The muon anomalous magnetic moment: a theory review

Kirill Melnikov

TTP KIT

Outline

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1) QED

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3) Hadronic light-by-light

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E821 and the new experiment at FNAL

The latest measurements of the muon anomalous magnetic moment in the Brookhaven experiment left us with an interesting puzzle: theoretical and experimental results for g-2 differ by about three standard deviations:

$$a_\mu^{\rm exp} = 116\ 592\ 089(63) \times 10^{-11}, \quad a_\mu^{\rm th} = 116\ 591\ 830(50) \times 10^{-11}$$

$$a_\mu^{\rm exp} - a_\mu^{\rm th} = (259 \pm 81) \times 10^{-11}$$

The new experiment at FNAL aims at reducing the experimental error by a factor of four. Assuming no changes in the central value, the above discrepancy will increase to 5.1 standard deviations. By accepted standards, this will qualify as a discovery.

Given the strong potential of the g-2 experiment to clarify the situation, it is important to scrutinize the theoretical prediction once again and ensure that the theoretical result is actually correct within the estimated uncertainty.

The budget

Let us briefly go through the various contributions to muon g-2 to identify what is relevant at the current level of precision.

QED $$a_\mu^{\mathrm{QED}} = 116584718.95(8) \times 10^{-11}$$

Electroweak $$a_\mu^{\mathrm{EW}} = (154 \pm 2) \times 10^{-11}$$

The LO hadronic vacuum polarization $$a_\mu^{\mathrm{HVP,LO}} = (6949 \pm 37.2 \pm 21.0) \times 10^{-11}$$

The NLO hadronic vacuum polarization $$a_\mu^{\mathrm{HVP,NLO}} = -98.4 \times 10^{-11}$$

The hadronic light-by-light scattering contribution $$a_\mu^{\mathrm{HLBL}} = (105 \pm 26) \times 10^{-11}$$

The grand total: $$a_\mu^{\mathrm{th}} = 116591830(50) \times 10^{-11}$$

The dominant errors are hadronic vacuum polarization and hadroinic light-by-light.
Everything else is -- at least nominally -- well-understood.

The message

I would like to emphasize that the current understanding of g-2 theory practically excludes the possibility the the current level of discrepancy between theory and experiment is a consequence of a major theoretical blunder. It is possible to use simple physical considerations to estimate and check the order of magnitude of all the relevant theory contributions. Although we still would welcome an independent check for various hadronic contributions (e.g. as provided by the lattice), it is highly improbable that these checks will give us a completely different version of the g-2 story.

The message therefore should be clear: given the current level of discrepancy and all the stress-tests that the theory of g-2 was subject to, we have to conclude that the discrepancy is caused by one of the three things (or their combination):

1) experimental issues;

2) a cocktail of small (one-sigmish) shifts in central values of theoretical and experimental results, all working in the same direction;

3) BSM physics.

The new experiment at Fermilab will be most likely be able to tell us what it is.

QED

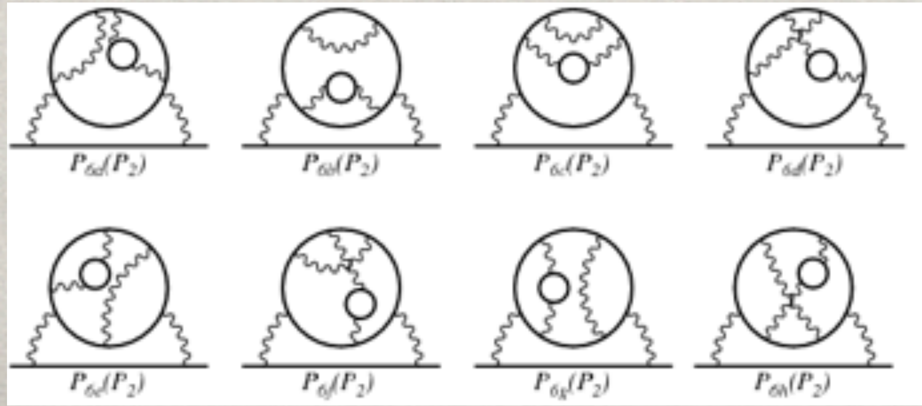
For almost all of us, the current computations of QED corrections to muon g-2 are not comprehensible. We do not know if they are right or wrong. Yet, we must take these results as they are reported and make our conclusions about the significance of the discrepancy in g-2 based on them. Given how large QED contribution is, a small change in it can cause a lot of trouble. So why are we so sure that QED results are OK?

