# Leading-order hadronic contributions to the lepton anomalous magnetic moments from the lattice


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for

$g-2$ @ ETMC

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# Outline

- Motivation
- Lattice calculation
  - General procedure [JHEP 1402 (2014) 099]
  - External scale problem [Phys.Rev.Lett. 107 (2011)]
  - HLO lepton anomalous magnetic moments [arXiv:1501.05110 [hep-lat]]
- Conclusions and outlook

# Motivation - hadronic corrections to electroweak observables

- hadronic corrections at leading order in the electroweak couplings from first principles
- focus today: anomalous magnetic moments of Standard Model leptons
- broad range of applications: hadronic vp-type NLO contributions to $g-2$, hadronic running of $\alpha_{\mathrm{QED}}$ and weak mixing angle $\sin(\theta_w)$, Adler function, $\Lambda_{\mathrm{QCD}}$, Lamb-shift in muonic hydrogen
- calculation completely in Euclidean space [Phys. Rev. Lett. 91 (2003) 052001]
- lattice calculations with multiple volumes, lattice spacings, quark-disconnected diagrams, pion masses close to or even at the physical point with contributions from up, down, strange and charm quarks
  $\longrightarrow$ reach for high precision lattice results with control over systematic uncertainties
- goal: match precision of phenomenological analyses using experimental data

- essential ingredient: hadronic vacuum polarization tensor and function from lattice QCD

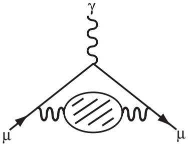
$$\Pi^{AB}_{\mu\nu}(Q) = \int d^4x \, \langle J^A_\mu(x)\, J^B_\nu(y)\rangle \, \mathrm{e}^{iQ(x-y)}$$
$$= \left(Q_\mu\, Q_\nu - \delta_{\mu\nu}\, Q^2\right)\, \Pi^{AB}(Q^2)$$

with $A,\, B \in \{\gamma, 3\}$ and $J^{A,B}_\mu$ vector current

- for our motivating examples:

$$a_l^{\mathrm{hlo}} \propto \int \frac{dQ^2}{Q^2}\, \Pi^{\gamma\gamma}_R(Q^2)\, w(Q^2/m_l^2)$$
$$\Delta^{\mathrm{hlo}}\alpha_{\mathrm{QED}}(Q^2) \propto \alpha_{\mathrm{QED}}(0)\, \Pi^{\gamma\gamma}_R(Q^2) \quad (\mathrm{photon\ self-energy})$$
$$\Delta^{\mathrm{hlo}}\sin(\theta_w)(Q^2) \propto \Pi^{\gamma\gamma}_R(Q^2),\, \Pi^{3\gamma}_R(Q^2) \quad (Z-\gamma \ \mathrm{mixing})$$

# Motivation - lepton anomalous magnetic moments




- radiative QCD corrections to the lepton-photon vertex, $a_l = (g-2)_l/2$
- $a_e$ experimentally and theoretically known to high precision [JHEP 1211 (2012) 113]

$$a_e(\mathrm{EXP}) = 115965218\,07.3\,(2.8) \cdot 10^{-13}$$
$$a_e(\mathrm{SM}) = 115965218\,17.8\,(7.6) \cdot 10^{-13}$$

- likewise for muon, but with 2-4 $\sigma$ discrepancy [Eur.Phys.J. C74 (2014) 10, 2981]

$$a_\mu(\mathrm{EXP}) = 11659208.9\,(6.3) \cdot 10^{-10}$$
$$a_\mu(\mathrm{SM}) = 11659180.4\,(4.2)\,(2.6) \cdot 10^{-10}$$

- new experiments for $(g-2)_\tau$ [Nucl.Phys.Proc.Suppl. 253-255 (2014) 103106, Nucl.Part.Phys.Proc. 260 (2015) 1215]
- hadronic LO dominant source of uncertainty in SM prediction
- at leading order (QED): universal calculation for all SM leptons

