Quantum Phase Transition in an Interacting Fermionic Chain.

Federico Bonetto

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In memory of a friend.

Work in collaboration with Vieri Mastropietro.

Publications:

- Benfatto G., Gallavotti: *JSP* **59**, 541 (1990).
- Benfatto G., Gallavotti G, Procacci, A, Scoppola B: *Comm. Math. Phys.* **160**, 93 (1994).

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- F. B., V. Mastropietro: *Ann. Henry Poincaré* (2014)

Introduction: Fermions in a periodic Potential.

Let ψ_x^+ and ψ_x^- the creation and annihilation operator for a Fermion in one dimension. Consider the Hamiltonian:

$$H_\lambda = -\int_0^L \psi_x^+ \partial_x^2 \psi_x^- dx + \int_0^L c(x) \psi_x^+ \psi_x^- dx + \lambda \int_0^L v(x-y) \psi_x^+ \psi_x^- \psi_y^+ \psi_y^- dxdy$$

with

$$c(x+1) = c(x) \qquad v(-x) = v(x), \qquad |v(x)| \le e^{-\kappa |x|}.$$

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When $\lambda = 0$ we can diagonalize the Hamiltonian using Bloch waves, i.e. the solution of the eigenvalue problem

$$-\partial_x^2\phi(k,x) + c(x)\phi(k,x) = \varepsilon(k)\phi(k,x)$$

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Setting

$$\psi_x^\pm = \frac{1}{L}\sum_{k=\frac{2\pi m}{L}} \phi(k,\pm x)\psi_k^\pm$$

we get

$$H_0 = \frac{1}{L}\sum_{k=\frac{2\pi m}{L}} \epsilon(k)\psi_k^+\psi_k^- .$$

The Schwinger Functions.

As usual we define

$$\langle O \rangle_{L,\beta} = \frac{\text{Tr}\, e^{\beta(H_\lambda - \mu N)} O}{\text{Tr}\, e^{\beta(H_\lambda - \mu N)}}$$

$$N = \int \psi_x^+ \psi_x^- dx$$

and μ is the *chemical potential*. Moreover we set

$$\langle O \rangle = \lim_{\beta \to 0} \lim_{L \to \infty} \langle O \rangle_{L,\beta}.$$

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Calling

$$\psi^\pm_{\mathbf{x}} = e^{(H_\lambda - \mu N)x_0} \psi^\pm_x e^{-(H_\lambda - \mu N)x_0}$$

where $\mathbf{x} = (x_0, x)$ the 2-points Schwinger function defined as

$$S_{\lambda,L,\beta}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{T} \psi^+_{\mathbf{x}}, \psi^-_{\mathbf{y}} \rangle_{L,\beta}$$

where $\mathbf{T}$ is the time-ordering operator.

Using the Bloch waves we can write

$$\hat{S}_{0,L,\beta}(\mathbf{k}) = \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathcal{D}} \phi(k,x)\phi(k,-y) e^{ik_0(x_0 - y_0)} S_{0,L,\beta}(\mathbf{x},\mathbf{y})$$

where

$$\mathcal{D} = \left\{ \mathbf{k} = (k_0, k) \,|\, k = \frac{2\pi m}{L},\ k_0 = \frac{2\pi}{\beta}(n + \frac{1}{2}) \right\}$$

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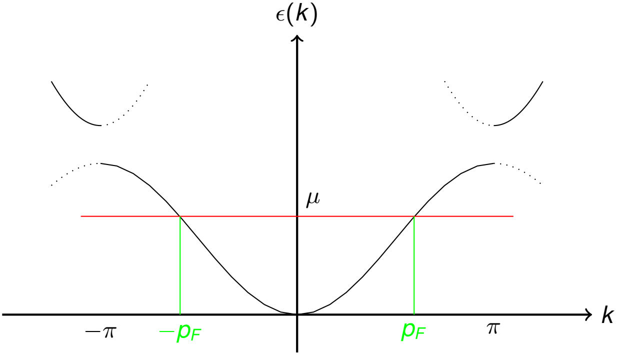
and

$$\hat{S}_0(k) = \lim_{\beta\to\infty} \lim_{L\to\infty} \hat{S}_{0,L,\beta}(k) = \frac{1}{-ik_0 + \epsilon(k) - \mu}.$$

The dispersion relation.



