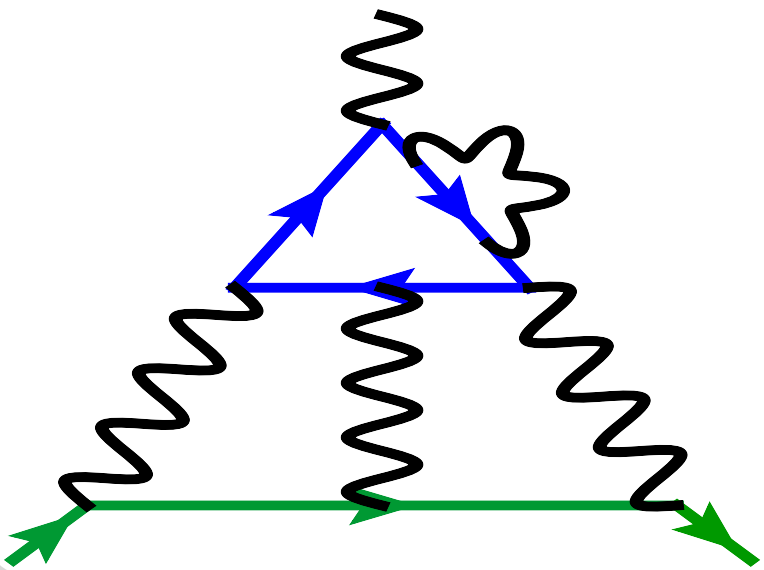
Karlsruhe Institute of Technology

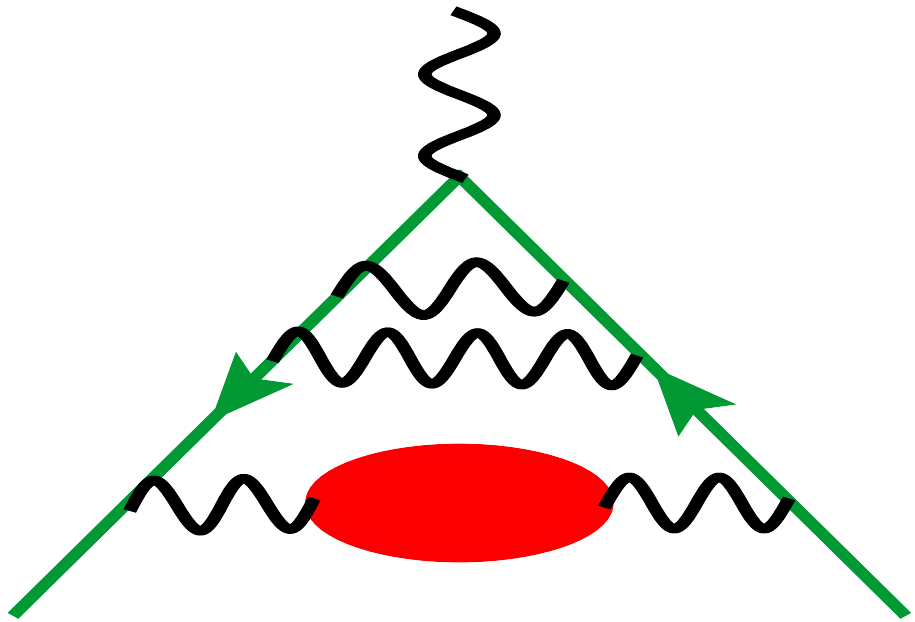
Higher order hadronic and leptonic contributions to the muon $g-2$

Matthias Steinhauser

in collaboration with A. Kurz, T. Liu, P. Marquard, A. Smirnov, V. Smirnov | FCCP, September 10-12 2015

TTP KARLSRUHE

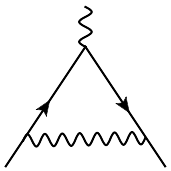




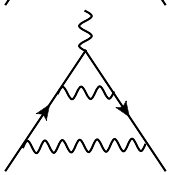
Higher order hadronic and leptonic contributions to the muon $g-2$

I. NNLO hadronic contribution
II. Electron contribution at 4 loops
III. Tau contribution at 4 loops

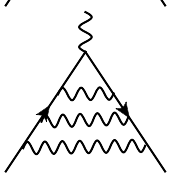
Muon $g-2$



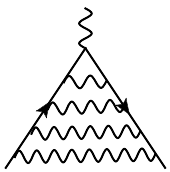
[Schwinger'48]



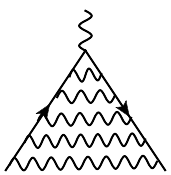
[Petermann'57;
Sommerfeld'58]



[Laporta,Remiddi'96;
. . . ; Passera'06]

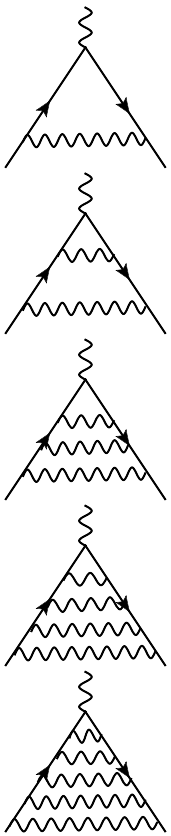


[Laporta'93;
Kinoshita,Nio'06;
Aoyama,Hayakawa,
Kinoshita,Nio'07'08]



[Aoyama,Hayakawa,Kinoshita,Nio'12]

Muon $g-2$



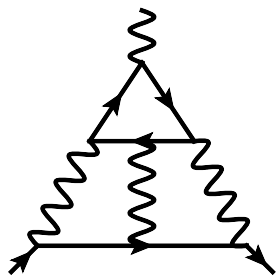
$$0.5\,\frac{\alpha}{\pi}$$

$$(-0.328\ldots+1.094\ldots|_e)\,\frac{\alpha^2}{\pi^2}$$

$$(1.181\ldots+22.868\ldots|_e)\,\frac{\alpha^3}{\pi^3}$$

$$(-1.910\ldots+132.682\ldots|_e)\,\frac{\alpha^4}{\pi^4}$$

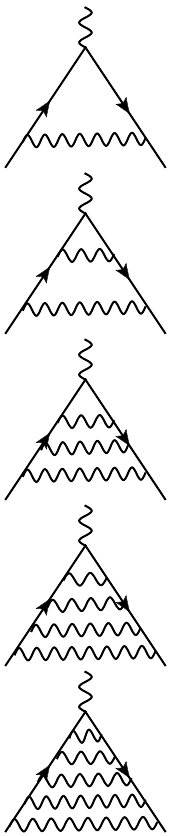
$$(9.168(571)+742.18(87)|_e)\,\frac{\alpha^5}{\pi^5}$$



Muon $g-2$

- $a_\mu(\mathrm{exp}) - a_\mu(\mathrm{SM}) \sim 249(87) \times 10^{-11}$
- $a_\mu(\text{4 loop QED}) \sim 381 \times 10^{-11}$