# Lattice calculation - twisted mass fermions with automatic $\mathcal{O}(a)$ improvement

- tmLQCD calculation with $N_f = 2+1+1$ dynamical sea quarks,

$$
\begin{aligned}
\mathcal{S}_{\mathrm{sea}} &= \sum_x \bar{\chi}_l(x)\,\left(D_W + i\mu_l\,\gamma_5\,\tau^3\right)\,\chi_l(x) \\
&\quad + \sum_x \bar{\chi}_h(x)\,\left(D_W + i\mu_\sigma\,\gamma_5\,\tau^1 + \mu_\delta\,\tau^3\right)\,\chi_h(x) \\
\chi_l &= (u,d)^T\,, \quad \chi_h = (c,s)^T
\end{aligned}
$$

- Osterwalder-Seiler valence quarks for strange and charm (conserved vector current) [Annals Phys. 110 (1978) 440, JHEP 0410 (2004) 070]

$$
\begin{aligned}
\mathcal{S}_{\mathrm{val}} &= \sum_{f=l,s,c} \sum_x \bar{\psi}_f(x)\,\left(D_W + i\mu_f\,\gamma_5\,\tau^3\right)\,\psi_f(x) \\
\psi_f &= (\psi_f^+, \psi_f^-)^T \\
D_W &= \frac{1}{2}\,\gamma_\mu\,\left(\nabla_\mu^f + \nabla_\mu^b\right) - \left(\frac{ar}{2}\,\nabla_\mu^b\,\nabla_\mu^f - m_{\mathrm{cr}}\right) + (m_0 - m_{\mathrm{cr}})
\end{aligned}
$$

with $m_0 \to m_{\mathrm{cr}}$ tuned to maximal twist; same $m_{\mathrm{cr}}$ for all quark flavors

# Lattice calculation - ensemble parameters

Gauge field ensembles generated by the European Twisted Mass Collaboration with $N_f = 2 + 1 + 1$ dynamical quarks

| Ensemble | $\beta$ | $a$[fm] | $\left(\frac{L}{a}\right)^3 \times \frac{T}{a}$ | $m_{PS}$[MeV] | $L$[fm] |
|---|---|---|---|---|---|
| D15.48 | 2.10 | 0.061 | $48^3 \times 96$ | 227 | 2.9 |
| D30.48 | 2.10 | 0.061 | $48^3 \times 96$ | 318 | 2.9 |
| D45.32sc | 2.10 | 0.061 | $32^3 \times 64$ | 387 | 1.9 |
| B25.32t | 1.95 | 0.078 | $32^3 \times 64$ | 274 | 2.5 |
| B35.32 | 1.95 | 0.078 | $32^3 \times 64$ | 319 | 2.5 |
| B35.48 | 1.95 | 0.078 | $48^3 \times 96$ | 314 | 3.7 |
| B55.32 | 1.95 | 0.078 | $32^3 \times 64$ | 393 | 2.5 |
| B75.32 | 1.95 | 0.078 | $32^3 \times 64$ | 456 | 2.5 |
| B85.24 | 1.95 | 0.078 | $24^3 \times 48$ | 491 | 1.9 |
| A30.32 | 1.90 | 0.086 | $32^3 \times 64$ | 283 | 2.8 |
| A40.32 | 1.90 | 0.086 | $32^3 \times 64$ | 323 | 2.8 |
| A50.32 | 1.90 | 0.086 | $32^3 \times 64$ | 361 | 2.8 |

## Lattice calculation - the hadronic vacuum polarization tensor

$$J_\mu^\gamma = \frac{2}{3}\,J_\mu^u - \frac{1}{3}\,J_\mu^d + \frac{2}{3}\,J_\mu^c - \frac{1}{3}\,J_\mu^s$$

- electromagnetic current = Noether current for vector flavor variation

$$\delta\psi(x) = iQ^{\mathrm{em}}\,\psi(x)\,, \quad \delta\bar{\psi}(x) = -i\bar{\psi}(x)\,Q^{\mathrm{em}}$$

  with electromagnetic charge matrix $Q^{\mathrm{em}} = \mathrm{diag}(e_u, e_d, \dots)$

$$J_\mu^f(x) = \frac{1}{2}\left(\bar{\psi}_f(x)\,(\gamma_\mu - 1)\,U_\mu(x)\,\psi_f(x+a\hat{\mu}) + \bar{\psi}_f(x+a\hat{\mu})\,(\gamma_\mu + 1)\,U_\mu(x)^\dagger\,\psi_f(x)\right)$$