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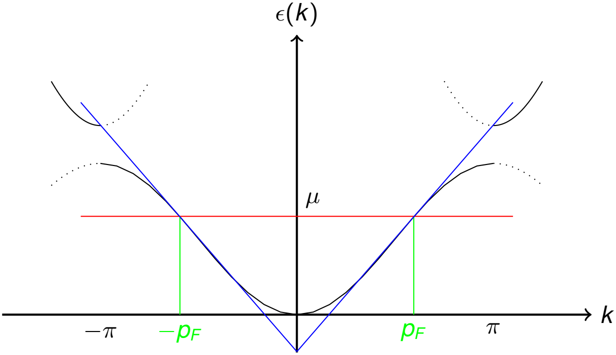

$$S_0(k) = \frac{1}{-ik_0 + \epsilon(k) - \mu} \simeq \frac{\vartheta(k)}{-ik_0 + v_F(k - p_F)} + \frac{\vartheta(-k)}{-ik_0 + v_F(k + p_F)}$$

$$S_\lambda(k) = \vartheta(k) \frac{\left(k_0^2 + v_F(\lambda)^2 (k + p_F(\lambda))^2\right)^{\eta(\lambda)}}{-ik_0 + v_F(\lambda)(k + p_F(\lambda))} (1 + R(\lambda)) + \ldots$$

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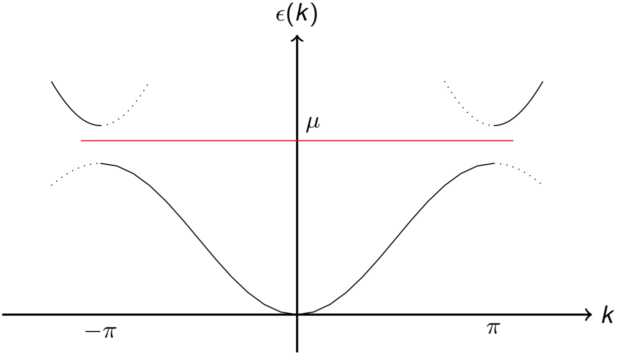
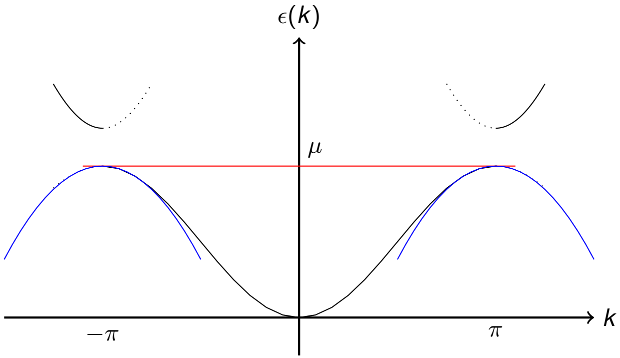


$$S_0(k) = \frac{1}{-ik_0 + \epsilon(k) - \mu} \simeq \frac{1}{-ik_0 + \alpha k^2 + r} \qquad r = \varepsilon(\pi) - \mu$$

$$S_\lambda(k) = \frac{1 + R(\lambda)}{-ik_0 + \alpha k^2 + r}$$



$$S_0(k) = \frac{1}{-ik_0 + \epsilon(k) - \mu} \simeq \frac{1}{-ik_0 - \alpha k^2}$$

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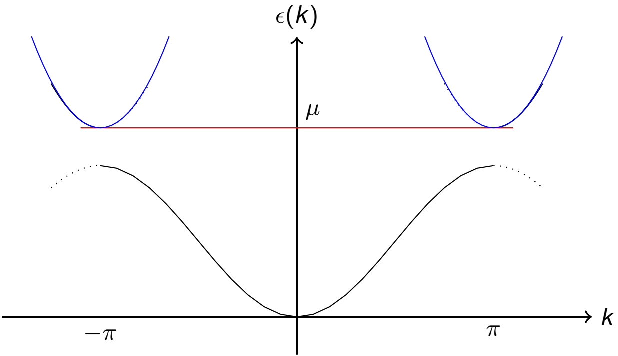

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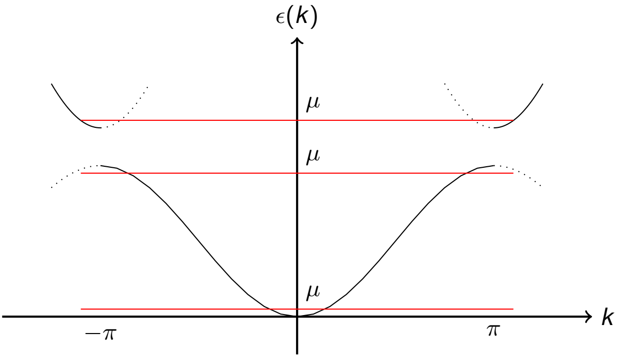

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Almost Filled band

To understand what happens when the band is almost filled we can neglect the existence of all band but the condution band.