[Aoyama,Hayakawa,Kinoshita,Nio'12]



$$0.5\,\frac{\alpha}{\pi}$$

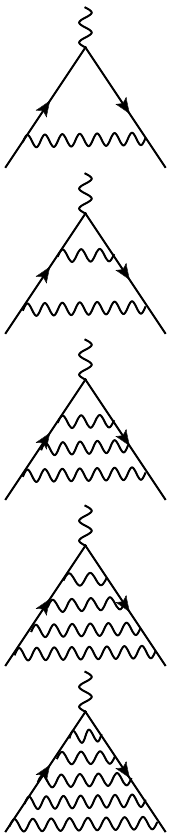
$$(-0.328\ldots + 1.094\ldots|_e)\,\frac{\alpha^2}{\pi^2}$$

$$(1.181\ldots + 22.868\ldots|_e)\,\frac{\alpha^3}{\pi^3}$$

$$(-1.910\ldots + 132.682\ldots|_e)\,\frac{\alpha^4}{\pi^4}$$

$$(9.168(571) + 742.18(87)|_e)\,\frac{\alpha^5}{\pi^5}$$

Muon $g-2$



$$0.5\,\frac{\alpha}{\pi}$$

$$(-0.328\ldots+1.094\ldots|_e)\,\frac{\alpha^2}{\pi^2}$$

$$(1.181\ldots+22.868\ldots|_e)\,\frac{\alpha^3}{\pi^3}$$

$$(-1.910\ldots+132.682\ldots|_e)\,\frac{\alpha^4}{\pi^4}$$

$$(9.168(571)+742.18(87)|_e)\,\frac{\alpha^5}{\pi^5}$$

- $a_\mu(\mathrm{exp})-a_\mu(\mathrm{SM})\sim 249(87)\times 10^{-11}$
- $a_\mu(\text{4 loop QED})\sim 381\times 10^{-11}$

[Aoyama,Hayakawa,Kinoshita,Nio'12]

- aim: check 4-loop result
- independent calculation
- different method

Contributions to muon $g-2$

- QED: $116\,584\,718.962\,(9)(19)(7)(78) \times 10^{-11}$ [Aoyama,Hayakawa,Kinoshita,Nio'12]
- hadronic VP: $6949.1(37.2)(21.0) - 984(6)(4) + 12.4(0.1) \times 10^{-11}$

  [Davier,Höcker,Malaescu,Zhang'10; Hagiwara,Liao,Martin,Nomura,Teubner'11; Jegerlehner,Szafron'11; Benayoun,David,DelBuono,Jegerlehner'12; ...; Kurz,Marquard,Liu,Steinhauser'14]]
- electroweak $154(2) \times 10^{-11}$ [Czarnecki,Krause,Marciano'96; Knecht,Peris,Perrottet,de Rafale'02; Czarnecki,Marciano,Vainshtein'03; Gnendiger,Stöckinger,Stöckinger-Kim'13]
- hadronic lbl: $116(40) \times 10^{-11}$ [Nyffeler'09; Melnikov,Vainshtein'14; Bijnens,Prades'15; ...]

Contributions to muon $g-2$

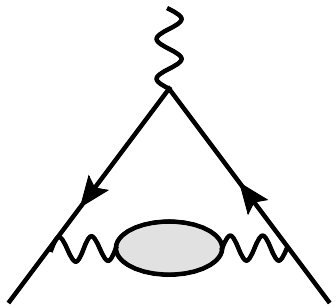
- QED: $116\,584\,718.962\,(9)(19)(7)(78) \times 10^{-11}$ [Aoyama,Hayakawa,Kinoshita,Nio'12]
- hadronic VP: $6949.1(37.2)(21.0) - 984(6)(4) + 12.4(0.1) \times 10^{-11}$

  [Davier,Höcker,Malaescu,Zhang'10; Hagiwara,Liao,Martin,Nomura,Teubner'11; Jegerlehner,Szafron'11; Benayoun,David,DelBuono,Jegerlehner'12; ...; Kurz,Marquard,Liu,Steinhauser'14]]
- electroweak $154(2) \times 10^{-11}$ [Czarnecki,Krause,Marciano'96; Knecht,Peris,Perrottet,de Rafale'02; Czarnecki,Marciano,Vainshtein'03; Gnendiger,Stöckinger,Stöckinger-Kim'13]
- hadronic lbl: $116(40) \times 10^{-11}$ [Nyffeler'09; Melnikov,Vainshtein'14; Bijnens,Prades'15; ...]

$g-2$
NNLO hadronic

Method

$$a_\mu^{(1)} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^{\infty} \mathrm{d}s \frac{R(s)}{s} K^{(1)}(s)$$

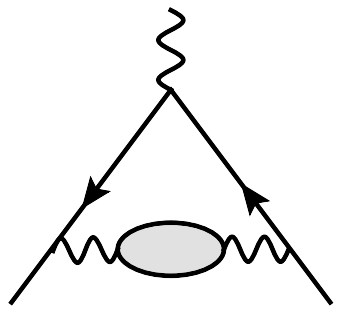


$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha^2/(3s)}$$

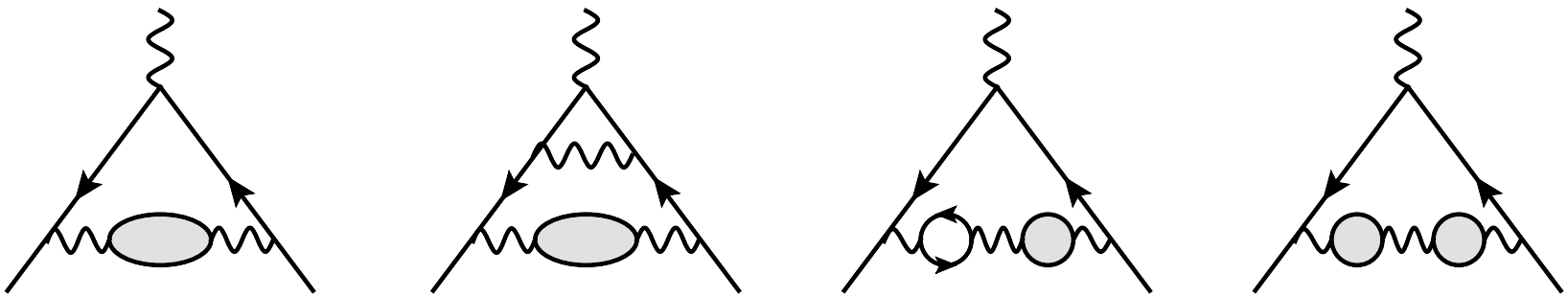
$$K^{(1)}(s) = \int_0^1 \mathrm{d}x \frac{x^2(1-x)}{x^2 + (1-x)\frac{s}{m_\mu^2}}$$

LO: exact result [Brodsky,de Rafael'68]
NLO, NNLO: expand for $s \gg m_\mu^2$
compilation of $R(s)$ from [Nomura,Teubner]

