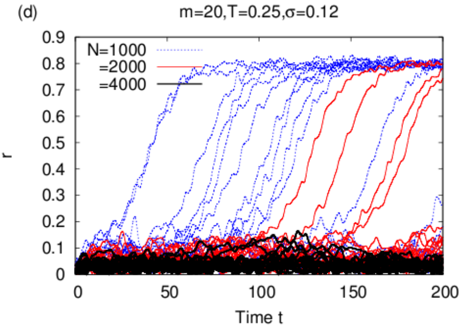
- The equation for λ has at most one solution with a positive real part and, when the solution exists, it is necessarily real.
- Neutral stability $\Rightarrow$ $\lambda = 0$ gives the stability surface $\sigma^{\text{inc}}(m, T)$.
- Similarly, one can define $\sigma^{\text{coh}}(m, T)$.
- The two surfaces enclose the first-order transition surface $\sigma_c(m, T)$.
- Taking proper limits, the surface $\sigma^{\text{inc}}(m, T)$ meets the critical lines on the (T, σ) and (m, T) planes.
- The intersection of the surface with the (m, σ) plane gives an implicit formula for $\sigma^{\text{inc}}_{\text{noiseless}}(m, \sigma)$.

Large deviation function



Mean-field metastability





Summary of the first part

- Kuramoto model from the point of view of equilibrium and non equilibrium statistical mechanics
- First-order phase transition in presence of inertia (full phase diagram).
- In absence of quenched randomness $\sigma = 0$ the stationary probability distribution is the Boltzmann-Gibbs product measure $\exp(-(K+U)/T) = \exp(-K/T)\exp(-U/T)$. The phase transition is characterized by the potential energy U only and it is the same for underdamped or overdamped dynamics.
- In presence of quenched randomness $\sigma \neq 0$ the system is out of equilibrium and the stationary measure is not a product measure and the phase transition depends on the damping coefficient.

Long-range interactions

At large inter particle distance r

$$V(r) \sim r^{-\alpha}$$
$$0 \leq \alpha \leq d$$

Energy per particle

$$\epsilon = \frac{E}{N} = \int_{\delta}^{R} d^d r \rho \frac{J}{r^\alpha} \propto \left[R^{d-\alpha} - \delta^{d-\alpha} \right]$$

- if $\alpha > d$ then $\epsilon \to const$ when $R \to \infty$.
- if $0 \leq \alpha \leq d$ then $\epsilon \sim V^{1-\alpha/d}$ $(V \sim R^d)$

Non additivity

$$E_{I+II} \neq E_I + E_{II}$$

Curie-Weiss Hamiltonian

$$H_{CW} = -\frac{J}{2N} \sum_{i,j} \sigma_i \sigma_j$$

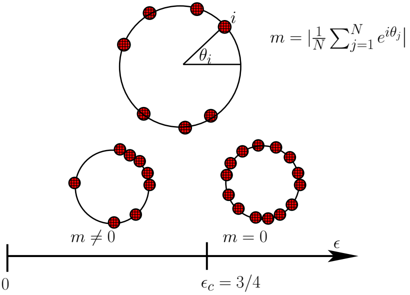
with $\sigma_i = \pm 1$.
Zero magnetization state $M = \sum_i \sigma_i = 0$

$$E_{I+II} = 0 \qquad E_I = E_{II} = -J/8N.$$

The HMF model

Antoni and SR, 1995

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} (1 - \cos(\theta_i - \theta_j))$$



Vlasov equation

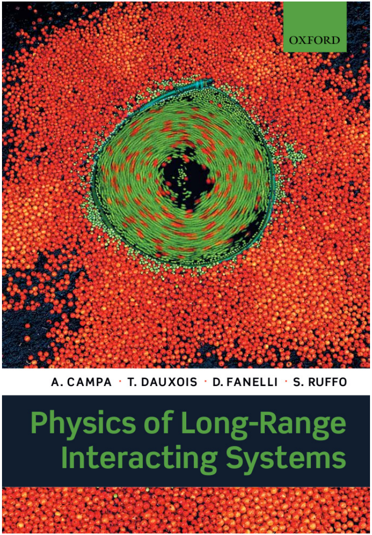
$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + U(\theta_i) \quad , \quad U(\theta_1, .., \theta_N) = \frac{1}{N} \sum_{i<j}^{N} V(\theta_i - \theta_j)$$

$$N \to \infty$$

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial \theta} - \frac{\partial \langle v \rangle}{\partial \theta} \frac{\partial f}{\partial p} = 0$$

$$\langle v \rangle(\theta, t) = \int d\theta' dp' V(\theta - \theta') f(\theta', p', t) \qquad \iint d\theta dp f = 1$$

