# Dispersion relation for hadronic light-by-light scattering

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# Outline

- Introduction: the hadronic light-by-light (HLbL) contribution to the anomalous magnetic moment of the muon. Dispersive approach to the HLbL tensor
- Lorentz structure of HLbL tensor: gauge invariance and crossing symmetry
- Master formula for the HLbL contribution to $(g-2)_\mu$
- Mandelstam representation for pion pole and pion box contributions
- Conclusions and outlook

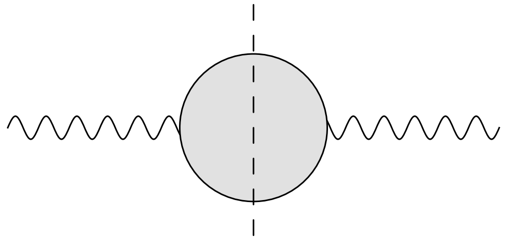
Colangelo, Hoferichter, Procura, Stoffer, arXiv: 1506.01386, JHEP, in print

# Introduction

- Limiting factor in the accuracy of SM predictions for $a_\mu = (g-2)_\mu/2$ is control over hadronic contributions, responsible for most of the theory uncertainty

- Hadronic vacuum polarization can be systematically improved

  - unitarity and analyticity relate it directly to $\sigma_{\rm tot}(e^+e^- \to \gamma^* \to {\rm hadrons})$



  - dedicated $e^+e^-$ program (BaBar, Belle II, BESIII, CMD3, KLOE2, SND)

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- Hadronic light-by-light (HLbL) is more problematic. Only model calculations have been performed so far and they are characterized by large uncertainties in the individual contributions

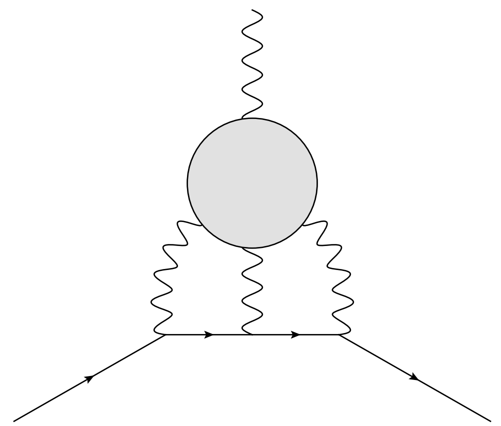

**Table 13**

Summary of the most recent results for the various contributions to $a_\mu^{\mathrm{LbL;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

| Contribution | BPP | HKS | KN | MV | BP | PdRV | N/JN |
|---|---|---|---|---|---|---|---|
| $\pi^0, \eta, \eta'$ | $85 \pm 13$ | $82.7 \pm 6.4$ | $83 \pm 12$ | $114 \pm 10$ | – | $114 \pm 13$ | $99 \pm 16$ |
| $\pi, K$ loops | $-19 \pm 13$ | $-4.5 \pm 8.1$ | – | – | – | $-19 \pm 19$ | $-19 \pm 13$ |
| $\pi, K$ loops + other subleading in $N_c$ | – | – | – | $0 \pm 10$ | – | – | – |
| Axial vectors | $2.5 \pm 1.0$ | $1.7 \pm 1.7$ | – | $22 \pm 5$ | – | $15 \pm 10$ | $22 \pm 5$ |
| Scalars | $-6.8 \pm 2.0$ | – | – | – | – | $-7 \pm 7$ | $-7 \pm 2$ |
| Quark loops | $21 \pm 3$ | $9.7 \pm 11.1$ | – | – | – | $2.3\pm$ | $21 \pm 3$ |
| Total | $83 \pm 32$ | $89.6 \pm 15.4$ | $80 \pm 40$ | $136 \pm 25$ | $110 \pm 40$ | $105 \pm 26$ | $116 \pm 39$ |

Jegerlehner and Nyffeler (2009)

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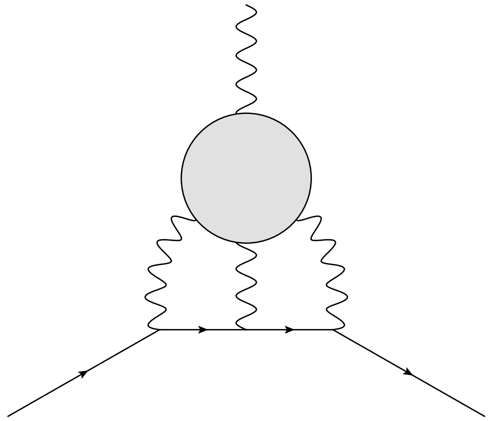