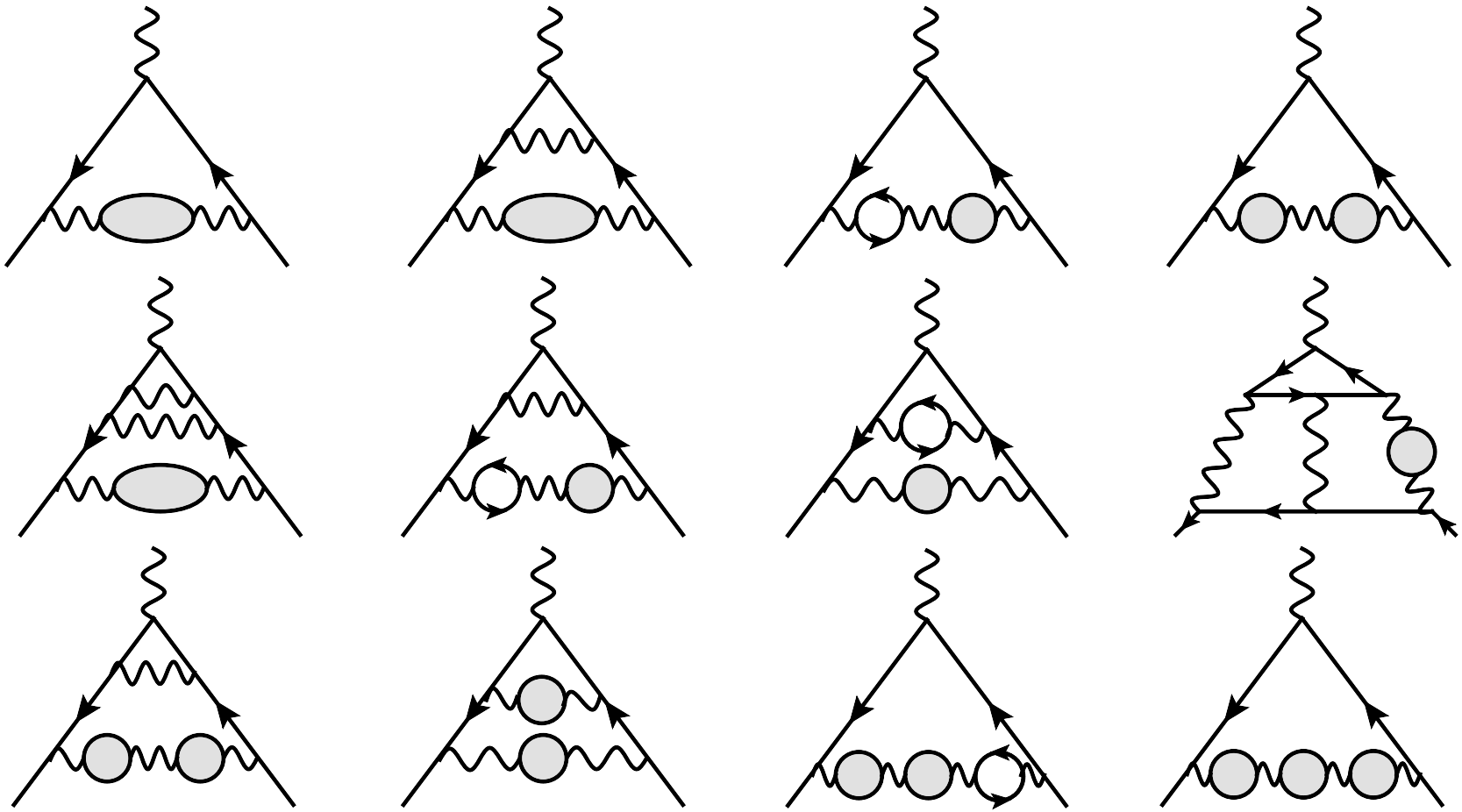
LO, NLO and NNLO contributions



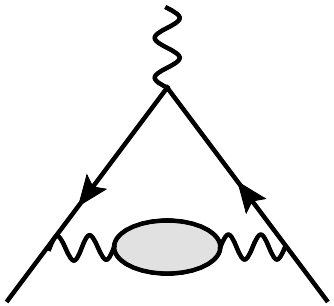
LO, NLO and NNLO contributions



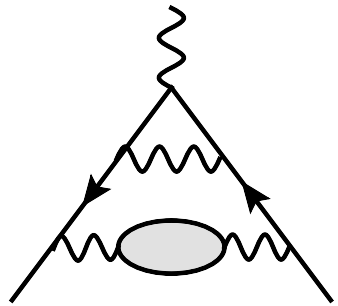
LO, NLO and NNLO contributions



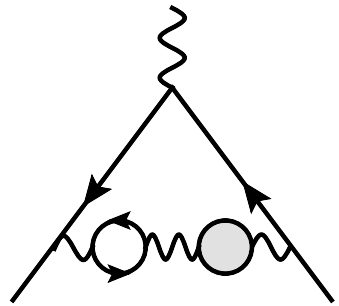
LO, NLO and NNLO contributions ($\times 10^{-11}$)