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Let thus $x \in \{1, 2, \ldots, L\}$ and consider the Hamiltonian

$$H_\lambda = -\sum_x \left[\frac{1}{2}(a^+_{x+1} a^-_x + a^+_x a^-_{x+1}) + h a^+_x a^-_x\right] - \lambda \sum_{x,y} v(x-y) a^+_x a^-_x a^+_y a^-_y$$

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This Hamiltonian can also be obtained via a Jordan-Wigner transformation from a spin chain model with Hamiltonian

$$H = -\sum_x \frac{1}{2}[S^1_x S^1_{x+1} + S^2_x S^2_{x+1}] - \lambda \sum_{x,y} v(x-y) S^3_x S^3_y - \bar{h} \sum_x S^3_x$$

where $(S^1_x, S^2_x, S^3_x) = \frac{1}{2}(\sigma^1_x, \sigma^2_x, \sigma^3_x)$ are Pauli matrices, $\bar{h}$ is the magnetic field.

Again we can write

$$a_x^{\pm} = \frac{1}{L} \sum_{k \in \widetilde{\mathcal{D}}} e^{\pm ikx} \hat{a}_k^{\pm}$$

where $\widetilde{\mathcal{D}} = \left\{ k \mid k = \frac{2\pi m}{L},\ -\pi \leq k < \pi \right\}$ and find

$$H_0 = \frac{1}{L} \sum_{k \in \bar{\mathcal{D}}} \varepsilon(k) \hat{a}_k^{+} \hat{a}_k^{-} \qquad \varepsilon(k) = -\cos k - h.$$

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The two point Schwinger function is given by

$$S_{0,L,\beta}(\mathbf{x}) = \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathcal{D}} e^{-i\mathbf{k}\mathbf{x}} \hat{S}_{0,\beta,L}(\mathbf{k})$$

with

$$\mathcal{D} = \left\{ \mathbf{k} = (k_0, k) \mid k = \frac{2\pi m}{L}, \ -\pi \le k < \pi, \ k_0 = \frac{2\pi}{\beta}(n + \frac{1}{2}) \right\}$$

and

$$\hat{S}_{0,L,\beta}(\mathbf{k}) = \frac{1}{-ik_0 + \cos k + h}.$$

To summarize:

- In the metallic phase $|h| < 1$ the Schwinger function $\hat{S}_0(\mathbf{k})$ is singular in correspondence of the Fermi points $(0, \pm p_F)$. For $|k|$ close to p_F we have

$$\hat{S}_0(\mathbf{k}) \sim \frac{1}{-ik_0 + v_F(|k| - p_F)} \qquad |k| \simeq p_F.$$

- At criticality when $|h| = 1$ the 2-point function $\hat{S}_0(\mathbf{k})$ is singular only at $(0, 0)$ and

$$\hat{S}_0(\mathbf{k}) \sim \frac{1}{-ik_0 + \frac{1}{2}k^2} \qquad k \simeq 0$$

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We will focus on $h \simeq -1$ and we will write

$$h = -1 + r$$

Convergence near $r = 0$.

Observe that

$$p_F = \arccos(1 - r) \simeq \sqrt{r} \qquad\qquad v_F = \sin p_F \simeq \sqrt{r}$$

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$$S_0(k) \simeq \frac{\vartheta(k)}{-ik_0 + v_F(k - p_F)} + \frac{\vartheta(-k)}{-ik_0 + v_F(k + p_F)}$$

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More precesely, the perturbative series in λ discussed by Benfatto yesterday converge in a neighbor of the origin of radius proportional to v_F.

Heuristic Analysis near $r = 0$.

By the change of variable $v_F k \leftrightarrow k$, one can see that a system with the above propagator is formally equivalent to a system with

$$v_F = 1 \qquad \tilde{\lambda} = \frac{\lambda}{v_F} \simeq \frac{\lambda}{\sqrt{r}}$$

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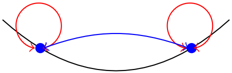

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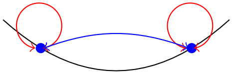

that is

$$\lambda_0 = \lambda(\hat{v}(0) - \hat{v}(2p_F)) \simeq \lambda r.$$

Thus our system looks formally equivalent to a system with effective coupling

$$\tilde{\lambda}_0 \simeq \lambda\sqrt{r}.$$

The problem with this argument is that the linear approximation is valid only very close to the Fermi points, that is

$$|k - p_F| \simeq \sqrt{r}$$

Away from the Fermi points the dispersion relation appears quadratic.