Braun and Hepp, 1977; Jabin, 2008; Villani, 2009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Vlasov equation on a lattice

$$H = \sum_j \frac{p_j^2}{2} + \frac{1}{2\tilde{N}} \sum_{j,k=1}^{N} \frac{v(q_j, q_k)}{|x_j - x_k|^\alpha}\,,\; x_j = ja\,,\; \tilde{N} = \sum_{i=1}^{N} i^{-\alpha} \propto N^{1-\alpha}\,,\; 0 \le \alpha$$

Continuum limit

$$\dot{q} = p \quad , \quad \dot{p} = -\frac{\partial V_x[f](q,t)}{\partial q}$$

where

$$V_x[f](q,t) = \kappa_\alpha \iiint dq'\,dp'\,dx'\,f(q',p';x',t)\frac{v(q,q')}{|x-x'|^\alpha}\ ,$$

$$\kappa_\alpha^{-1} = \int_{-1/2}^{+1/2} dx/|x|^\alpha$$

Vlasov equation

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial q} - \frac{\partial V'_x[f](q,t)}{\partial q}\frac{\partial f}{\partial p} = 0.$$

Linearized Vlasov equation

Expansion around a homogeneous state

$$f(q,p;x,t) = f_0(p) + \delta f(q,p;x,t)$$

$$\partial_t(\delta f) + p\partial_q(\delta f) - \partial_p f_0(p)\partial_q V_x[\delta f](q,t) = 0.$$

Fourier expansion

$$\delta f_t(q,p;x) = \sum_k e^{\lambda_k t}\hat{f}_k(q,p)\exp(2i\pi kx)$$

Dispersion relation

$$\hat{f}_k(q,p) - c_k(\alpha)\partial_p f_0(p)\frac{e^{-\lambda_k \frac{q}{p}}}{p}\int_{q_0}^{q} e^{\lambda_k \frac{q'}{p}} V[\hat{f}_k](q')dq' = 0$$

where

$$c_k(\alpha) = \kappa_\alpha \int_{-1/2}^{+1/2} \frac{e^{2i\pi ky}}{|y|^\alpha}dy \quad , \quad V[\hat{f}_k](q) = \iint dq'dp'\hat{f}_k(q',p')v(q,q')$$

α-HMF model

Tamarit, Anteneodo (2000), Campa, Giansanti Moroni (2000), Mori (2011,2012)

$$v(q,q') = -\cos(q-q') \quad , \quad V[\hat{f}_k](q) = -\kappa_\alpha \left(m_x[\hat{f}_k]\cos q + m_y[\hat{f}_k]\sin q \right),$$

$$m_x[\hat{f}_k] = \iint dq'\,dp'\,\hat{f}_k(q',p')\cos q' \quad , \quad m_y[\hat{f}_k] = \iint dq'\,dp'\,\hat{f}_k(q',p')\sin q'$$

Dispersion relation

$$\left(1 + \pi c_k(\alpha)\int dp \frac{f_0'(p)}{p\left(1+\frac{\lambda_k{}^2}{p^2}\right)}\right)^2 + \left(\pi c_k(\alpha)\int dp \frac{f_0'(p)}{p^2\left(1+\frac{\lambda_k{}^2}{p^2}\right)}\right)^2 = 0.$$

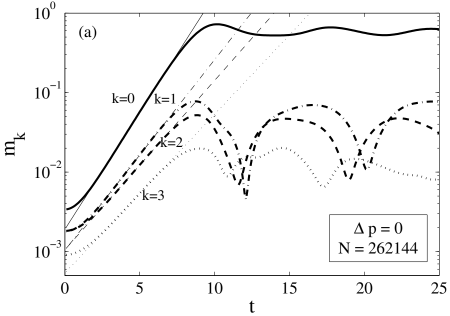
Waterbag distribution

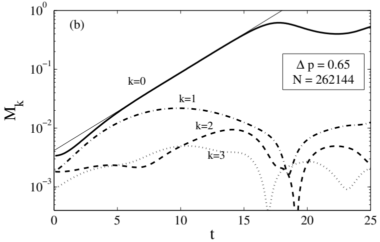
$$f_0(p) = \frac{1}{2\pi}\frac{1}{2\Delta p}\left(\Theta(p+\Delta p) - \Theta(p-\Delta p)\right),$$