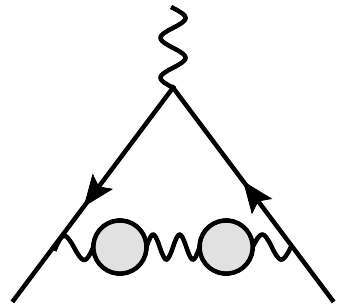
6949



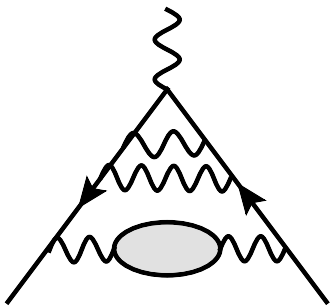
-209.0



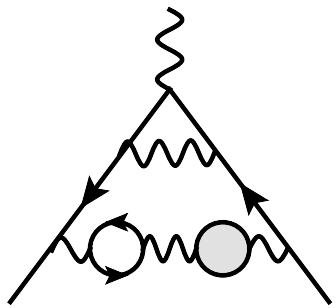
106.8



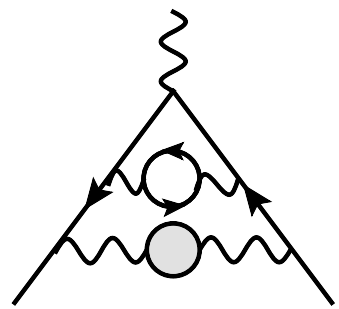
3.5

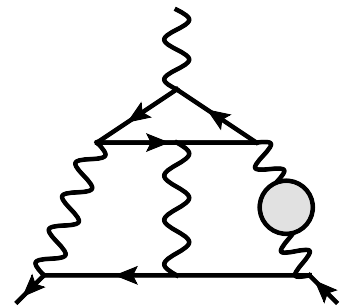


8.0

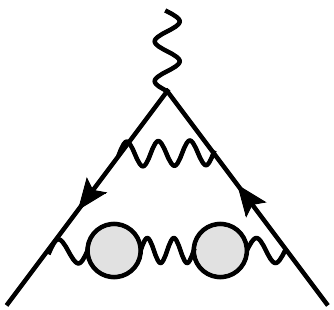


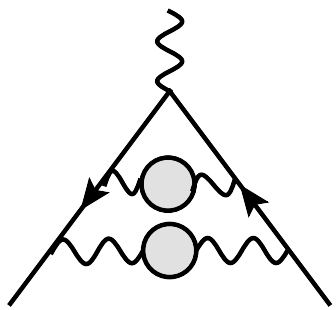
-4.1



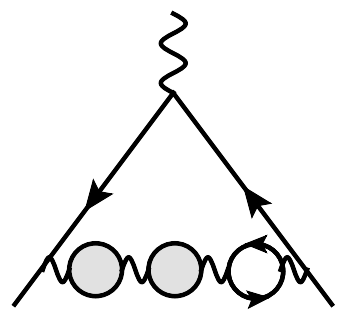


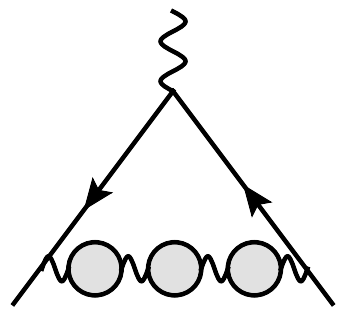
9.1





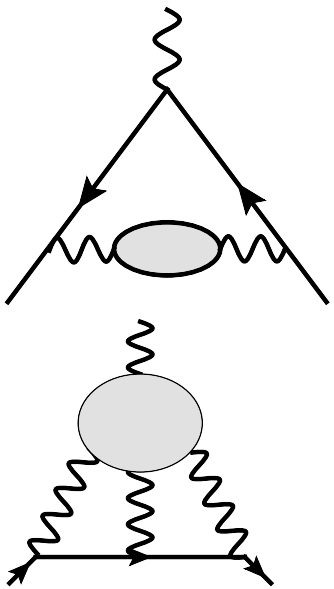
-0.6





0.005

Results a_μ^{had}



LO	6949(37.2)(21.0)
NLO	−98.4(6)(4)
NNLO	12.4(0.1)
LO lbl	116(40)
NLO lbl	3(2)

$(\times 10^{-11})$

LO: [Davier,Höcker,Malaescu,Zhang'10; Hagiwara,Liao,Martin,Nomura,Teubner'11; Jegerlehner,Szafron'11; Benayoun,David,DelBuono,Jegerlehner'12]

NLO: [Krause'96; Hagiwara,Martin,Nomura,Teubner'03; Hagiwara,Liao,Martin,Nomura,Teubner'11; Greynat, de Rafael'12]

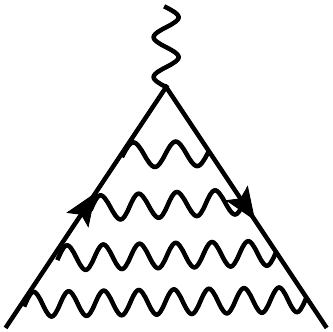
NNLO: [Kurz,Marquard,Liu,Steinhauser'14]

LO lbl: [Nyffeler'09; Melnikov,Vainshtein'14; Bijnens,Prades'15; . . .]

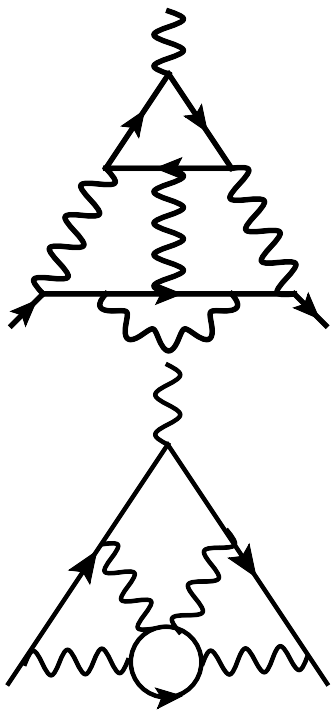
NLO lbl: [Colangelo,Hoferichter,Nyffeler,Passera,Stoffer'14]

$g-2$ – QED
Electrons at 4 loops

QED – electron loops



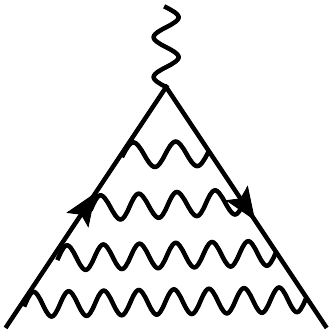
$$(-1.910\ldots + 132.685\ldots|_e)\ \frac{\alpha^4}{\pi^4}$$



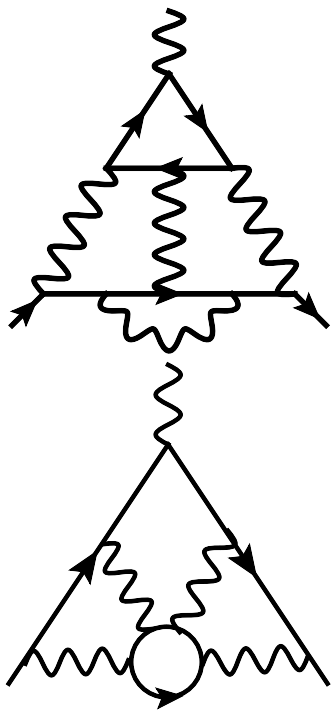
➪ “light-by-light”-type

➪ non-“light-by-light”-type

QED – electron loops



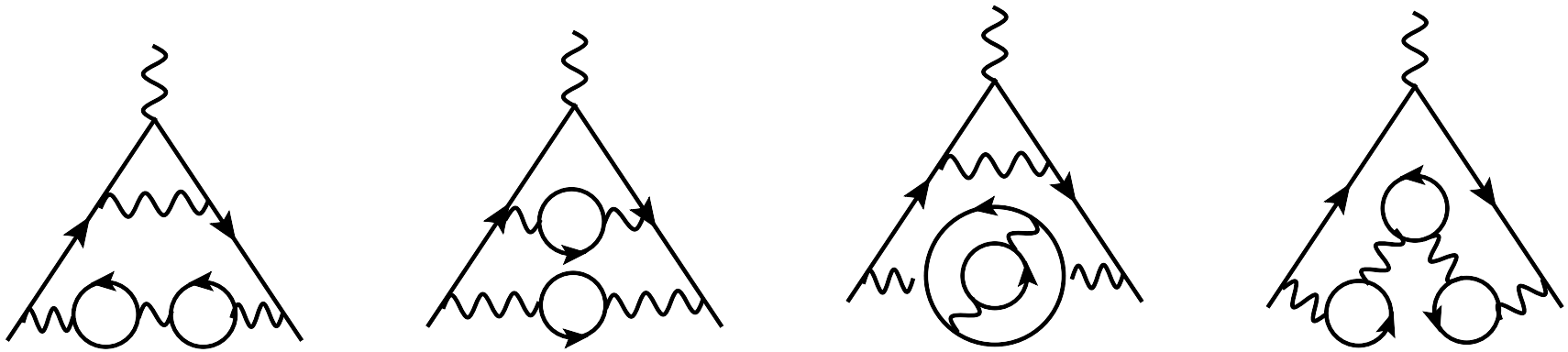
$$(-1.910\ldots + 132.685\ldots|_e)\ \frac{\alpha^4}{\pi^4}$$