The answer is simple: for the current level of discrepancy we only need the three-loop QED contribution -- which is well-known -- and enhanced contributions at four loops --that are well-understood.

An interesting thing though is that there are two enhancement parameters:

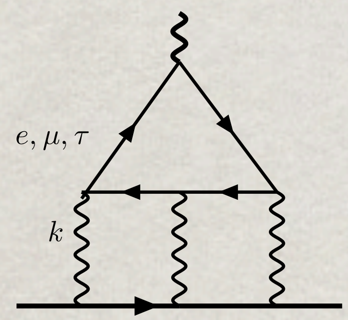
$$\ln \frac{m_\mu}{m_e} \sim 5, \qquad \pi^2 \sim 9$$

The logarithm is the consequence of RG-like running either of the fine structure constant or of more complex objects that can be thought of as some ``effective operators''. The π^2 enhancement shows up because several (even number) of poles of the muon propagator can contribute to the final result.



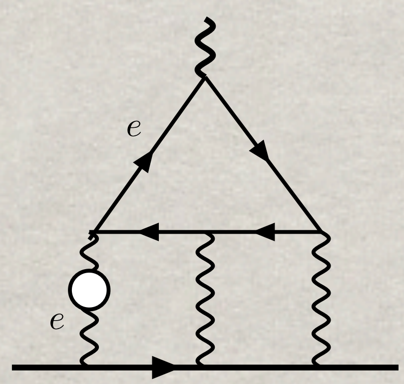
QED

1) The three-loop contribution is very well-established. An important point is that the dominant contribution is provided by the light-by-light scattering diagram with the electron loop. All other contributions are much less important, changing the complete 3-loop result by about 4%.



$$\sim \left(\frac{\alpha}{\pi}\right)^3 \frac{2\pi^2}{3} \ln \frac{m_\mu}{m_e}$$

2) At the four loops, the pattern repeats itself and the dominant contribution comes from same diagrams but with additional insertion of the lepton vacuum polarization. The part of the four-loop result that is not enhanced is not important at the current level of precision.



$$a_\mu^{(4)} = \left(\frac{\alpha}{\pi}\right)^4 (132.68|_{\rm lbl,vp} - 1.75) \qquad a_\mu^{(4),{\rm approx}} = 117.4 \left(\frac{\alpha}{\pi}\right)^4$$

$$a_\mu^{(4)} - a_\mu^{(4),{\rm approx}} \approx 40 \times 10^{-11} \ll a_\mu^{\rm th} - a_\mu^{\rm exp}$$

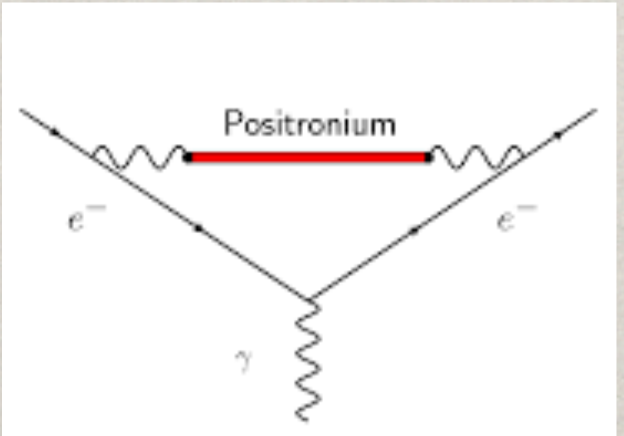
QED

3) The complete five-loop result was recently obtained by Aoyama, Hayakawa, Kinoshita and Nio . The result is well within the range of the earlier estimates of five loop results that utilizes the known mechanisms to enhance higher order results.