- lattice vacuum polarization tensor

$$\Pi_{\mu\nu}^{\gamma\gamma}(x,y) = \langle J_\mu^\gamma(x)\,J_\nu^\gamma(y)\rangle - a^{-3}\,\delta_{xy}\,\delta_{\mu\nu}\,\langle S_\nu(y)\rangle$$

$$S_\nu^f(y) = \frac{1}{2}\left[\bar{\psi}(y)\,(\gamma_\nu - r)\,U_\nu(y)\,\psi(y+a\hat{\nu}) - \bar{\psi}(y+a\hat{\nu})\,(\gamma_\nu + r)\,U_\nu(y)^\dagger\,\psi(y)\right]$$

- exactly transverse at finite lattice spacing

$$\partial_\mu^b\,\Pi_{\mu\nu}^{\gamma\gamma}(x,y) = 0$$

# Lattice calculation - the hadronic vacuum polarization function

- 4-dimensional Fourier transform in Euclidean space

$$\Pi_{\mu\nu}(Q) = \sum_x \Pi_{\mu\nu}(x,y)\,\mathrm{e}^{iQ\,(x+a\hat{\mu}/2-y-a\hat{\nu}/2)}$$

- transversality in momentum space

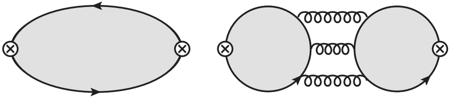
$$\hat{Q}_\mu\,\Pi_{\mu\nu}(Q) = 0\,, \quad \hat{Q}_\mu = 2/a\,\sin\left(aQ_\mu/2\right)\,, \quad Q_\mu = 2\pi/L_\mu \cdot n_\mu$$

- vacuum polarization function from projected real part

$$\Pi(\hat{Q}^2) = \mathrm{Re}\left\{ \sum_{\mu,\nu,Q\in[\hat{Q}^2]} P_{\mu\nu}(Q)\,\Pi_{\mu\nu}(Q) \left( \sum_{\mu,\nu,Q\in[\hat{Q}^2]} P_{\mu\nu}(Q)\,P_{\mu\nu}(Q) \right)^{-1} \right\}$$

$$P_{\mu\nu}(Q) = \hat{Q}_\mu\,\hat{Q}_\nu - \delta_{\mu\nu}\,\hat{Q}^2$$

- separation into (dominant) quark-connected and (small) quark-disconnected part



# Lattice calculation - example data for the vacuum polarization function

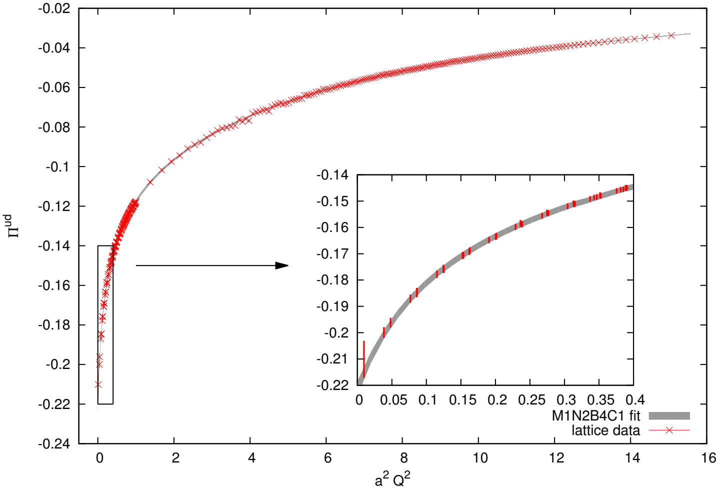



Figure : Hadronic vacuum polarization function (light quarks, connected) for $a = 0.078\,\mathrm{fm}$, $L = 2.5\,\mathrm{fm}$, $m_{PS} = 274\,\mathrm{MeV}$