➪ “light-by-light”-type $\approx 95\%$

➪ non-“light-by-light”-type

n_l^2 and n_l^3 terms: 2 and 3 e^- loops



- gauge invariant subset
- results $\sim c_0 + c_1 \log(m_e/m_\mu) + c_2 \log^2(m_e/m_\mu) + \ldots + \frac{m_e}{m_\mu}[\ldots]$
- strategy:

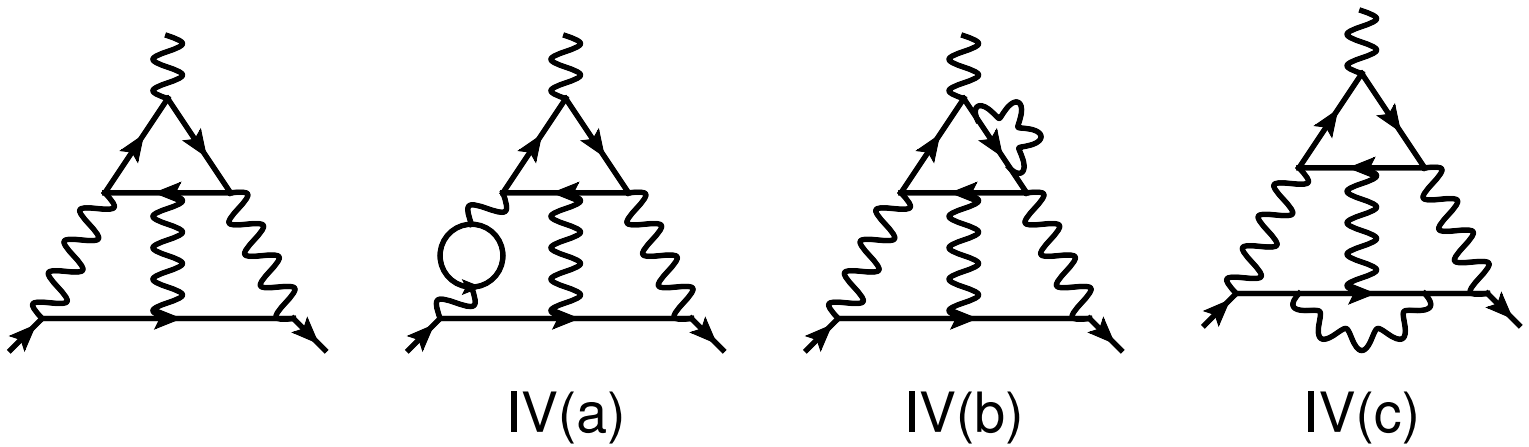
  1.) $m_e = 0, m_\mu^{\rm OS}, \overline{\alpha}(\mu)$ ➪ $\log(\mu/m_\mu)$

  2.) $\overline{\alpha}(\mu) \to \alpha^{\rm OS}$ ➪ $\log(\mu/m_e)$

  } ➪ $\log(m_e/m_\mu)$
- analytic results

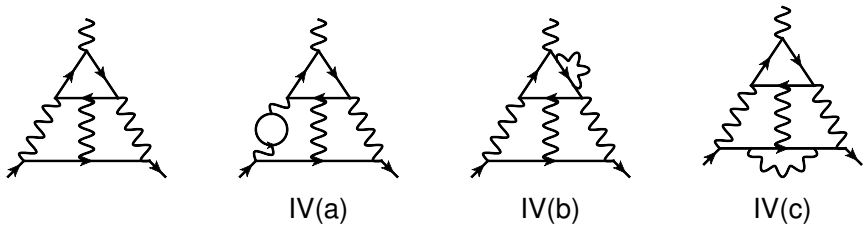
[Laporta'93; Aguilar,Greynat,De Rafael'08; ... Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]

“light-by-light”-type



- $m_e = 0$ not possible ➪ apply methods of region [Beneke,Smirnov’98; Smirnov]

“light-by-light”-type



- $m_e = 0$ not possible ➪ apply methods of region [Beneke,Smirnov’98; Smirnov]
- example: 3-loop diagram
  loop momenta $k, l, r \sim m_\mu$ (“hard”) or $k, l, r \sim m_e$ (“soft”)

  e.g.: k soft, $p^2 = m_\mu^2$ ➪ $\frac{1}{m_\mu^2-(p-k)^2} = \frac{1}{2pk-k^2} \to \frac{1}{2pk}$

➪
- 3-loop OS integrals (3× hard)
- 3-loop “linear” integrals (3× soft)
- 2-loop OS × 1-loop tadpole (2× hard, 1× soft)
- 1-loop OS × 2-loop tadpole (1× hard, 2× soft)
- 1-loop OS × 2-loop “linear” (1× hard, 2× soft)

“light-by-light”-type

- asymptotic expansion for $x \equiv m_e/m_\mu \ll 1$

3 loops

$$
\begin{aligned}
a_\mu \;=\; & a_\mu^{(0,0)} + a_\mu^{(0,1)} \ln \frac{m_\mu}{m_e} \\
& + \frac{m_e}{m_\mu} \left(a_\mu^{(1,0)} + a_\mu^{(1,1)} \ln \frac{m_\mu}{m_e} \right) \\
& + \frac{m_e^2}{m_\mu^2} \left(a_\mu^{(2,0)} + a_\mu^{(2,1)} \ln \frac{m_\mu}{m_e} + a_\mu^{(2,2)} \ln^2 \frac{m_\mu}{m_e} \right. \\
& \qquad \left. + a_\mu^{(2,3)} \ln^3 \frac{m_\mu}{m_e} \right) \\
& + \ldots
\end{aligned}
$$

$$\ln \frac{m_\mu}{m_e} \approx 5.332$$

$$\frac{m_e}{m_\mu} \approx 0.005 \qquad \frac{m_e}{m_\mu} \ln \frac{m_\mu}{m_e} \approx 0.026 \qquad \frac{m_e}{m_\mu} \ln^2 \frac{m_\mu}{m_e} \approx 0.137$$

$$\frac{m_e^2}{m_\mu^2} \ln^3 \frac{m_\mu}{m_e} \approx 0.004 \qquad \frac{m_e^2}{m_\mu^2} \ln^4 \frac{m_\mu}{m_e} \approx 0.019$$

