Proof of a 43-Year-Old Prediction by Wilson on Anomalous Scaling for a Hierarchical Composite Field

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Partly joint work with Ajay Chandra (U. of Warwick)
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1. Introduction
2. Model and results
3. Key ideas in the proof

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Composite fields: $\mathcal{O} = 1, \phi, \phi\partial\phi, \phi^2, (\partial^2\phi)\phi^3, \ldots$
expected to satisfy OPE

$$\mathcal{O}_A(x)\mathcal{O}_B(y) = \sum_C \mathcal{C}_{AB,C}(y-x)\mathcal{O}_C(x)$$

inside correlations as asymptotic series when $y \to x$.

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Conclusion of the talk:

Progress on these difficult questions is possible if one focuses on natural hierarchical models, e.g., the p-adic model of A.A.-Chandra-Guadagni arXiv 2013.

Hierarchial models

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Long history in QFT and statistical mechanics:

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“Then, at Michael's urging, I work out what happens near four dimensions for the approximate recursion formula, and find that d-4 acts as a small parameter. Knowing this it is then trivial, given my field theoretic training, to construct the beginning of the epsilon expansion for critical exponents.” K. G. Wilson, interview in Physics of Scales Activities, July 6, 2002.

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HM idea is ubiquitous in mathematics: see 2007 blog post "Dyadic Models" by Terence Tao. For example in harmonic analysis:

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artificial: tree is embedded in $\mathbb{R}^d$.

natural: tree is considered intrinsically without reference to any embedding in $\mathbb{R}^d$.

Hierarchical models

$$\text{artificial} \neq \text{bad}$$

Sometimes one can prove a result on Euclidean model by reduction to embedded HM.

Example 1: work of Dyson on long-range 1d Ising.

Example 2: work of Bramson-Zeitouni for extrema of massless free field (embedded HM is branching random walk).

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W2) $\langle \phi^2_{\mathbf{x}}, \phi^2_{\mathbf{y}} \rangle^{\mathrm{T}} \sim |\mathbf{x} - \mathbf{y}|^{-2[\phi^2]}$ with $[\phi^2] > 2[\phi]$, namely, ϕ^2 has an anomalous dimension.

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In A.A.-Chandra-Guadagni arXiv 2013 we proved an integrated version of **W2** therefore justifying Wilson's prediction and invalidating the claim by Gawedzki-Kupiainen when $m = 2$.

1) Hierarchical continuum

Let p be an integer > 1 (in fact a prime number).

Let $\mathbb{L}_k$, $k \in \mathbb{Z}$, be the set of boxes $\prod_{i=1}^{d} \left[a_i p^k, (a_i + 1)p^k\right)$ for $a_1, \ldots, a_d \in \mathbb{N}$. The cubes in $\mathbb{L}_k$ forms a partition of the octant $[0, \infty)^d$.

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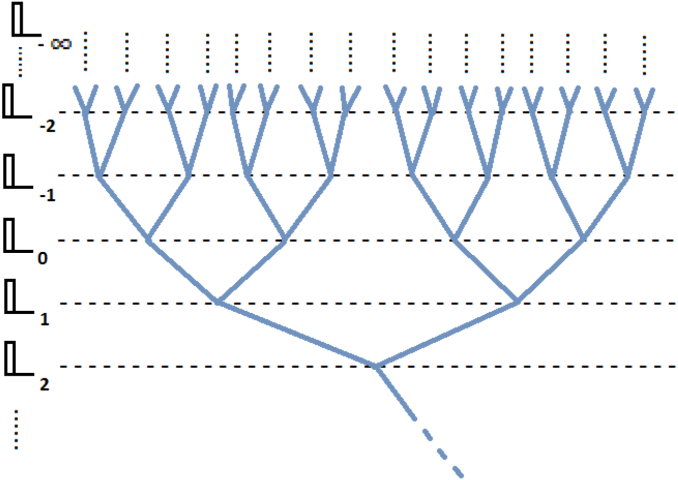
Then $\mathbb{T} = \cup_{k \in \mathbb{Z}} \mathbb{L}_k$ naturally has the structure of a doubly infinite tree organized in layers or generations $\mathbb{L}_k$:



Picture for $d = 1$, $p = 2$

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Define the substitute for continuum $\mathbb{Q}_p^d :=$ set of leafs at infinity "$\mathbb{L}_{-\infty}$".