The eigenvalue of the k-th Fourier mode is given by

$$\lambda_k = \sqrt{\frac{c_k(\alpha)}{2} - \Delta p^2}.$$

Zero-mode dominance-I





$$m_k = \left| \int e^{-ikx} e^{iq} f(q,p;x,t) dq\ dp \right| \approx \frac{1}{N} \left| \sum_j e^{-ikx_j} e^{iq_j} \right| ,\ \alpha = 0.8$$

α-Kuramoto

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{\widetilde{N}} \sum_{j=1, j\neq i}^{N} \frac{\sin(\theta_j - \theta_i)}{|x_j - x_i|^\alpha}$$

ω_i is a quenched random variable with distribution $g(\omega)$, Rogers and Wille (1996); Chowdhury and Cross (2010).
In the continuum limit, the local density $\rho(\theta; \omega, x, t)$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}\left(\rho \frac{\partial \theta}{\partial t}\right)$$

$$\frac{\partial \theta(\omega, x, t)}{\partial t} = \omega + \kappa(\alpha) K \int d\theta' d\omega' dx' \frac{\sin(\theta' - \theta)}{|x' - x|^\alpha} \rho(\theta'; \omega', x', t) g(\omega')$$

Linear stability analysis of the homogeneous state

$$\rho(\theta; \omega, x, t) = \frac{1}{2\pi} + \delta\rho(\theta; \omega, x, t)$$

Dispersion relation

$$1 - \frac{c_k(\alpha) K}{2} \int_{-\infty}^{\infty} d\omega \frac{g(\omega)}{(\lambda_k \pm i\omega)} = 0$$

Stability of the incoherent state

If $g(\omega)$ is symmetric around the mean and non increasing then λ_k is either real positive or zero.
The dispersion relation rewrites

$$1 - c_k(\alpha)K \int_0^\infty d\omega \frac{\lambda_k}{\lambda_k^2 + \omega^2} g(\omega) = 0$$

The limit $\lambda_k \to 0$ gives the critical couplings

$$K_c^{(k)} = \frac{2}{c_k(\alpha)\pi g(0)}$$

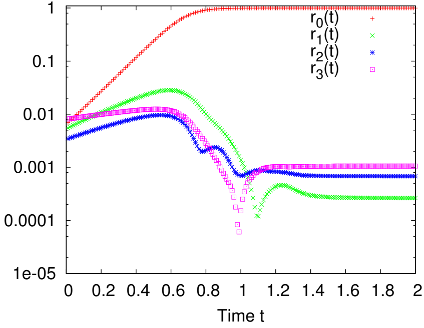
$$K_c^{(0)} < K_c^{(1)} < K_c^{(2)} < \dots.$$

and the growth rates λ_k

$$\frac{2K}{\pi g(0) K_c^{(k)}} \int_0^\infty d\omega \; \frac{\lambda_k}{\lambda_k^2 + \omega^2} g(\omega) = \frac{K}{K_c^{(k)}} e^{\lambda_k^2/2} \mathrm{Erfc}\left[\frac{\lambda_k}{\sqrt{2}}\right] = 1.$$

for Gaussian $g(\omega)$.

Zero-mode dominance



$$\alpha = 0.5\,, K = 15\,, K_c^{(0)} \approx 1.59577\,, K_c^{(1)} \approx 4.26696\,, K_c^{(2)} \approx 6.53664\,, K_c^{(3)}$$

so that the Fourier modes $0, 1, 2, 3$ are all linearly unstable.

Summary

- Kuramoto model from the point of view of equilibrium and non equilibrium statistical mechanics
- First-order phase transition in presence of inertia (full phase diagram)
- Long-range interactions, the HMF model and the Vlasov equation
- Lattices with long-range interactions: α-HMF and α-Kuramoto