“light-by-light”-type

- asymptotic expansion for $x \equiv m_e/m_\mu \ll 1$

3 loops

$$
\begin{aligned}
a_\mu \;=\; & a_\mu^{(0,0)} + a_\mu^{(0,1)} \ln \frac{m_\mu}{m_e} \\
& + \frac{m_e}{m_\mu} \left(a_\mu^{(1,0)} + a_\mu^{(1,1)} \ln \frac{m_\mu}{m_e} \right) \\
& + \frac{m_e^2}{m_\mu^2} \left(a_\mu^{(2,0)} + a_\mu^{(2,1)} \ln \frac{m_\mu}{m_e} + a_\mu^{(2,2)} \ln^2 \frac{m_\mu}{m_e} \right. \\
& \qquad \left. + a_\mu^{(2,3)} \ln^3 \frac{m_\mu}{m_e} \right) \\
& + \ldots \\
= \; & (20.52801865 + 0.42645778 + 0.00655695 + \ldots) \left(\frac{\alpha}{\pi}\right)^3
\end{aligned}
$$

$$\ln \frac{m_\mu}{m_e} \approx 5.332$$

$$\frac{m_e}{m_\mu} \approx 0.005 \qquad \frac{m_e}{m_\mu} \ln \frac{m_\mu}{m_e} \approx 0.026 \qquad \frac{m_e}{m_\mu} \ln^2 \frac{m_\mu}{m_e} \approx 0.137$$

$$\frac{m_e^2}{m_\mu^2} \ln^3 \frac{m_\mu}{m_e} \approx 0.004 \qquad \frac{m_e^2}{m_\mu^2} \ln^4 \frac{m_\mu}{m_e} \approx 0.019$$

“light-by-light”-type

- asymptotic expansion for $x \equiv m_e/m_\mu \ll 1$

4 loops

$$
\begin{aligned}
a_\mu \;=\; & a_\mu^{(0,0)} + a_\mu^{(0,1)} \ln\frac{m_\mu}{m_e} + a_\mu^{(0,2)} \ln^2\frac{m_\mu}{m_e} \\
& + \frac{m_e}{m_\mu}\left(a_\mu^{(1,0)} + a_\mu^{(1,1)} \ln\frac{m_\mu}{m_e} + a_\mu^{(1,2)} \ln^2\frac{m_\mu}{m_e} \right) \\
& + \frac{m_e^2}{m_\mu^2}\left(a_\mu^{(2,0)} + a_\mu^{(2,1)} \ln\frac{m_\mu}{m_e} + a_\mu^{(2,2)} \ln^2\frac{m_\mu}{m_e} \right. \\
& \qquad \left. + a_\mu^{(2,3)} \ln^3\frac{m_\mu}{m_e} + a_\mu^{(2,4)} \ln^4\frac{m_\mu}{m_e} \right) \\
& + \ldots
\end{aligned}
$$

$$\ln\frac{m_\mu}{m_e} \approx 5.332$$

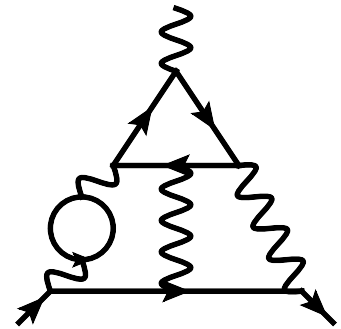
$$\frac{m_e}{m_\mu} \approx 0.005 \qquad \frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e} \approx 0.026 \qquad \frac{m_e}{m_\mu}\ln^2\frac{m_\mu}{m_e} \approx 0.137$$

$$\frac{m_e^2}{m_\mu^2}\ln^3\frac{m_\mu}{m_e} \approx 0.004 \qquad \frac{m_e^2}{m_\mu^2}\ln^4\frac{m_\mu}{m_e} \approx 0.019$$

Techniques

- generation of amplitudes with `qgraf` [Nogueira'91]
- traces, projectors, ... `FORM` [Vermaseren,...]
- asymptotic expansion: `asy` [Pak,Smirnov'10] and in-house program
  basic idea: alpha parametrization
- all 3-loop integrals are known analytically
  ➪ all 4-loop CTs known analytically
- each individual Feynman amplitude:
  ➪ analytic linear combination of master integrals
- 4-loop on-shell integrals:
  $\approx 70 = 40_{\text{ana/high prec.}} + 30_{\text{num}}$ [Marquard,Smirnov,Smirnov,Steinhauser'15]
- 4-loop "linear" integrals:
  $\approx 70 = 20_{\text{ana/high prec.}} + 50_{\text{num}}$
- numerical integration with `FIESTA` [Smirnov'13]

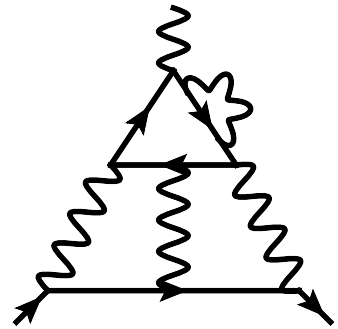
“light-by-light” expansion in m_e/m_μ



$$
\begin{aligned}
& A_2^{(8),\mathrm{IV(a0)}} \\
= \quad & 7.5018 \pm 0.0026 + 14.8808\ell_x + 6.5797\ell_x^2 \\
& + x\left[6.29 \pm 0.46 - 14.6216\ell_x + 8.7729\ell_x^2\right] \\
& + x^2\left[-16.81 \pm 0.43 + 30.0172\ell_x - 6.5069\ell_x^2 + 7.6489\ell_x^3 - 0.8889\ell_x^4\right] \\
& + x^3\left[-48.31 \pm 0.24 - 4.8739\ell_x + 13.1595\ell_x^2\right] \\
= \quad & 7.5018 \pm 0.0026 - 79.3384 + 187.0352 \\
& + [0.0304 \pm 0.0022 + 0.3770 + 1.2061] \\
& + [-0.000393 \pm 0.000010 - 0.003743 - 0.004326 - 0.02711 - 0.01680] \\
& + [-0.0000055 + 0.0000029 + 0.0000423] \\
= \quad & [115.1986 \pm 0.0026] + [1.6135 \pm 0.0022] + [-0.052378 \pm 0.000010] \\
& \qquad + [0.000040] \\
= \quad & 116.7598 \pm 0.0034
\end{aligned}
$$