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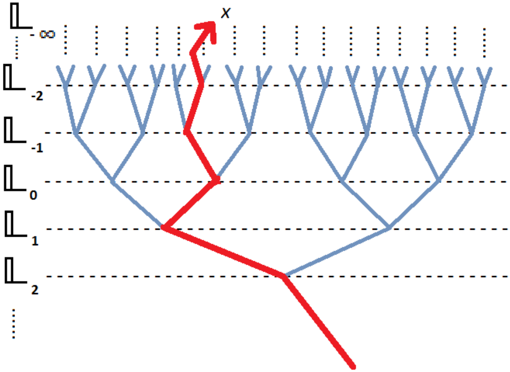
More precisely, this is the set of upward paths in the tree.



A path representing some $x \in \mathbb{Q}_p^d$

A point $x \in \mathbb{Q}_p^d$ encoded by sequence $(a_n)_{n \in \mathbb{Z}}$, $a_n \in \{0, 1, \ldots, p-1\}^d$.
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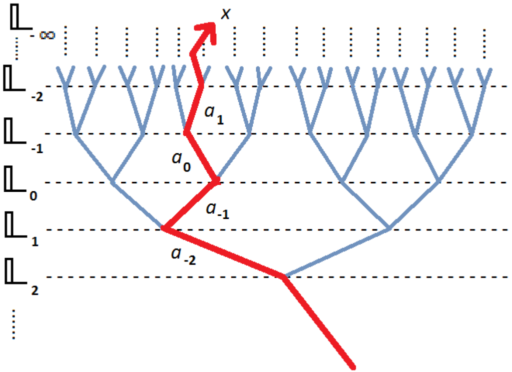

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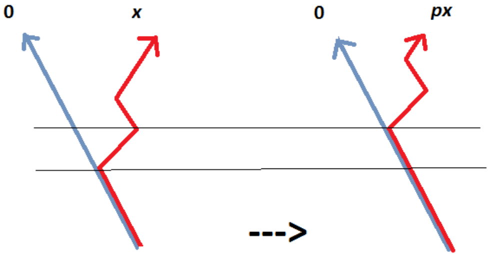

Likewise $p^{-1}x$ is downward shift and so on for defining $p^k x$, $k \in \mathbb{Z}$.

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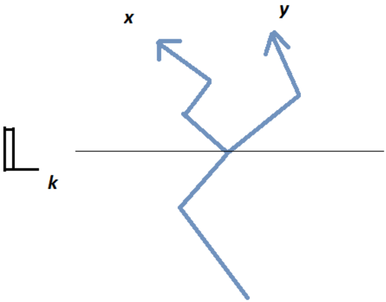

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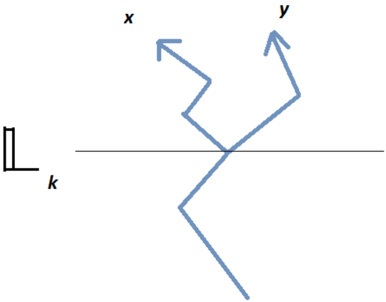

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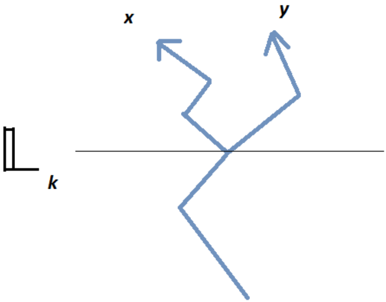

also define $|x| := |x - 0|$. Because of the strange notation

$$|px| = p^{-1}|x|$$

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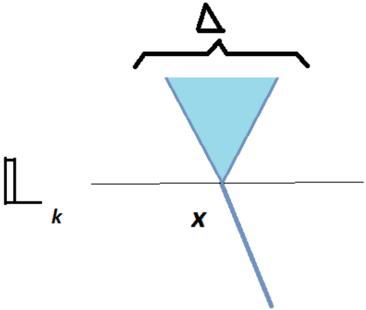

3) Lebesgue measure

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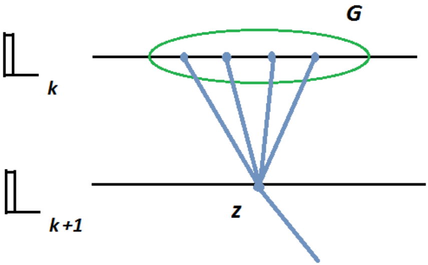
Construction: take product of uniform probability measures on $(\{0,1,\ldots,p-1\}^d)^{\mathbb{N}}$ for $\overline{B}(0,1)$ and similarly for other balls of radius 1, then collate.