$$a_\mu^{(5)} = 753.29 \left(\frac{\alpha}{\pi}\right)^5 \approx 5 \times 10^{-11} \qquad a_\mu^{(5),\mathrm{approx}} \sim (500 - 1000) \times \left(\frac{\alpha}{\pi}\right)^5$$

4) We had an interesting discussion about positronium contribution to g-2 recently. The positronium contribution appears at the five-loop order (i.e. it is $\mathcal{O}(\alpha^5)$) but it requires a summation of infinitely many Feynman diagrams so it is hard to imagine that this is part of regular perturbative series. But it is...(the g-2 is an Euclidean observable).

Interestingly, this is a direct illustration of ``duality'' between resonances and continuum that is crucial for estimates of hadronic properties using the method of QCD sum rules.



$$a_\mu^{\mathrm{PS}} = \frac{\alpha^5}{8\pi}\zeta(3)$$

QED summary

In summary : we know enough about QED contributions to the muon magnetic anomaly to say with absolute confidence that the current discrepancy between theory and experiment is not caused by it.

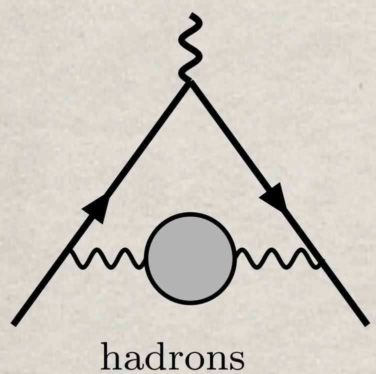
Even if one is willing to doubt the validity of four- and five-loop QED results without an independent confirmation, this can only change the current results by $O(40) \times 10^{-11}$, which is about one half of the current standard deviation.

However, even this scenario should be considered as an extremely improbable.

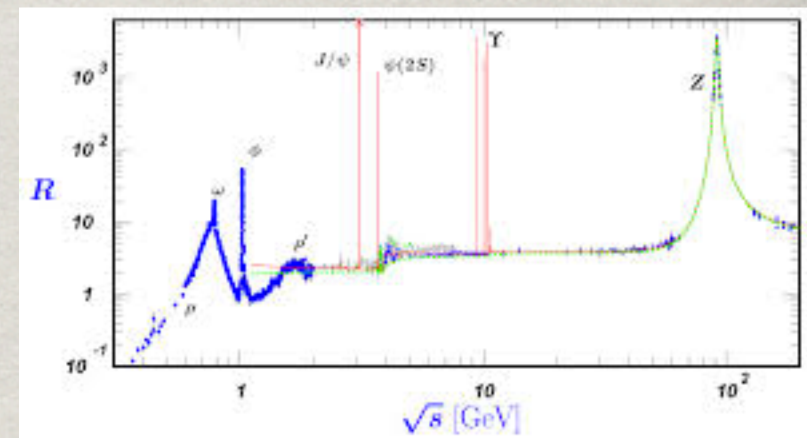
Hadronic vacuum polarization

Hadronic vacuum polarization contribution is obtained by integrating the cross section for e+e- -> hadrons weighted with a kernel computable in perturbation theory.

$$a_\mu^{\text{hvp}} = \frac{\alpha}{3\pi}\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s}R^{\text{hadr}}(s)a_\mu^{(1)}(s) \qquad R(s) = \frac{\sigma_{e^+e^-\to \text{hadr}}}{\sigma_{e^+e^-\to\mu^+\mu^-}}$$



Given the measurement of e+e- -> hadrons, it is straightforward to obtain its contribution to the muon magnetic anomaly. But it is also possible to estimate it.