## Lattice calculation - example data for the vacuum polarization function

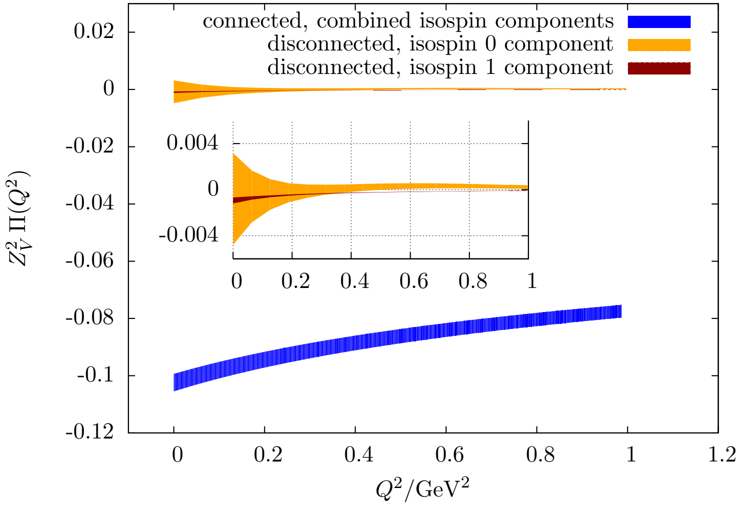
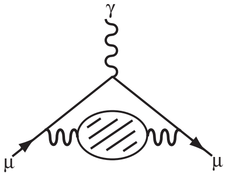



Figure : Hadronic vacuum polarization function (light quarks) for $a = 0.078\,\mathrm{fm}$, $L = 2.5\,\mathrm{fm}$, $m_{PS} = 393\,\mathrm{MeV}$; blue curve: quark-connected part, yellow and red curves: quark-disconnected parts

# Lattice calculation - $a_l^{\mathrm{hlo}}$

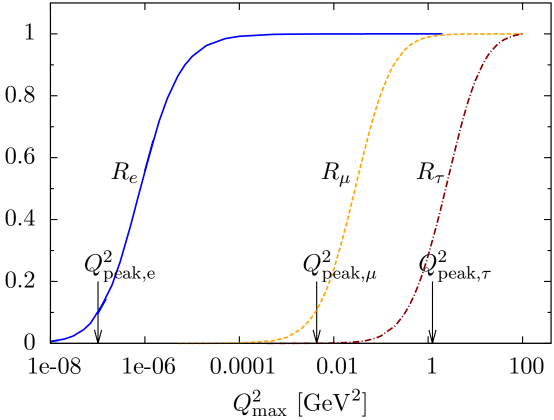



$$a_l^{\mathrm{hlo}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w\left(Q^2/m_l^2\right)\, \Pi_{\mathrm{R}}(Q^2)$$

- requires subtracted vacuum polarization function $\Pi_{\mathrm{R}}(Q^2) = \Pi(Q^2) - \Pi(Q^2 = 0)$
- requires $\Pi(Q^2)$ for arbitrarily low momenta, in particular $Q^2 < Q^2_{\mathrm{min}} = (2\pi/L)^2 \sim 0.25\,\mathrm{GeV}^2$
- requires $\Pi(Q^2)$ for arbitrarily large momenta, in particular $Q^2 > Q^2_{\mathrm{max}} = 16/a^2 \sim 100\,\mathrm{GeV}^2$
- lepton mass $m_l$ introduced as external scale, not tied to lattice parameters
- lattice artifacts

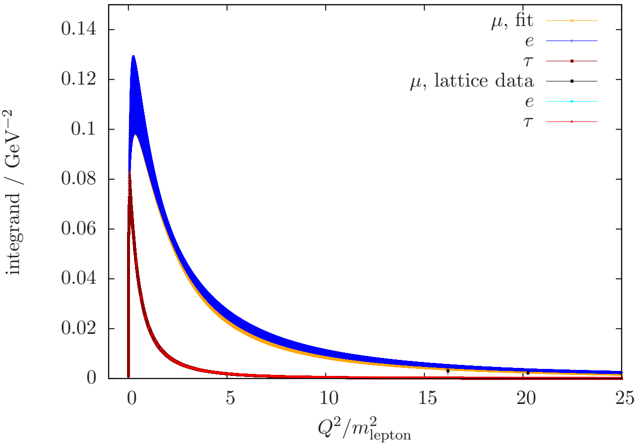
# Lattice calculation - integral saturation

$$R_l(Q^2_{\max}) = \frac{a_l^{\rm hvp}(Q^2_{\max})}{a_l^{\rm hvp}(100\,{\rm GeV}^2)}\,,$$




example data for $a = 0.078\,{\rm fm}$, $L = 2.5\,{\rm fm}$, $m_{PS} = 393\,{\rm MeV}$