4) Massless Gaussian measure



To any G group of offsprings of site $\mathbf{z} \in \mathbb{L}_{k+1}$ associate centered Gaussian vector $(\zeta_{\mathbf{x}})_{\mathbf{x}\in G}$ with $p^d \times p^d$ covariance matrix with $1 - p^{-d}$ on diagonal and $-p^{-d}$ everywhere else. These vectors are set to be independent for different groups or layers.

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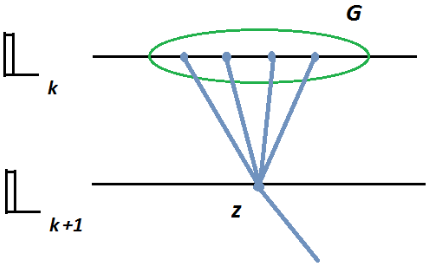

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Massless Gaussian field $\phi(x)$, $x \in \mathbb{Q}_p^d$ with engineering scaling dimension $[\phi]$ is

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only formal since ϕ not defined pointwise. Need random distributions.

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Topology generated by the set of all seminorms.

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Probability theory on $S'(\mathbb{Q}_p^d)$ is very nice!

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5. Analytic RG and dynamical systems methods we introduced deliver exactly that.
6. $S'(\mathbb{Q}_p^d) \times S'(\mathbb{Q}_p^d) \simeq S'(\mathbb{Q}_p^d)$ so same tools work for joint law of pair of distributional random fields, e.g., $(\phi, N[\phi^2])$.

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Cut-off Gaussian measure μ_{C_r} is law of

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Cut-off Gaussian measure μ_{C_r} is law of

$$\phi_r(x) = \sum_{k=lr}^{\infty} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

Sample paths are functions that are locally constant at scale L^r.
Gaussian measures are scaled copies of each other.

7) p-adic BMS model

$d = 3$, $[\phi] = \frac{3-\epsilon}{4}$, $L = p^l$ RG step

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Sample paths are functions that are locally constant at scale L^r.

Gaussian measures are scaled copies of each other.

If law of $\phi(\cdot)$ is μ_{C_0}, then law of $L^{-r[\phi]}\phi(L^r\cdot)$ is μ_{C_r}.

Introduce fixed parameters g, μ and cut-off dependent couplings $g_r = L^{-(3-4[\phi])r} g$ and $\mu_r = L^{-(3-2[\phi])r} \mu$.

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Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{g_r : \phi^4 :_{C_r} (x) + \mu_r : \phi^2 :_{C_r} (x)\} d^3x$$

and define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-V_{r,s}(\phi)} d\mu_{C_r}(\phi)$$

Let $\phi_{r,s}$ random variable in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define square field $N_r[\phi_{r,s}^2]$ which is deterministic $S'(\mathbb{Q}_p^3)$-valued function of $\phi_{r,s}$ given by

$$N_r[\phi_{r,s}^2](j) = Z_2^r \int_{\mathbb{Q}_p^3} \{Y_2 :\phi_{r,s}^2:_{C_r}(x) - Y_0 L^{-2r[\phi]}\}\, j(x)\, d^3x$$

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Our main result concerns the limit law of the pair $(\phi_{r,s}, N_r[\phi_{r,s}^2])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ regardless of the order of limits.
Will need approximate fixed point coupling

$$\bar{g}_* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})}$$

8) Results

Theorem 1: A.A.-Chandra-Guadagni 2013

$\exists\rho$, $\exists L_0$, $\forall L \geq L_0$, $\exists\epsilon_0 > 0$, $\forall\epsilon(0,\epsilon_0]$, $\exists\eta_{\phi^2} > 0$, $\exists$ functions $\mu(g)$, $Y_0(g)$, $Y_2(g)$ on interval $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$ such that if one sets $\mu = \mu(g)$, $Y_0 = Y_0(g)$, $Y_2 = Y_2(g)$ then law of $(\phi_{r,s}, N_r[\phi^2_{r,s}])$ converges weakly and in the sense of moments to that of a pair $(\phi, N[\phi^2])$ such that:

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$$(L^{-k[\phi]}\phi(L^k\cdot), L^{-k[\phi^2]}N[\phi^2](L^k\cdot)) \stackrel{d}{=} (\phi, N[\phi^2])$$

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$$\langle\phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} < 0$$

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3) $\langle N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}), N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} = 1$

Mixed correlations satisfy in sense of distributions

$$\langle\phi(L^{-k}x_1)\cdots\phi(L^{-k}x_n)N[\phi^2](L^{-k}y_1)\cdots N[\phi^2](L^{-k}y_m)\rangle$$

$$= L^{-(n[\phi]+m[\phi^2])k}\langle\phi(x_1)\cdots\phi(x_n)N[\phi^2](y_1)\cdots N[\phi^2](y_m)\rangle$$

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The law $\nu_{\phi\times\phi^2}$ of $(\phi, N[\phi^2])$ is independent of g: universality

Theorem 2: A.A.-Chandra-Guadagni 2013

$\nu_{\phi\times\phi^2}$ is fully scale invariant, i.e., invariant under action of $p^{\mathbb{Z}}$ instead of just $L^{\mathbb{Z}}$. Moreover, $\mu(g)$ and η_{ϕ^2} independent of RG step L.

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Two point functions given as distributions by

$$\langle \phi(x)\phi(y)\rangle = \frac{c_1}{|x-y|^{2[\phi]}}$$

$$\langle N[\phi^2](x)\ N[\phi^2](y)\rangle = \frac{c_2}{|x-y|^{4[\phi]+\eta_{\phi^2}}}$$

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Note $4[\phi] = 3-\epsilon$ so $4[\phi]+\eta_{\phi^2} = 3-\frac{1}{3}\epsilon + o(\epsilon) -> $ still $L^{1,\mathrm{loc}}$!

Theorem 3: A.A. 2015

Let ψ_i denote ϕ or $N[\phi^2]$. Then for every mixed correlation $\exists$ smooth function $\langle\psi_1(z_1)\cdots\psi_n(z_n)\rangle$ on $(\mathbb{Q}_p^3)^n\backslash\mathrm{Diag}$ which is locally integrable (even on Diag) such that

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for all test functions $f_1,\ldots,f_n\in S(\mathbb{Q}_p^3)$.

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for all test functions $f_1,\ldots,f_n \in S(\mathbb{Q}_p^3)$. Moreover the pointwise correlations satisfy the possibly new but certainly nice $L^{1,\mathrm{loc}}$ bound

$$|\langle\psi_1(z_1)\cdots\psi_n(z_n)\rangle| \le$$

$$O(1)\times\prod_{i=1}^{n}|z_i-(\text{nearest neighbor of } z_i)|^{-[\psi_i]}$$

Usually rigorous RG for couplings which are constant in space

$$\int \{g : \phi^4 : (x) + \mu : \phi^2 : (x)\} d^d x$$

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Extended RG is rigorous nonperturbative version of local RG:
Wilson-Kogut PR 1974, Drummond-Shore PRD 1979,
Jack-Osborn NPB 1990,...
used in generalizations of Zamolodchikov's c-"Theorem",
investigations of scale vs. conformal invariance, AdS/CFT,...

1st step: rescale to unit lattice

$$\mathcal{S}^{\mathrm{T}}_{r,s}(f) := \log$$

$$\frac{\int d\mu_{C_r}(\phi) \exp\left(-\int_{\Lambda_s}\{g_r :\phi^4:_r(x) + \mu_r :\phi^2:_r\}dx + \int \phi(x)f(x)dx\right)}{\int d\mu_{C_r}(\phi) \exp\left(-\int_{\Lambda_s}\{g_r :\phi^4:_r(x) + \mu_r :\phi^2:_r\}dx\right)}$$

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$$= \log \frac{\int d\mu_{C_0}(\phi) \mathcal{I}^{(r,r)}[f](\phi)}{\int d\mu_{C_0}(\phi) \mathcal{I}^{(r,r)}[0](\phi)}$$

with

$$\mathcal{I}^{(r,r)}[f](\phi) =$$

$$\exp\left(-\int_{\Lambda_{s-r}} \{g : \phi^4 :_0 (x) + \mu : \phi^2 :_0\} d^3x\right.$$

$$\left. + L^{(3-[\phi])r} \int \phi(x) f(L^{-r}x) d^3x \right)$$

2nd step: define extended RG

Fluctuation covariance $\Gamma := C_0 - C_1$.