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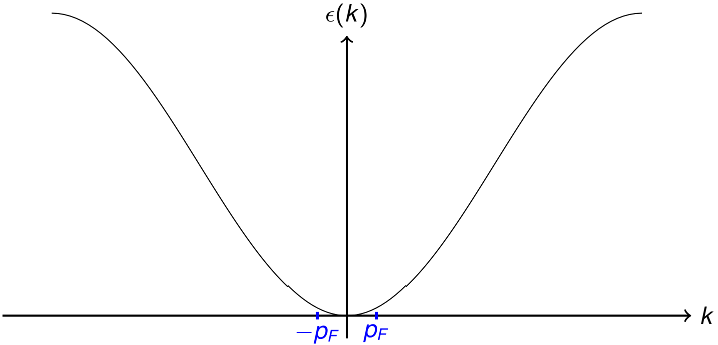
Moreover, the theory with quadratic dispersion relation is, prima facie, non renormalizable so that the assumption that the $\lambda_0 \simeq \lambda\sqrt{r}$ is not justified.

Momentum regions.



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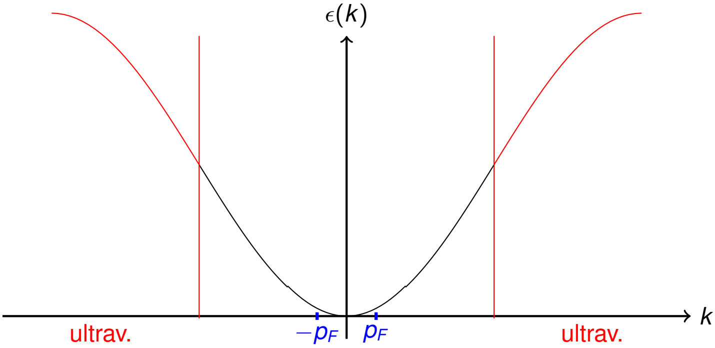

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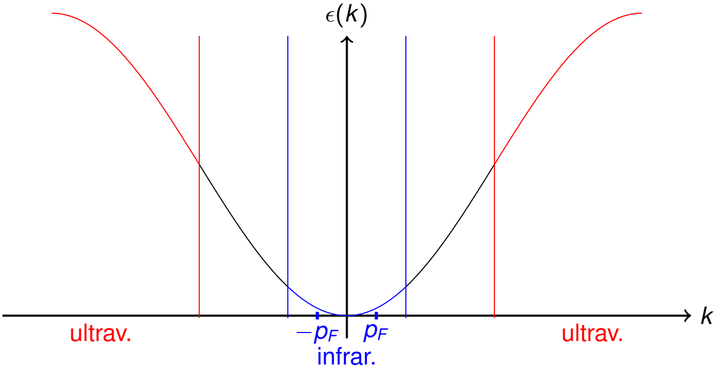

Theorem

Given the Hamiltonian H_λ with $h = -1 + r$ with $|r| < 1$, there exists $\varepsilon > 0$ and $C > 0$ (independent from L, β, r) such that, if $|\lambda| < \varepsilon$ then the Fourier transform of $S_{L,\beta}(\mathbf{x})$ can be written in the following way.

Results: main theorem.

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1. *For $r > 0$ (metallic phase),*

$$\hat{S}_{L,\beta}(\mathbf{k}) = \frac{[k_0^2 + \alpha(\lambda)^2(\cos k - 1 + \nu(\lambda))^2]^{\eta(\lambda)}}{-ik_0 + \alpha(\lambda)(\cos k - 1 + \nu(\lambda))}(1 + \lambda R_S(\lambda, \mathbf{k}))$$

where

$$\nu(\lambda) = r + \lambda r R_\nu(\lambda) \qquad \alpha(\lambda) = 1 + \lambda R_\alpha(\lambda)$$
$$\eta(\lambda) = b\lambda^2 r + \lambda^3 r^{\frac{3}{2}} R_\eta(\lambda) \tag{1}$$

with $b > 0$ a constant and $|R_i| \le C$ for $i = S, \nu, \alpha$ and η.

Theorem

With the same hipothese as above we have:

2. *For $r = 0$ (critical point)*

$$\hat{S}_{L,\beta}(\mathbf{k}) = \frac{1 + \lambda R_S(\lambda, \mathbf{k})}{-ik_0 + \alpha(\lambda)(\cos(k) - 1)}$$

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3. *For $r < 0$ (insulating phase)*

$$|\hat{S}_{L,\beta}(\mathbf{k})| \leq \frac{C}{|r|}$$

Moreover $\hat{S}(\mathbf{k}) = \lim_{\beta\to\infty} \lim_{L\to\infty} \hat{S}_{L,\beta}(\mathbf{k})$ exists and is reached uniformly in λ.

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Identical results hold for $h = 1 - r$ thank to a hole-particle symmetry.