  - a reliable uncertainty estimate is still an open issue

- How to reduce model dependence? Recent strategies for an improved calculation :

  - lattice QCD
  - dispersion theory to make the evaluation as data driven as possible

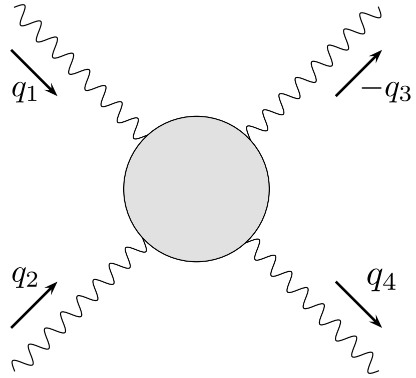
# Our strategy

- Exploit fundamental principles :
  - gauge invariance and crossing symmetry
  - unitarity and analyticity
- Relate HLbL to experimentally accessible quantities through a dispersive approach
- Much more challenging task than for the hadronic vacuum polarization due to the complexity of the HLbL tensor, which is the key object of our analysis

(for dispersive treatment of the HLbL contribution to Pauli FF, see talk by Pere)

# The HLbL tensor

- The fully off-shell HLbL tensor :




$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x\, d^4y\, d^4z\, e^{-i(q_1\cdot x + q_2\cdot y + q_3\cdot z)} \langle 0 | T\{ j^\mu_{\rm em}(x) j^\nu_{\rm em}(y) j^\lambda_{\rm em}(z) j^\sigma_{\rm em}(0)\} | 0 \rangle$$

- Mandelstam variables:

$$s = (q_1 + q_2)^2,\; t = (q_1 + q_3)^2,\; u = (q_2 + q_3)^2$$

- For the evaluation of $a_\mu^{\rm HLbL}$ one photon will be taken on shell ( $q_4^2 = 0$ )

# Lorentz structure of HLbL tensor

- Based on Lorentz covariance the HLbL tensor can be decomposed in 138 structures

$$
\begin{aligned}
\Pi^{\mu\nu\lambda\sigma} &= g^{\mu\nu}g^{\lambda\sigma}\,\Pi^1 + g^{\mu\lambda}g^{\nu\sigma}\,\Pi^2 + g^{\mu\sigma}g^{\nu\lambda}\,\Pi^3 \\
&+ \sum_{\substack{i=2,3,4\\ j=1,3,4}} \sum_{\substack{k=1,2,4\\ l=1,2,3}} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma\,\Pi^4_{ijkl} \\
&+ \sum_{\substack{i=2,3,4\\ j=1,3,4}} g^{\lambda\sigma} q_i^\mu q_j^\nu\,\Pi^5_{ij} + \sum_{\substack{i=2,3,4\\ k=1,2,4}} g^{\nu\sigma} q_i^\mu q_k^\lambda\,\Pi^6_{ik} + \sum_{\substack{i=2,3,4\\ l=1,2,3}} g^{\nu\lambda} q_i^\mu q_l^\sigma\,\Pi^7_{il} \\
&+ \sum_{\substack{j=1,3,4\\ k=1,2,4}} g^{\mu\sigma} q_j^\nu q_k^\lambda\,\Pi^8_{jk} + \sum_{\substack{j=1,3,4\\ l=1,2,3}} g^{\mu\lambda} q_j^\nu q_l^\sigma\,\Pi^9_{jl} + \sum_{\substack{k=1,2,4\\ l=1,2,3}} g^{\mu\nu} q_k^\lambda q_l^\sigma\,\Pi^{10}_{kl}
\end{aligned}
$$

- In 4 space-time dimensions there are 2 linear relations among these 138 structures

Eichmann, Fischer, Heupel, Williams (2014)

- Scalar functions encode the hadronic dynamics and depend on 6 kinematic variables

- This set of functions is hugely redundant: Ward identities imply 95 linear relations between these scalar functions (kinematic zeros)

# Lorentz structure of HLbL tensor

- Following Bardeen and Tung (1968) - "BT"- we contracted the HLBL tensor with

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4}$$

  - 95 structures project to zero

- removed the $1/q_1 \cdot q_2$ and $1/q_3 \cdot q_4$ poles by taking appropriate linear combinations