We can represent the spectral density as the sum of three terms:

1) chirally-enhanced two-pion contribution $\quad a_\mu^{\pi\pi} \approx 400\times 10^{-11}, \quad 4m_\pi^2 < s < m_\rho^2/2$

2) vector mesons $\quad \sigma_{e^+e^-\to V} = \frac{12\pi^2\Gamma_{V\to e^+e^-}}{m_V}\delta(s-m_V^2) \quad a_\mu^{\rho,\omega,\phi} \approx 5514\times 10^{-11}$

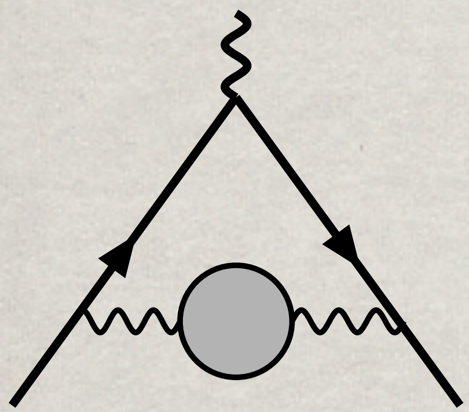
3) continuum $\quad s_0 = 1\ \text{GeV}^2, \quad a_\mu^{\text{cont}} = 1240\times 10^{-11}$

$$a_\mu^{\text{th}} \approx 7160\times 10^{-11}$$

Since it is very easy to obtain the result in the right ballpark, one should be very cautious when judging the success (or lack of it) of prospective lattice results on hadronic vacuum polarization contribution.

Hadronic vacuum polarization

This estimate compares very well with the results of detailed analyses that use measurements of e+e- -> hadrons annihilation cross section.

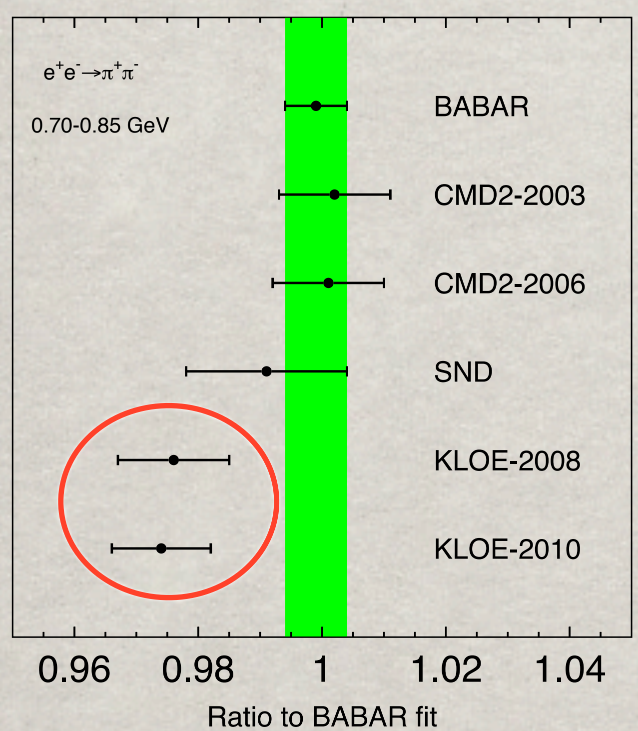


hadrons

$$a_\mu^{\rm hvp} = \frac{\alpha}{3\pi}\int\limits_{s_0}^{\infty} \frac{{\rm d}s}{s} R^{\rm hadr}(s) a_\mu^{(1)}(s) \qquad R(s) = \frac{\sigma_{e^+e^-\to {\rm hadr}}}{\sigma_{e^+e^-\to\mu^+\mu^-}}$$