# Lattice calculation - integrands




- example data for $a = 0.061\,\mathrm{fm}$, $m_{PS} = 227\,\mathrm{MeV}$, $L = 2.9\,\mathrm{fm}$
- $Q^2/m_l^2 = 5$ means $1.3 \times 10^{-6}\,\mathrm{GeV}^2$ (electron), $0.056\,\mathrm{GeV}^2$ (muon), $16\,\mathrm{GeV}^2$ (tau)

## Lattice calculation - momentum dependence of Π

- fit momentum dependence and extrapolate to zero momentum based on analyticity at the origin, spectrum, resonance features, ...
- interpolate for all momenta up to $Q^2_{\mathrm{max}} = 16/a^2$
- polynomial, VMD, Padé functions [Phys.Rev. D86 (2012) 054509], MNBC-scheme (used here), pQCD
  $\longrightarrow$ systematic approach with correct convergence properties in the limits $m_{PS} \to m_\pi$ and $V \to \infty$
- MNBC fit ansatz for each flavor $l$, $s$, $c$

$$\Pi_{\mathrm{low}}(Q^2) = \sum_{i=1}^{M} \frac{f_i^2}{m_i^2 + Q^2} + \sum_{j=0}^{N-1} a_j (Q^2)^j$$

$$\Pi_{\mathrm{high}}(Q^2) = \log(Q^2) \sum_{k=0}^{B-1} b_k (Q^2)^k + \sum_{l=0}^{C-1} c_l (Q^2)^l$$

$$\Pi(Q^2) = (1 - \Theta(Q^2 - Q^2_{\mathrm{match}}))\Pi_{\mathrm{low}}(Q^2) + \Theta(Q^2 - Q^2_{\mathrm{match}})\Pi_{\mathrm{high}}(Q^2)$$

with Heaviside step function $\Theta$ and $\{(f_i, m_i) \,|\, i = 1, \ldots, M\}$ from fits to time-dependent single-flavor quark vector current correlators

## Lattice calculation - modified observables

$$a_l^{\mathrm{hlo}} = \alpha^2 \int_0^\infty \frac{d\hat{Q}^2}{\hat{Q}^2}\, w\left(\hat{Q}^2/(am_l)^2\right)\, \Pi_{\mathrm{R}}(\hat{Q}^2)\,, \quad \hat{Q} = aQ$$

- introduction of external scale $m_l$ implies the need of the lattice spacing $a$ to have $am_l$ in lattice units
- dimensionless $a_\mu^{\mathrm{hlo}}$ becomes effectively dependent on QCD scales in the lattice calculation
- new family of observables

$$a_{\bar{l}}^{\mathrm{hlo}} = \alpha^2 \int_0^\infty \frac{d\hat{Q}^2}{\hat{Q}^2}\, w\left(\frac{\hat{Q}^2}{H^2}\frac{H_{\mathrm{phys}}^2}{m_l^2}\right) \Pi_{\mathrm{R}}(\hat{Q}^2)$$

- $H$ hadronic scale on the lattice such that simultaneously

$$\lim_{m_{PS}\to m_\pi} H = H_{\mathrm{phys}} \quad \text{and} \quad \lim_{m_{PS}\to m_\pi} a_{\bar{l}}^{\mathrm{hlo}} = a_l^{\mathrm{hlo}}$$

- solution to external scale problem, $a_{\bar{l}}^{\mathrm{hlo}}$ becomes dimensionless lattice quantity
- our (empirical) choice due to strong correlation in pion mass dependence: $H = m_V$, light vector meson mass with $m_V \to m_\rho$

## Lattice calculation - extrapolation and systematic uncertainties

extrapolation to the physical point, $m_{PS} \to m_\pi$ and continuum $a \to 0$ using

$$a_l^{\mathrm{hlo}}(m_{PS},\, a) = A + B_1\, m_{PS}^2 \left(+B_2\, m_{PS}^4 + \ldots\right) + C\, a^2$$

- finite size effects - variation of range in $m_{PS}\, L$
- chiral extrapolation - variation of range in $m_{PS}$
- vector meson fit ranges - variation of $[t_{\mathrm{min}},\, t_{\mathrm{max}}]$
- dependence on fit function for $\Pi(Q^2)$ / finite volume effects - variation of number of $MN$ terms
- quark-disconnected contribution - at present level of accuracy no statistically significant impact
- sea - valence quark mass matching uncertainties - variation of valence quark mass, negligible