- This procedure introduces kinematic singularities in the scalar functions : degeneracies in these BT Lorentz structures as $q_1 \cdot q_2 \to 0, q_3 \cdot q_4 \to 0$

$$\sum_k c_k^i T_k^{\mu\nu\lambda\sigma} = q_1 \cdot q_2 X_i^{\mu\nu\lambda\sigma} + q_3 \cdot q_4 Y_i^{\mu\nu\lambda\sigma}$$

# Lorentz structure of HLbL tensor

- Following Tarrach (1975) we extended BT set to incorporate $X_i^{\mu\nu\lambda\sigma}$, $Y_i^{\mu\nu\lambda\sigma}$ ("BTT")

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures are manifestly gauge invariant
- crossing symmetry is manifest (only 7 genuinely different structures, the remaining ones being obtained by crossing)
- the BTT scalar functions are free of kinematic singularities and zeros : their analytic structure is dictated by dynamics only, suitable for a dispersive treatment

# Master formula for $a_\mu^{\mathrm{HLbL}}$

- Differentiating the Ward identity with respect to $q_4$ ,

$$\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_4-q_1-q_2) = -q_4^\sigma \frac{\partial}{\partial q_4^\rho}\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,q_4-q_1-q_2)$$

one obtains the relation

$$a_\mu^{\mathrm{HLbL}} = -\frac{1}{48m_\mu}\mathrm{Tr}\left((\not p+m_\mu)[\gamma^\rho,\gamma^\sigma](\not p+m_\mu)\Gamma_{\rho\sigma}^{\mathrm{HLbL}}(p)\right)$$

with

$$\Gamma_{\rho\sigma}^{\mathrm{HLbL}}(p) = e^6\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\gamma^\mu\frac{(\not p+\not q_1+m_\mu)}{(p+q_1)^2-m_\mu^2}\gamma^\lambda\frac{(\not p-\not q_2+m_\mu)}{(p-q_2)^2-m_\mu^2}\gamma^\nu \times\frac{1}{q_1^2q_2^2(q_1+q_2)^2}\frac{\partial}{\partial q_4^\rho}\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,q_4-q_1-q_2)\Big|_{q_4=0}$$

# Master formula for $a_\mu^{HLbL}$

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$$a_\mu^{\mathrm{HLbL}} = -\frac{1}{48 m_\mu}\mathrm{Tr}\left((\not{p}+m_\mu)[\gamma^\rho,\gamma^\sigma](\not{p}+m_\mu)\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p)\right)$$

- Since there are no kinematic singularities in the BTT scalar functions,

$$\begin{aligned}
a_\mu^{\mathrm{HLbL}} = &-\frac{e^6}{48 m_\mu}\int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{1}{q_1^2 q_2^2 (q_1+q_2)^2}\frac{1}{(p+q_1)^2-m_\mu^2}\frac{1}{(p-q_2)^2-m_\mu^2}\\
&\times \mathrm{Tr}\left((\not{p}+m_\mu)[\gamma^\rho,\gamma^\sigma](\not{p}+m_\mu)\gamma^\mu(\not{p}+\not{q}_1+m_\mu)\gamma^\lambda(\not{p}-\not{q}_2+m_\mu)\gamma^\nu\right)\\
&\times \sum_{i=1}^{54}\left(\frac{\partial}{\partial q_4^\rho}T^i_{\mu\nu\lambda\sigma}(q_1,q_2,q_4-q_1-q_2)\right)\Bigg|_{q_4=0}\Pi_i(q_1,q_2,-q_1-q_2)
\end{aligned}$$

# Master formula for $a_\mu^{HLbL}$

- Only 12 linear combinations of the scalar functions contribute to $a_\mu^{\mathrm{HLbL}}$ :

$$a_\mu^{\mathrm{HLbL}} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- we determined the integration kernel functions $\hat{T}_i$

- five out of eight integrals can be performed analytically

- Wick rotation of $q_1$ , $q_2$ and $p$ (allowed even in the presence of anomalous cuts)

# Master formula for $a_\mu^{HLbL}$

- Obtained a general master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\, Q_1^3\, Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau)\bar{\Pi}_i(Q_1, Q_2, \tau)$$

where $T_i$ are known integration kernels and the scalar functions $\bar{\Pi}_i$ are linear combinations of the BTT $\Pi_i$

- $Q_i^2 = -q_i^2$ are Euclidean momenta and $Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1Q_2\tau$

- Generalization of the formula for the pion pole in Knecht and Nyffeler (2002)