Corresponding Gaussian measure is law of fluctuation field

$$\zeta(x) = \sum_{0 \le k < l} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

L-blocks are independent.

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L-blocks are independent.Write

$$\int \mathcal{I}^{(r,r)}[f](\phi)\ d\mu_{C_0}(\phi) = \int \int \mathcal{I}^{(r,r)}[f](\zeta + \psi)\ d\mu_{\Gamma}(\zeta) d\mu_{C_1}(\psi)$$

$$= \int \mathcal{I}^{(r,r+1)}[f](\phi)\ d\mu_{C_0}(\phi)$$

with new integrand

$$\mathcal{I}^{(r,r+1)}[f](\phi) = \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\ d\mu_{\Gamma}(\zeta)$$

In fact, we extract vacuum renormalization, so correct definition is

$$\mathcal{I}^{(r,r+1)}[f](\phi) = e^{-\delta b(\mathcal{I}^{(r,r)}[f])} \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\ d\mu_\Gamma(\zeta)$$

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Then repeat: $\mathcal{I}^{(r,r)} \to \mathcal{I}^{(r,r+1)} \to \mathcal{I}^{(r,r+2)} \to \cdots \to \mathcal{I}^{(r,s)}$

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Need strong control on

$$\mathcal{S}^{\mathrm{T}}(f) = \lim_{\substack{r\to-\infty \\ s\to\infty}} \sum_{r\le q<s} \left(\delta b(\mathcal{I}^{(r,q)}[f]) - \delta b(\mathcal{I}^{(r,q)}[0])\right)$$

Need to use Brydges-Yau lift

$$\begin{array}{ccc} \vec{V}^{(r,q)} & \xrightarrow{RG_{\mathrm{ext}}} & \vec{V}^{(r,q+1)} \\ \downarrow & & \downarrow \\ \mathcal{I}^{(r,q)} & \longrightarrow & \mathcal{I}^{(r,q+1)} \end{array}$$

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$$

$$
\mathcal{I}^{(r,q)}(\phi) = \prod_{\substack{\Delta \in \mathbb{L}_0 \\ \Delta \subset \Lambda_{s-q}}} \left[e^{f_\Delta \phi_\Delta} \times \right.
$$

$$
\left\{ \exp\left(-\beta_{4,\Delta} : \phi_\Delta^4 :_{C_0} - \beta_{3,\Delta} : \phi_\Delta^3 :_{C_0} - \beta_{2,\Delta} : \phi_\Delta^2 :_{C_0} - \beta_{1,\Delta} : \phi_\Delta^1 :_{C_0} \right) \right.
$$

$$
\times \left(1 + W_{5,\Delta} : \phi_\Delta^5 :_{C_0} + W_{6,\Delta} : \phi_\Delta^6 :_{C_0} \right)
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$$+ R_\Delta(\phi_\Delta) \Big\} \Big]$$

Dynamical variable is $\vec{V} = (V_\Delta)_{\Delta\in\mathbb{L}_0}$ with

$$V_\Delta = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta)$$

RG_{ext} acts on space $\mathcal{E}_{\mathrm{ext}}$ which essentially is

$$\prod_{\Delta \in \mathbb{L}_0} \{\mathbb{C}^7 \times C^9(\mathbb{R}, \mathbb{C})\}$$

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Stable subspaces

$\mathcal{E}_{\mathrm{bulk}} \subset \mathcal{E}_{\mathrm{ext}}$: data constant in space
$\mathcal{E} \subset \mathcal{E}_{\mathrm{bulk}}$: even potentials, i.e, g, μ only and R even function.
Let RG be action of RG_{ext} inside $\mathcal{E}$.