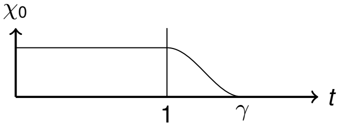
Grassmann Variables.

Let

$$g_{M,L,\beta}(\mathbf{x}-\mathbf{y}) = \frac{1}{\beta L}\sum_{\mathbf{k}\in\mathcal{D}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\chi_0(\gamma^{-M}|k_0|)}{-ik_0+\cos k+h}$$

where

$$\chi_0(t) =$$



Let

$$\mathcal{D}_\beta = \mathcal{D}\cap \operatorname{supp}\chi_0(\gamma^{-M}|k_0|) = \{\mathbf{k}\in\mathcal{D}\,|\,|k_0|<\gamma^{M+1}\}$$

We consider the anticommuting Grassmannian variables

$$\{\psi^{\pm}_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{D}_\beta}$$

that generate a Grasmannian Algebra $\mathcal{G}$.

On $\mathcal{G}$ we define the Grassmann integration, that is the the linear operator, defined as

$$\int \Big[\prod_{\mathbf{k}\in\mathcal{D}_\beta} d\psi^+_{\mathbf{k}} d\psi^-_{\mathbf{k}} \Big] \prod_{\mathbf{k}\in\mathcal{D}_\beta} \psi^-_{\mathbf{k}} \psi^+_{\mathbf{k}} = 1$$

while

$$\int \Big[\prod_{\mathbf{k}\in\mathcal{D}_\beta} d\psi^+_{\mathbf{k}} d\psi^-_{\mathbf{k}} \Big] Q(\psi^-, \psi^+) = 0$$

if the monomial $Q(\psi^-, \psi^+)$ does not contains all of the variables $\{\psi^\pm_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{D}_\beta}$.

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We define the *Grassmanian fields*

$$\psi_{\mathbf{x}}^{\pm} = \frac{1}{\beta L} \sum_{\mathbf{k}\in\mathcal{D}_\beta} e^{\pm i\mathbf{k}\mathbf{x}} \psi_{\mathbf{k}}^{\pm} \qquad\qquad \mathbf{x} \in \Gamma_\beta \times \Lambda$$

while the *Gaussiam Grassmann measure* is defined as

$$P(d\psi) = \Big[\prod_{\mathbf{k}\in\mathcal{D}_\beta} \beta L d\psi_{\kappa}^- d\psi_{\mathbf{k}}^+ \hat{g}^{(\le M)}(\mathbf{k}) \Big] \exp\Big\{ -\frac{1}{\beta L} \sum_{\mathbf{k}\in\mathcal{D}_\beta} (\hat{g}^{(\le M)}(\mathbf{k}))^{-1} \psi_{\mathbf{k}}^+ \psi_{\mathbf{k}}^- \Big\}$$

We introduce the generating functional $\mathcal{W}_M(\phi)$ defined in terms of the following Grassmann integral

$$e^{-\mathcal{W}_M(\phi)} = \int P(d\psi) e^{-\mathcal{V}(\psi)+(\psi,\phi)}$$

where

$$(\psi,\phi) = \int d\mathbf{x}[\psi^+_\mathbf{x}\phi^-_\mathbf{x} + \psi^-_\mathbf{x}\phi^+_\mathbf{x}] \tag{2}$$

$$\mathcal{V}(\psi) = \lambda \int d\mathbf{x}d\mathbf{y}v(\mathbf{x}-\mathbf{y})\psi^+_\mathbf{x}\psi^-_\mathbf{x}\psi^+_\mathbf{y}\psi^-_\mathbf{y} + \bar{\nu}\int d\mathbf{x}\psi^+_\mathbf{x}\psi^-_\mathbf{x} \tag{3}$$

Here

$$\int d\mathbf{x} \qquad \text{stands for} \qquad \sum_{x\in\Lambda} a \sum_{x_0\in\Gamma_\beta}$$

and $v(\mathbf{x}-\mathbf{y}) = \delta(x_0-y_0)v(x-y)$.

Calling $\lim_{M\to\infty} g_{M,L,\beta}(\mathbf{x}) = g_{L,\beta}(\mathbf{x})$ and we observe that

$$g_{L,\beta}(\mathbf{x}) = S_{0,L,\beta}(\mathbf{x})$$

wherever $S_{0,L,\beta}(\mathbf{x})$ is continuous.

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Finally we define

$$S^M_{L,\beta}(\mathbf{x}-\mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}}\mathcal{W}_M(\phi)|_{\phi=0}.$$

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The above Grassmann integral can be used to compute the thermodynamical properties of the model with Hamiltonian H_λ.