$$x = m_e/m_\mu \approx 1/206.7682843 \qquad \ell_x = \log(x) \approx -5.3$$

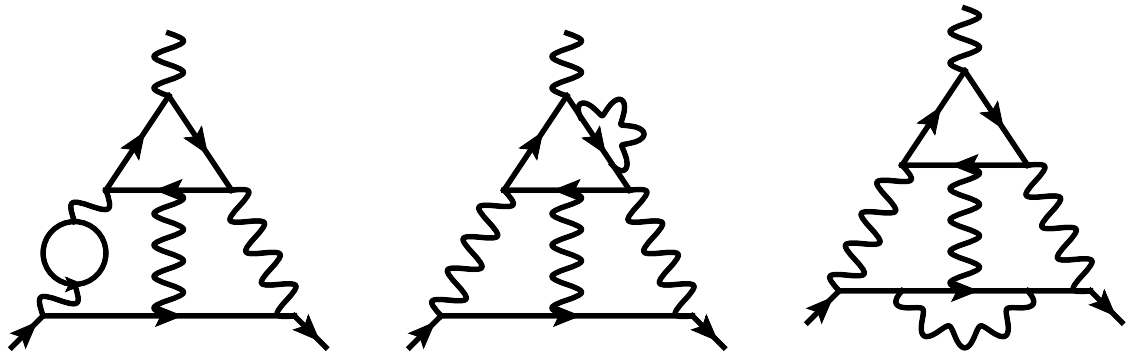
“light-by-light” expansion in m_e/m_μ



$$
\begin{aligned}
& A_2^{(8),\mathrm{IV(b)}} \\
= \quad & 27.395 \pm 0.014 + (4.93482 \pm 0.00003)\ell_x \\
& + x\,[-0.81 \pm 1.22 + 59.0235\ell_x] \\
& + x^2\left[142.5 \pm 7.6 + 40.6546\ell_x + 20.5582\ell_x^2 - 9.6167\ell_x^3 + 0.8333\ell_x^4\right] \\
& + x^3\left[62.11 \pm 2.89 + 132.7421\ell_x - 40.9406\ell_x^2\right] \\
= \quad & 27.395 \pm 0.014 + (-26.3105 \pm 0.0002) \\
& + [-0.0039 \pm 0.0059 - 1.5219] \\
& + [0.003334 \pm 0.0001769 - 0.005070 + 0.01367 + 0.03409 + 0.01575] \\
& + [0.000007 \pm 0. - 0.000080 - 0.000132] \\
= \quad & [1.084 \pm 0.014] + [-1.5259 \pm 0.0059] + [0.06177 \pm 0.00018] + [-0.0002047] \\
= \quad & -0.380 \pm 0.016
\end{aligned}
$$

$$x = m_e/m_\mu \approx 1/206.7682843 \qquad \ell_x = \log(x) \approx -5.3$$

“light-by-light” results



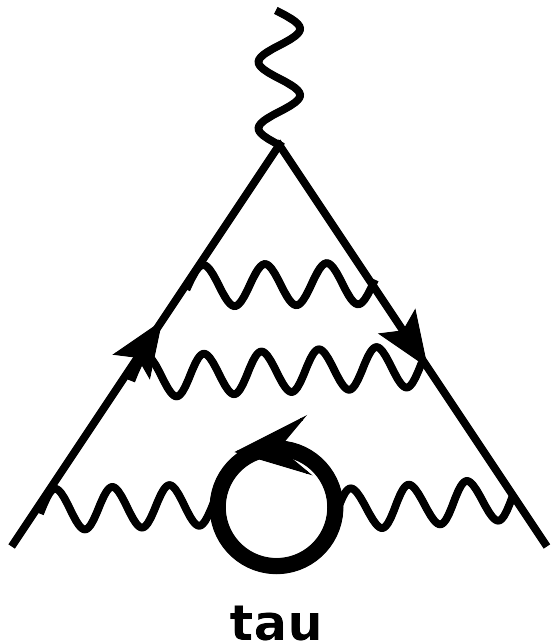
IV(a) IV(b) IV(c)

	[Kurz,Liu et al.'15]	[Kinoshita,Nio'04; Aoyama et al.'08]
IV(a0) ee	116.76 ± 0.02	116.759183 ± 0.000292
IV(a1) $e\mu$	2.69 ± 0.14	2.697443 ± 0.000142
IV(a2) μe	4.33 ± 0.17	4.328885 ± 0.000293
IV(a)	123.78 ± 0.22	123.78551 ± 0.00044
IV(b)	-0.38 ± 0.08	-0.4170 ± 0.0037
IV(c)	2.94 ± 0.30	2.9072 ± 0.0044

- IV(a0): see also [Calmet,Peterman'75; Chlouber,Samuel'77]: 117.4 ± 0.5
- numerical uncertainty: $1.2 \times 10^{-11} \ll a_\mu(\mathrm{exp}) - a_\mu(\mathrm{SM}) \sim 249(87) \times 10^{-11}$

Heavy-lepton contribution

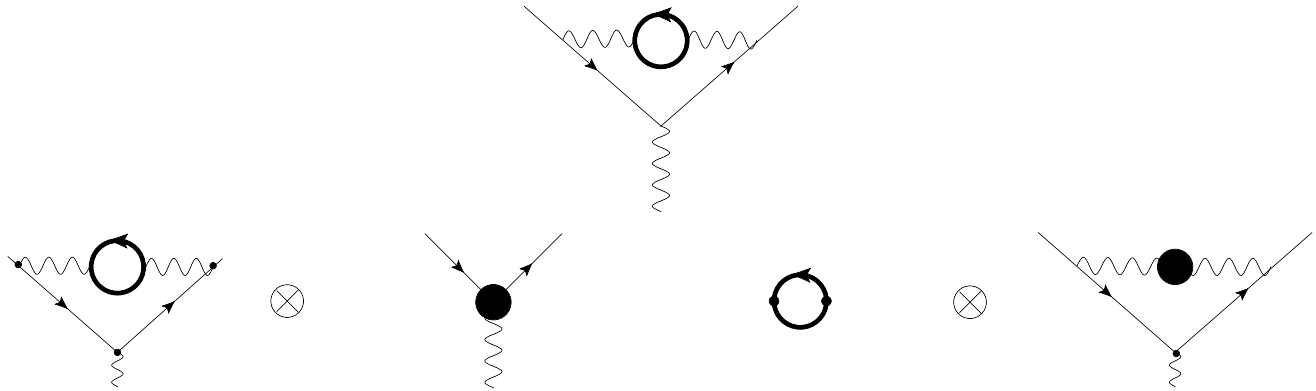
- $\left(\frac{\alpha}{\pi}\right)^4 \frac{m_\mu^2}{m_\tau^2} \approx \left(\frac{\alpha}{\pi}\right)^5$
- analytic asymptotic expansion for $m_\mu \ll m_\tau$