# Results - muon anomalous magnetic moment

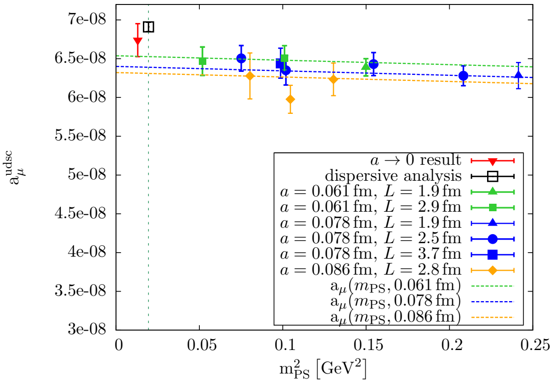



Figure : Combined chiral and continuum extrapolation of four-flavor muon anomalous magnetic moment, comparison with dispersive analysis from [Nucl. Phys. Proc. Suppl. 181-182 (2008) 2631]

$$a_\mu^{\rm hlo} = 6.78(24) \cdot 10^{-8} \quad (N_f = 2+1+1)$$
$$a_\mu^{\rm hlo} = 6.91(05) \cdot 10^{-8} \quad \text{(dispersive analysis)}$$

## Results - Standard Model lepton anomalous magnetic moments

Results for all three SM leptons

| | $a_e^{\mathrm{hlo}}$ | $a_\mu^{\mathrm{hlo}}$ | $a_\tau^{\mathrm{hlo}}$ |
|---|---|---|---|
| tmLQCD | $1.782(64)(45) \cdot 10^{-12}$ | $6.78(24)(18) \cdot 10^{-8}$ | $3.41(8)(5) \cdot 10^{-6}$ |
| disp. analyses | $1.866(10)(05) \cdot 10^{-12}$ [1] | $6.91(01)(05) \cdot 10^{-8}$ [2] | $3.38(4) \cdot 10^{-6}$ [3] |

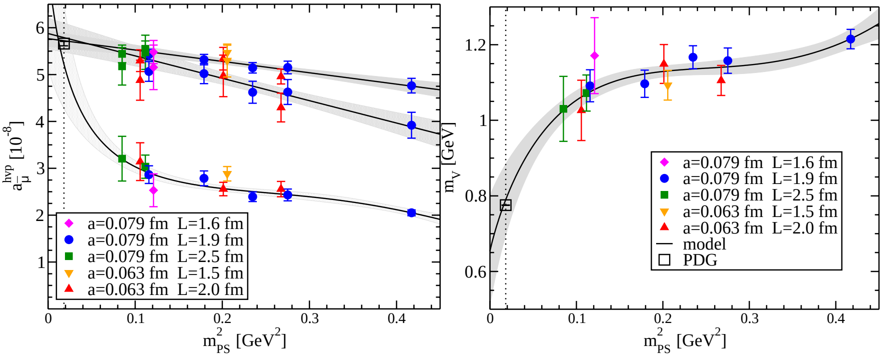
- [1] = [Nucl.Phys. B867 (2013) 236-243], [2] = [Eur.Phys.J. C71 (2011) 1632], [3] = [Mod.Phys.Lett. A22 (2007) 159-179]
- including systematic uncertainties from chiral extrapolation, continuum limit and finite volume
- excluding quark-disconnected diagrams

## Conclusions and outlook

- calculated hadronic leading-order contribution to anomalous magnetic moment for all three Standard Model leptons
- agreement with phenomenological analyses
- probes the lattice vacuum polarization in different momentum regions
- control over pion mass dependence due to modified observables $a_{\bar{l}}^{\mathrm{hlo}}$
- combination with $\rho$-resonance analysis (ongoing)
- calculation with ETMC gauge field ensembles featuring up, down, strange and charm quark at the physical point (planned)
- higher statistical precision for $\Pi$ with exact deflation
- effects due to $u - d-$mass splitting, QED-effects