The starting point of the analysis is the following decomposition of the propagator

$$g_{M,L,\beta}(\mathbf{x}) = g^{(>0)}(\mathbf{x}) + g^{(\leq 0)}(\mathbf{x})$$

where

$$g^{(\leq 0)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_0(\gamma^{-M}|k_0|)\chi_{\leq 0}(\mathbf{k})}{-ik_0 + \cos k + h}$$

Here

$$\chi_{\leq 0}(\mathbf{k}) = \chi_0\left(\sqrt{k_0^2 + (\cos k - 1 + r)^2}\right).$$

Observe that $\chi_{\leq 0}(\mathbf{k})$ is a smooth version of the characteristic function of the set

$$A_0 = \{\mathbf{k} \,|\, |-ik_0 + (\cos k - 1 + r)| \leq 1\}.$$

Ultraviolet Integration.

The starting point of the analysis is the following decomposition of the propagator

$$g_{M,L,\beta}(\mathbf{x}) = g^{(>0)}(\mathbf{x}) + g^{(\leq 0)}(\mathbf{x})$$

where

$$g^{(\leq 0)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_0(\gamma^{-M}|k_0|)\chi_{\leq 0}(\mathbf{k})}{-ik_0 + \cos k + h}$$

Here

$$\chi_{\leq 0}(\mathbf{k}) = \chi_0\left(\sqrt{k_0^2 + (\cos k - 1 + r)^2}\right).$$

Observe that $\chi_{\leq 0}(\mathbf{k})$ is a smooth version of the characteristic function of the set

$$A_0 = \{\mathbf{k} \,|\, |-ik_0 + (\cos k - 1 + r)| \leq 1\}.$$

By using the addition property of Grassmann integrations we can write

$$e^{-\mathcal{W}(\phi)} = \int P(d\psi^{(\leq 0)}) \int P(d\psi^{(\leq 0)}) e^{-\mathcal{V}(\psi^{(>0)}+\psi^{(>0)})+(\psi^{(>0)}+\psi^{(\leq 0)},\phi)}.$$

After integrating the field $\psi^{(>0)}$ one obtains

$$e^{-\mathcal{W}(\phi)} = e^{-\beta L F_0} \int P(d\psi^{(\leq 0)}) e^{-\mathcal{V}^{(0)}(\psi^{(\leq 0}, \phi)}$$

where

$$\mathcal{V}^{(0)}(\psi, \phi) = \sum_{n+m\geq 1} \int d\underline{\mathbf{x}} \int d\underline{\mathbf{y}} \prod_{i=1}^{n} \psi_{\mathbf{x}_i}^{\varepsilon_i} \prod_{j=1}^{m} \phi_{\mathbf{x}_j}^{\sigma_j} W_{n,m}^{(0)}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$$

where $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\underline{\mathbf{y}} = (\mathbf{y}_1, \ldots, \mathbf{y}_m)$.

After integrating the field $\psi^{(>0)}$ one obtains

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where $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\underline{\mathbf{y}} = (\mathbf{y}_1, \ldots, \mathbf{y}_m)$.

We know that $W_{n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ are given by convergent power series in λ for λ small enough and they decay faster than any power in any coordinate difference. Finally, the limit $M \to \infty$ of $\mathcal{V}^{(0)}(\psi, \phi)$ exists and is reached uniformly in β, L.

Multiscale Decomposition.

Thus we are left with the integration over $\psi^{(\leq 0)}$. The idea in order to to perform this integration is to decompose $\psi_{\mathbf{x}}^{(\leq 0)}$ as

$$\psi_{\mathbf{x}}^{(\leq 0)} = \sum_{h=0}^{-\infty} \psi_{\mathbf{x}}^{(h)}$$

where $\psi_{\mathbf{x}}^{(h)}$ depends only on the momenta **k** such that

$$|-ik_0 + \cos k - 1 + r| \simeq \gamma^h.$$

Thus we are left with the integration over $\psi^{(\leq 0)}$. The idea in order to to perform this integration is to decompose $\psi^{(\leq 0)}_{\mathbf{x}}$ as

$$\psi^{(\leq 0)}_{\mathbf{x}} = \sum_{h=0}^{-\infty} \psi^{(h)}_{\mathbf{x}}$$

where $\psi^{(h)}_{\mathbf{x}}$ depends only on the momenta $\mathbf{k}$ such that

$$|-ik_0 + \cos k - 1 + r| \simeq \gamma^h.$$

To do this, consider the sets

$$A_h = \{\mathbf{k}|\, \gamma^{h-1} \leq |ik_0 + (\cos(k) - 1 + r)| \leq \gamma^h\}$$

and write

$$\hat{g}^{(h)}(\mathbf{k}) = \frac{I_{A_h}(\mathbf{k})}{ik_0 + (\cos(k) - 1 + r)}$$

where I_{A_h} is the characteristic function of A_h.