3rd step: stabilize the bulk

Show $\forall q \in \mathbb{Z}$, that

$$\lim_{r \to -\infty} \vec{V}^{(r,q)}[0]$$

exists, i.e.,

$$\lim_{r \to -\infty} RG^{q-r}\left(\vec{V}^{(r,r)}[0]\right)$$

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exists.Bulk evolution is

$$\left\{\begin{array}{lllllllll} g' & = & L^{\epsilon} g & - & A_1 g^2 & + & \cdots & & \\ \mu' & = & L^{\frac{3+\epsilon}{2}} \mu & - & A_2 g^2 & - & A_3 g \mu & + & \cdots \\ R' & = & \mathcal{L}^{(g,\mu)}(R) & + & \cdots & & & & \end{array}\right.$$

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Tadpole graph with mass insertion

$$A_3 = 12 L^{3-2[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(0,x)^2 \, d^3x$$

is main culprit for anomalous dimension.

Irwin's proof $->$ stable manifold W^{s}
Restrict dynamics to W^{s} $->$ contraction $->$ IR fixed point v_*.
Construct unstable manifold W^{u} and show intersects W^{s} transversely exactly at v_*.

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Restrict dynamics to W^{s} $->$ contraction $->$ IR fixed point v_*.
Construct unstable manifold W^{u} and show intersects W^{s} transversely exactly at v_*.
Here, $\vec{V}^{(r,r)}[0]$ independent of r: strict scaling limit of a fixed lattice theory.
Just a matter of choosing it on W^{s} $->$ $\mu(g)$ critical mass.
Thus

$$\forall q \in \mathbb{Z}, \qquad \lim_{r \to -\infty} \vec{V}^{(r,q)}[0] = v_*$$

Tangent spaces at fixed point: E^{s} and E^{u}.
$E^{\mathrm{u}} = \mathbb{C}e_{\mathrm{u}}$, with e_{u} eigenvector of $D_{v_*}RG$ for eigenvalue $\alpha_{\mathrm{u}} = L^{3-2[\phi]} \times Z_2 =: L^{3-[\phi^2]}$.

4th step: control the deviations

Namely, for all q bound the deviations $\vec{V}^{(r,q)}[f] - \vec{V}^{(r,q)}[0]$ uniformly in r.

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1) $\sum_{\mathbf{x}\in G} \zeta_{\mathbf{x}} = 0$ a.s. $->$ deviation is 0 for q less than the constancy scale of f.

2) deviation resides in unit box at origin for q more than radius of support of f with respect to origin $->$ geometric decay for q large.

4th step: control the deviations

Namely, for all q bound the deviations $\vec{V}^{(r,q)}[f] - \vec{V}^{(r,q)}[0]$ uniformly in r.

1) $\sum_{\mathbf{x}\in G} \zeta_{\mathbf{x}} = 0$ a.s. $->$ deviation is 0 for q less than the constancy scale of f.

2) deviation resides in unit box at origin for q more than radius of support of f with respect to origin $->$ geometric decay for q large. For ϕ^2 source term add

$$Y_2 Z_2^r \int :\phi^2:_{C_r}(x) j(x) d^3x$$

in potential. $\mathcal{S}^{\mathrm{T}}_{r,s}(f,j)$ depends on two test functions. After rescaling to unit lattice, we get

$$Y_2 \alpha_{\mathrm{u}}^r \int :\phi^2:_{C_0}(x) j(L^{-r}x) d^3x$$

to be combined with μ in the now space-dependent mass $(\beta_{2,\Delta})_{\Delta\in\mathbb{L}_0}$.

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$\Psi(v,w)$ is jointly holomorphic in v and w.

Essential for probability theory interpretation of $(\phi, N[\phi^2])$ as pair of $S'(\mathbb{Q}_p^3)$-valued random variables.

What next?

1) Prove pointwise OPE

2) Proved smeared OPE, i.e., show $N[\phi^2]$ deterministic local function of ϕ

3) Prove OS positivity: UV cut-off by convolution with compactly supported mollifier $+$ exclusion corridor. Show theory is the same as without corridor $->$ extended RG for boundaries, domain walls, etc.

4) Heteroclinic RG trajectory

5) Investigate conformal invariance

6) Transpose all this to the Euclidean setting: all hinges on developing Euclidean extended RG.