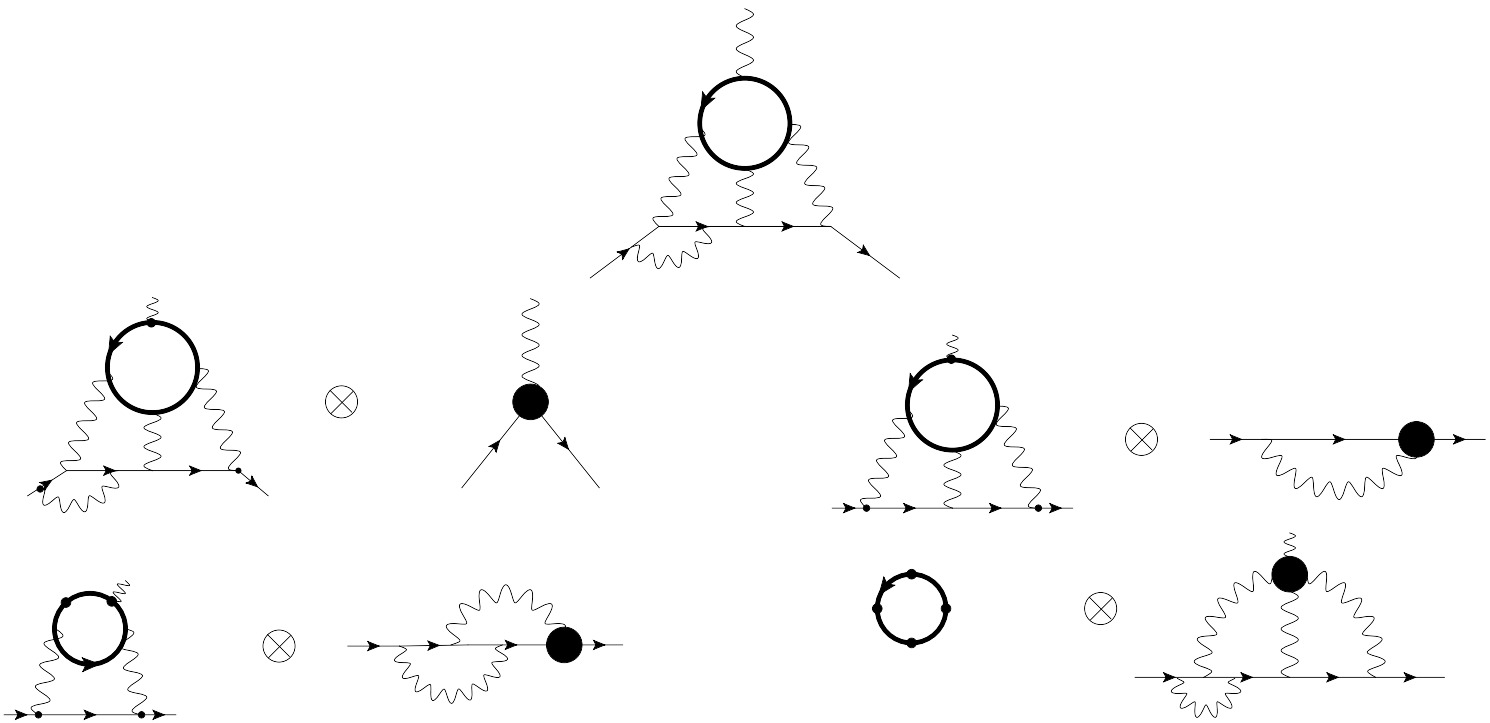


2 loops: [Elend'66; Passera'04'06]

3 loops: [Samuel,Li'91; Laporta,Remiddi'93; Laporta'93; Czarnecki'98; Kühn'03; Friot,Greynat,De Rafael'05; Passera'04'06]

Graphical asymptotic expansion





Results

- analytic result

[Kurz,Liu,Marquard,Steinhauser]

$$A_{2,\mu}^{(8)}(m_\mu/m_\tau) \quad = \quad \sum_n c_n \left(\frac{m_\mu}{m_\tau}\right)^n$$

	$10^2 \cdot A_{2,\mu}^{(8)}(m_\mu/m_\tau)$	
	[Kurz,Liu,Marquard,Steinhauser'13]	[Aoyama,Hayakawa,Kinoshita,Nio'12]
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	−0.6292808(6)	−0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	−2.316756(5)	−2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)	0.612661(5)	0.6106(31)
IV(c)	−1.83010(1)	−1.823(11)

- $A_{2,\mu}^{(8)}(m_\mu/m_\tau)$

$$= 0.0421670 + 0.0003257 + 0.0000015 + \mathcal{O}\left(\frac{m_\mu}{m_\tau}\right)^8$$

$$= 0.0424942 + \mathcal{O}\left(\frac{M_\mu}{M_\tau}\right)^8$$

- $0.04234(12)$ [Aoyama,Hayakawa,Kinoshita,Nio'12]

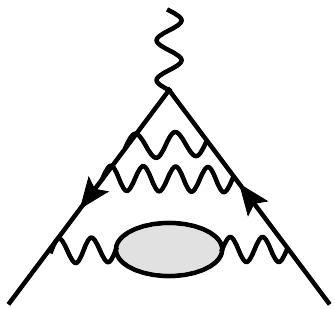
Summary

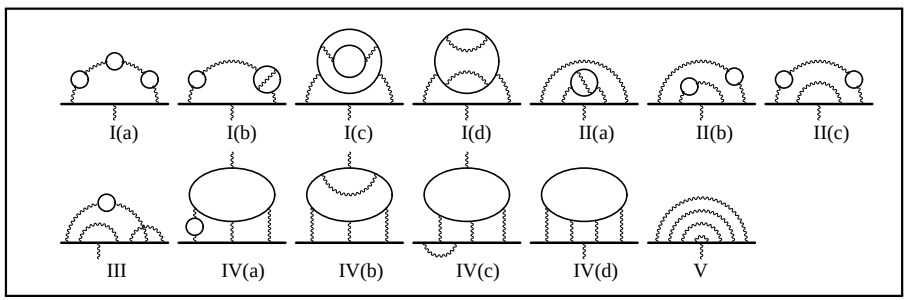
- 4-loop lbl-type contribution $m_e \neq 0$
  systematic improvement of numerical accuracy possible
- 4-loop m_τ contribution

$$10^{11} \times a_\mu\Big|_{\tau\text{loops}} = 42.13 + 0.45 + 0.12$$

$$10^{11} \times a_\mu\Big|_{\text{univ.}} = 116\,140\,973.21 - 177\,230.51 + 1\,480.42 - 5.56 + 0.06$$

- 2 closed electron loops: known analytically for $m_e \to 0$
- hadronic vacuum polarization contribution at NNLO



[Aoyama,Hayakawa,Kinoshita,Nio'12]

group	$A_2^{(8)}(m_\mu/m_e)$	$A_2^{(8)}(m_\mu/m_\tau)$	$A_3^{(8)}$
I(a)	7.74547 (42)	0.000032 (0)	0.003209 (0)
I(b)	7.58201 (71)	0.000252 (0)	0.002611 (0)
I(c)	1.624307 (40)	0.000737 (0)	0.001807 (0)
I(d)	-0.22982 (37)	0.000368 (0)	0
II(a)	-2.77888 (38)	-0.007329 (1)	0
II(b)	-4.55277 (30)	-0.002036 (0)	-0.009008 (1)
II(c)	-9.34180 (83)	-0.005246 (1)	-0.019642 (2)
III	10.7934 (27)	0.04504 (14)	0
IV(a)	123.78551 (44)	0.038513 (11)	0.083739 (36)
IV(b)	-0.4170 (37)	0.006106 (31)	0
IV(c)	2.9072 (44)	-0.01823 (11)	0
IV(d)	-4.43243 (58)	-0.015868 (37)	0