# Mandelstam representation of the $\Pi_i$

- Analytic properties of scalar functions relevant for the evaluation of $a_\mu^{\mathrm{HLbL}}$ : right- and left-hand cuts, double spectral regions (box topologies)

- Very complex analytic structure: approximations are required. We order the contributions according to the mass of intermediate states: the lightest states are expected to be the most important (in agreement with model calculations)

- Here we consider the 2 lowest-lying contributions: one- and two-pion intermediate states in all channels

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

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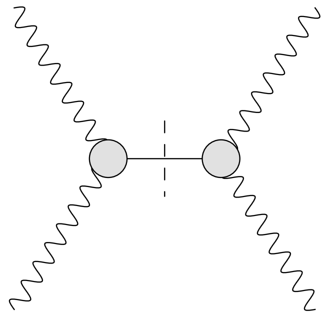
one-pion intermediate state :



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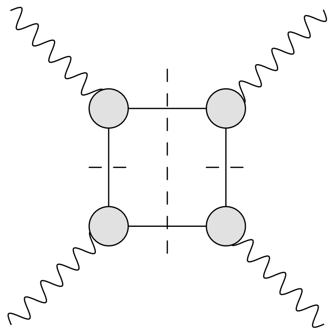
two-pion intermediate state in both channels :



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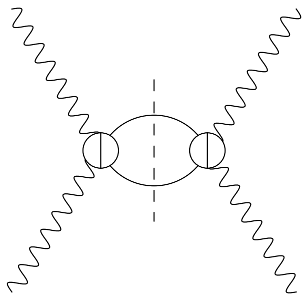
two-pion intermediate state in the direct channel: 

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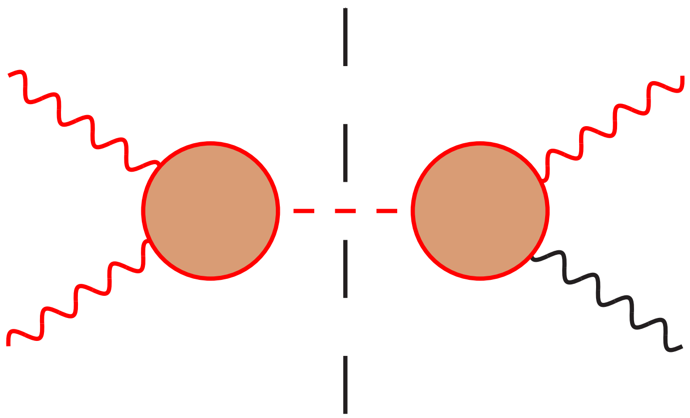
higher intermediate states: neglected so far

# The pion pole contribution

- From the unitarity relation with only $\pi^0$ intermediate state, the pole residues in each channel are given by products of doubly-virtual and singly-virtual pion transition form factors ( $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$ )



- Pion transition FFs are input for a numerical analysis of the master formula: formulation of a dispersive framework in Hoferichter et al. (2014)

# Pion box contribution

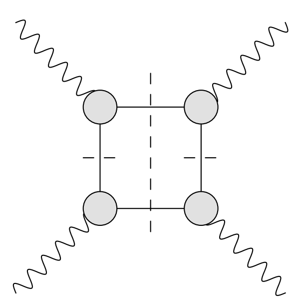
- Defined by simultaneous two-pion cuts in two channels

- Discontinuities as a dispersive integral over double spectral functions

$$\Pi_i = \frac{1}{\pi^2}\int ds'dt' \frac{\rho_i^{st}(s',t')}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

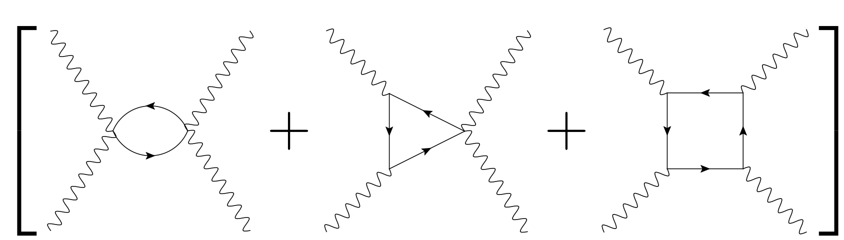
- Dependence on $q_i^2$ carried by the pion vector FFs for each off-shell photon

- sQED loop projected onto the BTT basis fulfills the same Mandelstam representation of the pion box, the only difference being the pion vector FFs :