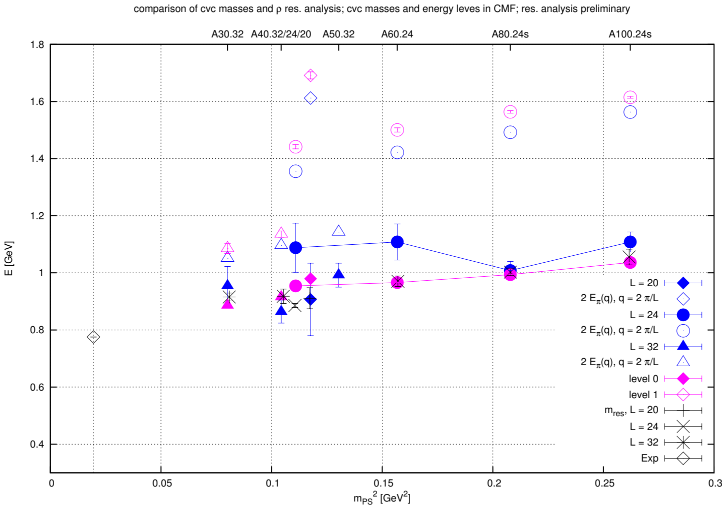
Thank you very much for your attention.

# Lattice calculation - external scales problem




model-dependent extrapolation of $a_\mu^{\mathrm{hlo}}$ (left) and $m_V$ (right) to the physical pion mass

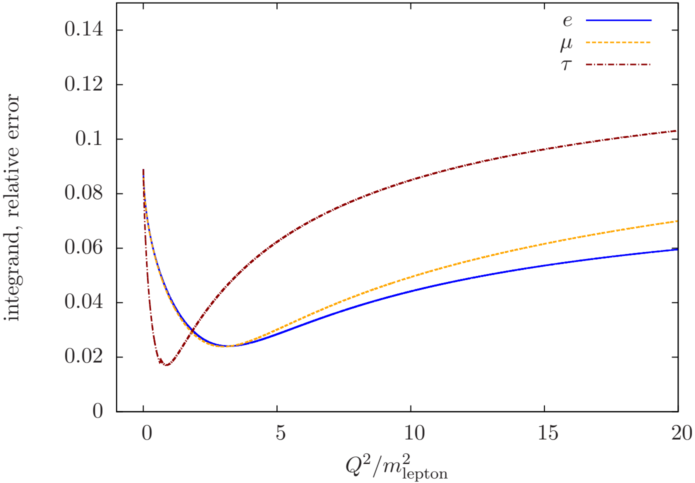
# Lattice calculation - combination with $\rho$-resonance analysis




comparison of energy levels from resonance analysis and vector meson channel for ensembles with $a = 0.086\,\mathrm{fm}$

preliminary data from Markus Werner, HISKP

# Lattice calculation - relative error of the integrand




example data for $a = 0.061\,\mathrm{fm}$, $L = 2.9\,\mathrm{fm}$, $m_{PS} = 318\,\mathrm{MeV}$

# Lattice calculation - systematic uncertainty for $a_l^{\rm hlo}$

- finite size effects - $3.35 < m_{PS} L < 5.93 \to m_{PS}\, L > 3.8$

$$a_\mu^{\rm hlo} = 6.73\,(25) \cdot 10^{-8}$$

| Ensemble | $L/\,{\rm fm}$ | $a_{\mu,{\rm ud}}^{\rm hvp}$ | $a_\mu^{\rm hvp}$ |
|---|---|---|---|
| B35.32 | 2.5 | $5.45\,(18) \cdot 10^{-8}$ | $6.35\,(19) \cdot 10^{-8}$ |
| B35.48 | 3.7 | $5.46\,(18) \cdot 10^{-8}$ | $6.44\,(19) \cdot 10^{-8}$ |

- chiral extrapolation - $227\,{\rm MeV} \le m_{PS} \le 491\,{\rm MeV} \to m_{PS} < 400\,{\rm MeV}$

$$a_\mu^{\rm hlo} = 6.76\,(26) \cdot 10^{-8}$$

- vector meson fit ranges - variation of $[t_{\rm min},\, t_{\rm max}]$

$$\Delta_V = 0.13 \cdot 10^{-8}$$

- dependence on fit function for $\Pi(Q^2)$ / finite volume effects - $M = 1, 2$ and $N = 2, 3, 4$

$$\Delta_{MNBC} = 0.09 \cdot 10^{-8}$$

- quark-disconnected contribution - at present level of accuracy no statistically significant impact