We can also define

$$g^{(h)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \hat{g}^{(h)}(\mathbf{k})$$

so that

$$g^{(\leq 0)}(\mathbf{x}) = \sum_{h \leq 0} g^{(h)}(\mathbf{x})$$

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so that

$$g^{(\leq 0)}(\mathbf{x}) = \sum_{h \leq 0} g^{(h)}(\mathbf{x})$$

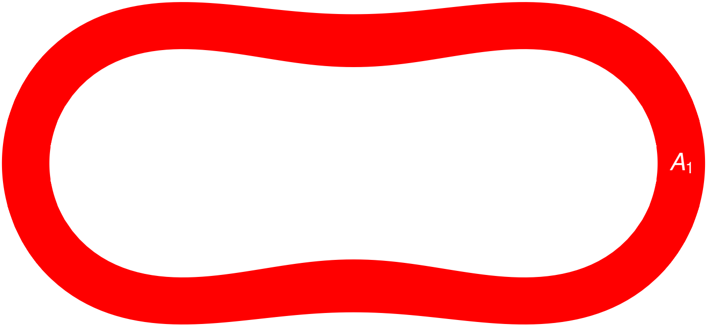
The index h is called the *scale* of the field $\psi^{(h)}$. When $r > 0$, two different regimes naturally appear in the analysis, separated by an energy scale depending on r and defined as

$$h^* = \inf\{h \,|\, \gamma^{h+1} > |r|\}.$$

Behavior of the scales: $\gamma = 2$ and $r = 0.27$.

$h > h^*$: $|A_h| = \gamma^{\frac{3}{2}h}$

$$\hat{g}^{(h)}(\mathbf{k}) \simeq \gamma^{-h}\hat{\tilde{g}}(\gamma^{-h}k_0, \gamma^{-\frac{h}{2}}k)$$
$$g^{(h)}(\mathbf{x}) \simeq \gamma^{\frac{h}{2}}\tilde{g}(\gamma^{h}x_0, \gamma^{\frac{h}{2}}x).$$



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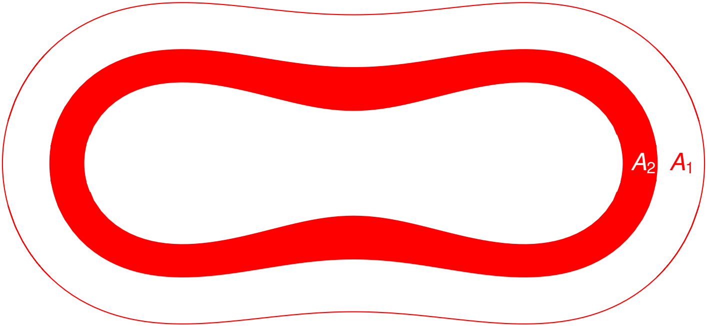
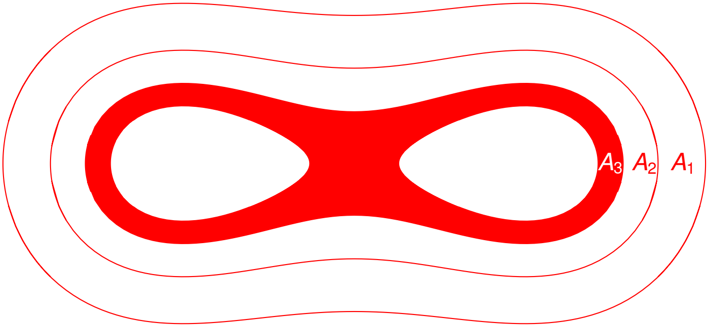


Behavior of the scales: $\gamma = 2$ and $r = 0.27$.

$h = h^*$: transition scale, both scaling are good.

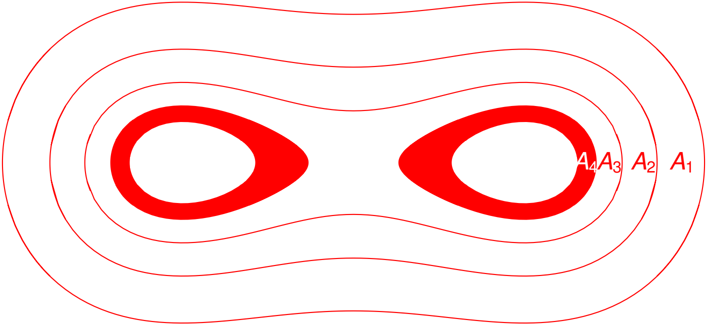


Behavior of the scales: $\gamma = 2$ and $r = 0.27$.