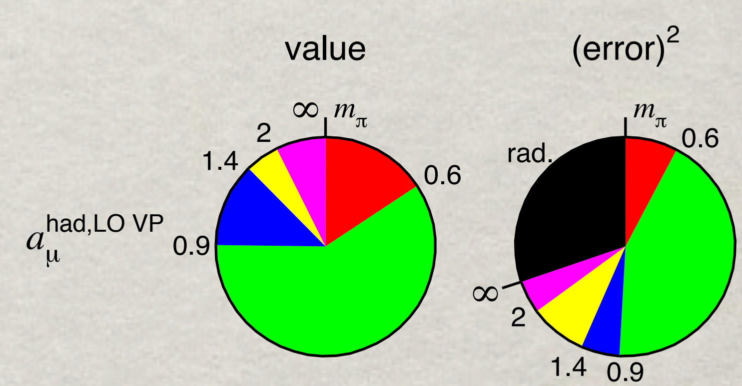
A ``recent" compilation gives $a_\mu^{\rm HVP,LO} = (6949 \pm 37.2 \pm 21.0) \times 10^{-11}$

Hagiwara et al.



Comparison of contributions to g-2 from around the rho meson as measured in different experiments (Davier, Malaescu)

The major contributions to this result and to the error by the energy region are shown in the pie diagram to the right.



In practice, important issues are:

1) compatibility of different data sets (CMD, SND, BABAR, KLOE);

2) usefulness of tau-data for hadronic vacuum polarization (my opinion -- not a good idea, one should not keep repeating the same mistake again and again);

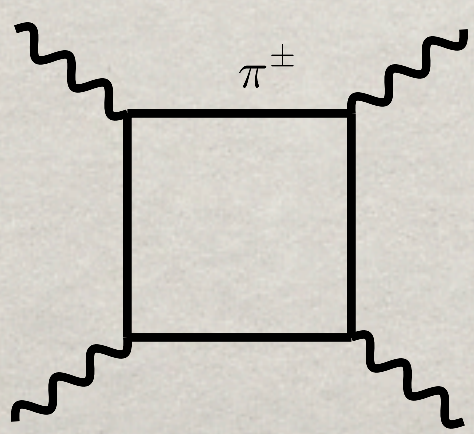
Hadronic light-by-light

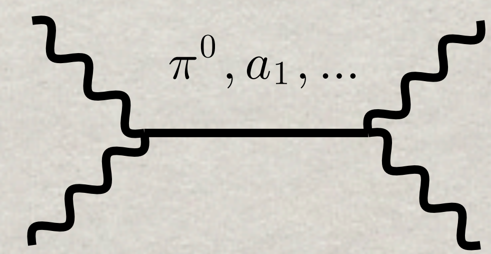
The hadronic light-by-light scattering contribution is estimated to be 105(26) x 10^{-11}. It is not large and the 25% error -- in principle -- can well be tolerated.

However, given the persistent g-2 discrepancy, the volatile history of the hadronic light-by-light contribution and the fact that its calculation is only vaguely constrained by experimental data, there seems to be an uneasy feeling towards it. So let me summarize a few things that we know about hadronic light-by-light scattering.

This contribution is non-perturbative -- the muon mass is small. There are two parameters that we can use to estimate it -- the (small) pion mass and the (large) number of colors.
We know with certainty that the large Nc is a fairly good parameter and the pion mass is not.
We also know that none of these parameters is really large or small -- this means that leading order estimates do not work well and one has to use models/extrapolations to catch important and relevant physics.

$$a_\mu^{\rm lbl} \sim \left(\frac{\alpha}{\pi}\right)^3 \left[c_1 \frac{m^2}{m_\pi^2} + c_2 N_c \frac{m^2}{\Lambda_{\rm QCD}^2} \right]$$