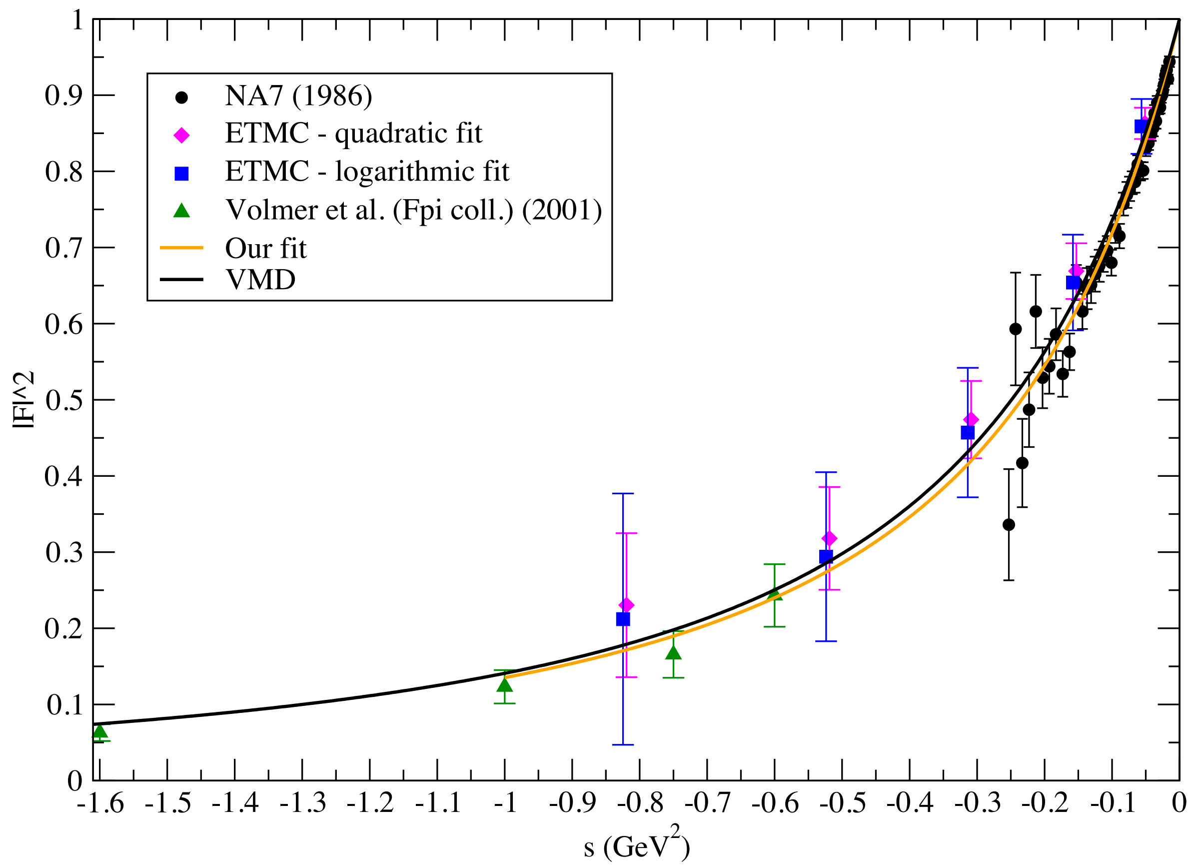
$$\equiv F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)$$

$\times$



# Numerics for the pion box contribution

- Pion vector form factor in the space-like region :

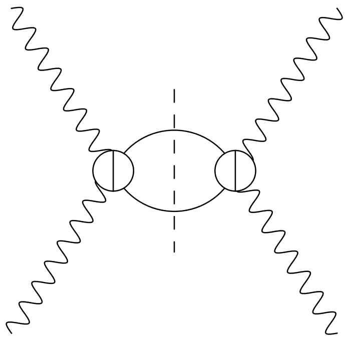



- Preliminary results:

$$a_\mu^{\pi\text{-box}} = -15.9 \cdot 10^{-11}, \qquad a_\mu^{\pi\text{-box, VMD}} = -16.4 \cdot 10^{-11}$$

# The remaining ππ contribution

- Two-pion cut only in the direct channel



- LH cut due to multi-particle intermediate states in the crossed channel neglected

- unitarity relates this to the helicity amplitudes for the subprocess $\gamma^*\gamma^{(*)} \to \pi\pi$