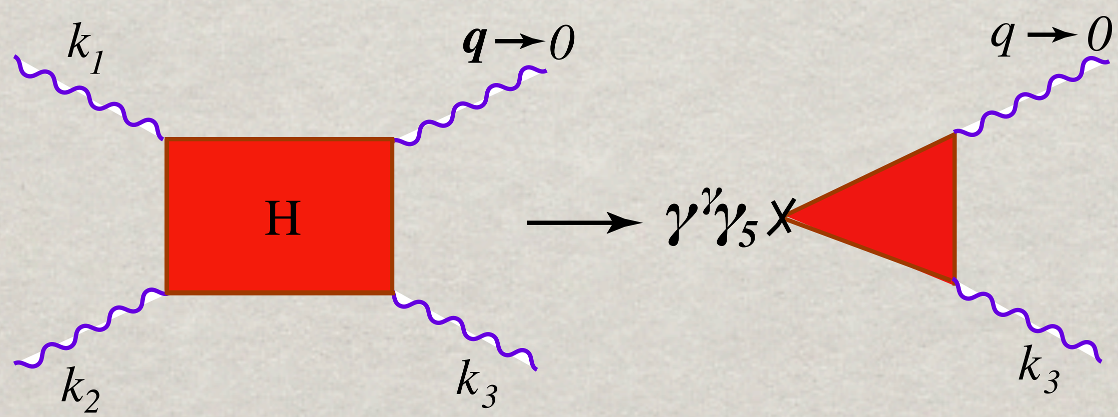
Hadronic light-by-light: exact results

There are two exact statements that can be made about hadronic light-by-light scattering.

1) Working in the large-Nc limit and assuming a large mass gap between the pion and the lightest vector meson, one can calculate the double-logarithmic contribution to HLBL contribution “exactly”.

$$a_\mu^{\rm lbl}[\pi^0] = \left(\frac{\alpha}{\pi}\right)^3 3\left(\frac{N_c}{3}\right)^2 \frac{m^2}{(4\pi F_\pi)^2} \ln^2 \frac{\Lambda}{m_\pi}$$

2) There is an OPE constraint that dictates the behavior of the four-photon scattering amplitude at large virtualities of the photons. This constraint prohibits any (transition) form factor in the (off-shell) pion-photon-photon interaction vertex where one of the photons is soft.



These constraints restrict the light-by-light scattering amplitude at small and large momentum transfers, making them useful for phenomenology.

Hadronic light-by-light

One can argue that the large Nc contribution is relatively robust. Indeed, the hadronic light-by-light scattering contribution is Euclidean and can be thought of as being ``dual" to a constituent (massive) quark loop. The quark mass is estimated by requiring that constituent quark loop reproduces the hadronic vacuum polarization contribution.

The stability of this approach can be probed in several ways. For example, one can repeat the same exercise including additional contributions with ``gluon" exchanges or modifying the theory in a way that combines a neutral pion and the constituent quark loop. Whatever one does, the result is quite stable.

$$a_\mu^{N_c}|_{\rm quark} \approx (120 - 150) \times 10^{-11}$$

Current estimates of the large-Nc contributions to HLBL part of g-2 utilize models of hadronic light-by-light scattering based on pseudo-scalar and pseudo-vector exchanges, subject to two constraints listed on a previous slide. The result for the large Nc part of HLBL contribution to g-2 is accepted to be

$$a_\mu^{\rm N_c} = (128 \pm 13) \times 10^{-11}$$

Hadronic light-by-light

The major current issue in the theory of hadronic light-by-light seems to be the chirally enhanced, sub-leading in Nc contribution (the ``charged pion" box). The problem can be stated very clearly by looking at three numbers : the true chiral pion loop and the pion loops dressed up to include higher-mass vector resonances in two different ways (vector meson dominance and hidden local symmetry)

$$a_\mu^\pi = -43 \times 10^{-11}, \quad a_\mu^{\mathrm{VDM}} = -4.5 \times 10^{-11}, \quad a_\mu^{\mathrm{HLS}} = -19 \times 10^{-11}$$

An explanation of this behavior is that the charged pion contribution is somehow sensitive to larger values of the loop momenta where the importance of ``hadronic" modifications actually kicks in. Also, one can learn from these results that first non-trivial power correction to the chiral limit is insufficient; one needs quite a number of such terms to obtain the correct result.

The accepted current value of the pion box contribution is:

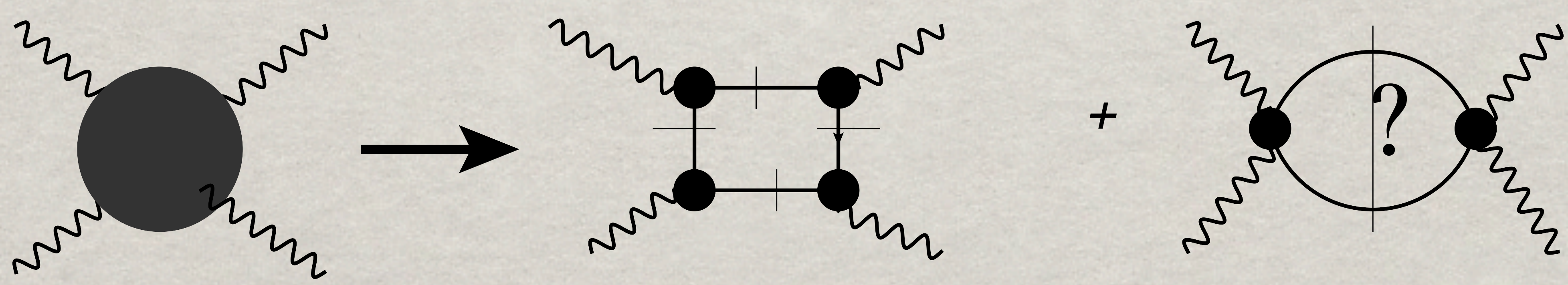
$$a_\mu^\pi \approx (-19 \pm 19) \times 10^{-11}$$

Hadronic light-by-light

To what extent new approaches suggested recently for computing hadronic light-by-light scattering contributions can address this issue?

One recent idea is to use dispersion relations to make computations of hadronic light-by-light scattering contributions less model-dependent. To this end, one represents the sub-leading Nc contributions as a sum of two terms: the first term (four-pion cut) is ``simple" and unique, the second term (two-pion cut) is ``difficult"; its analysis relies on partial wave expansion.

Note, however, that the first term (four-pion) is the same in VMD and HLS models (and any other model that does not do horrible things to a pion form factor). The difference is in the second (problematic !) term. The pion polarizability --- a potentially large contribution discussed recently -- is also in the second (two-pion-two-photon) term.



Conclusions

I believe that after O(10) years of searching for reasons for the discrepancy between the E821 result and theoretical prediction for muon g-2, we can say with confidence that missed SM effects as large as 260×10^{-11} can be excluded.

The three logical possibilities are then 1) coherent combination of small effects in theory and experiment that reduces the discrepancy to an ``acceptable" level; 2) some experimental issue or 3) physics beyond the Standard Model.

The first option can be made less and less probable by systematically reducing all the relevant uncertainties. On the experimental side, this will happen thanks to the new FNAL experiment. On the theory side, this may happen as well, but it is more problematic. The new experiments (Novosibirsk and BEPC) will help to understand hadronic vacuum polarization better. But improving understanding of hadronic light-by-light -- and in particular the part subleading in Nc - is very difficult.

The second option -- experimental issue with E821 result -- will be resolved thanks to an independent measurement at FNAL.

The third option is, arguably, the most exciting; the likely candidate is still the supersymmetry with relatively small chargino and neutralino masses and relatively large value of tan(beta). Here the interplay with direct measurements at the LHC is crucial but so far there is no contradiction between the LHC data and the masses of electroweakinos required to explain the muon magnetic anomaly. The phase-space for other theoretical possibilities -- e.g. dark photons -- is being continuously squeezed.