- no double spectral region: partial wave expansion is possible. S-wave contribution discussed in Colangelo, Hoferichter, Procura, Stoffer (2014). BTT formalism facilitates the generalization to D-waves

- Goal: reconstruct dispersively helicity partial waves for $\gamma^*\gamma^* \to \pi\pi$ . Treat ππ rescattering using the Omnes method (inclusion of resonance effects)

# Conclusions and Outlook

- Dispersive approach to HLbL scattering based on general principles: gauge invariance and crossing symmetry, unitarity and analyticity

- Derivation of a set of structures according to Bardeen-Tung-Tarrach (BTT) such that the scalar functions are free of kinematic singularities and zeros

- Derivation of a master formula for $a_\mu^{\mathrm{HLbL}}$ in terms of BTT functions

- Single- and double-pion intermediate states are taken into account

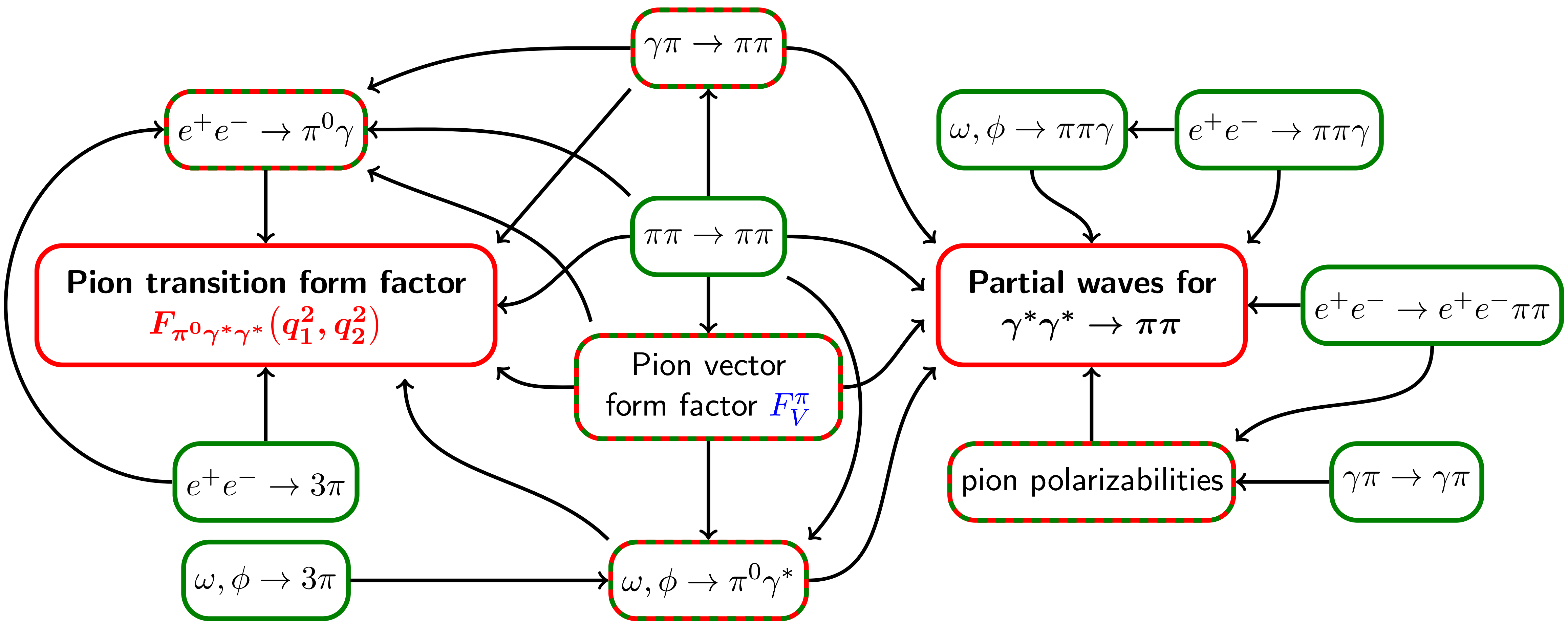
- Future work: model estimates for higher intermediate states (with more than 2 pions). Investigate and incorporate high-energy constraints

- First step towards a reduction of model dependence of HLbL: within a dispersive framework, relations with experimentally accessible (or dispersively reconstructed) quantities

# Additional slides

# A roadmap for HLbL

GC, Hoferichter, Kubis, Procura, Stoffer `arXiv:1408.2517` (PLB '14)




Artwork by M. Hoferichter