$h < h^*$: $|A_h| = \gamma^{2h}$

$$\hat{g}^{(h)}(\mathbf{k}) = \hat{g}^{(h)}_{-1}(\mathbf{k}) + \hat{g}^{(h)}_{-1}(\mathbf{k}) \qquad \hat{g}^{(h)}_{\omega} \simeq \gamma^{-h}\hat{\bar{g}}(\gamma^{-h}k_0, v_F\gamma^{-h}k)$$

$$g^{(h)}_{\omega} \simeq v_F^{-1}\gamma^{h}\bar{g}(\gamma^{h}x_0, v_F^{-1}\gamma^{h}k)$$



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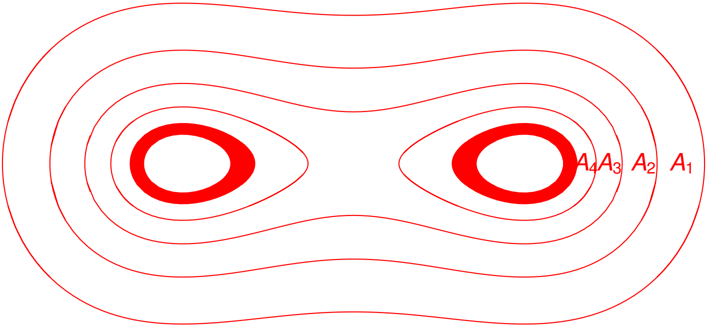

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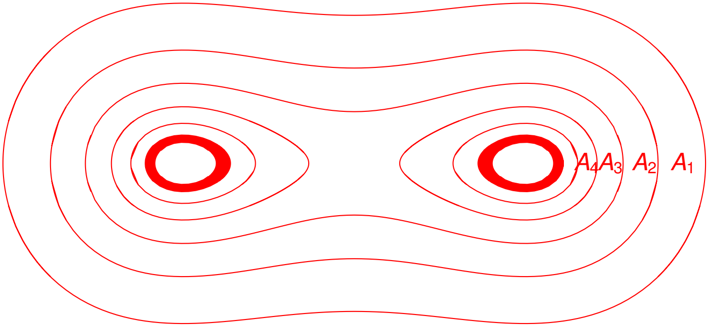

We saw that after the ultraviolet integration we have

$$e^{-\mathcal{W}(0)} = e^{-\beta L F_0} \int P(d\psi^{(\leq 0)}) e^{-\mathcal{V}^{(0)}(\psi^{(\leq 0)})}$$

where

$$P(d\psi^{(\leq 0)}) \qquad \longleftrightarrow \qquad g^{(\leq 0)}(\mathbf{k})$$

with

$$g^{(\leq 0)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{\leq 0}(\mathbf{k})}{D_0(\mathbf{k})} \qquad\qquad \chi_{<0}(\mathbf{k}) = \chi_0\left(\gamma |D_0(\mathbf{k})|\right)$$

and

$$D_0(\mathbf{k}) = |-ik^0 + (\cos(k) - 1 + r)|$$

Moreover $\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \mathcal{V}^{(0)}(\psi, 0)$ is the *effective potential* on scale 0 and can be written has

$$\mathcal{V}^{(0)}(\psi) = \sum_{n\geq 1} \int d\underline{\mathbf{x}} \int d\underline{\mathbf{y}} W_{2n}^{(0)}(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \prod_{i=1}^{n} \psi^{+}_{\mathbf{x}_i} \psi^{-}_{\mathbf{y}_i} = \sum_{n\geq 1} \mathcal{V}_{2n}^{(0)}(\psi).$$

With a dimensional analysis of the perturbation theory we get:

- $\mathbf{n} = \mathbf{1}, \mathbf{2}$: relevant;
- $\mathbf{n} = \mathbf{3}$: marginal;
- $\mathbf{n} > \mathbf{3}$: irrelevant;

We define

$$\mathcal{V}^{(0)} = \mathcal{L}_1 \mathcal{V}^{(0)} + \mathcal{R}_1 \mathcal{V}^{(0)}$$

with $\mathcal{R}_1 = 1 - \mathcal{L}_1$ and $\mathcal{R}_1$ is defined in the following way;

1. for $n \geq 4$
$$\mathcal{R}_1 \mathcal{V}^{(0)}_{2n} = \mathcal{V}^{(0)}_{2n};$$
2. for $n = 3, 2$
$$\mathcal{R}_1 \mathcal{V}^{(0)}_4(\psi) = \int \prod_{i=1}^{4} d\mathbf{x}_i W^{(0)}_4(\underline{\mathbf{x}}) \psi^+_{\mathbf{x}_1} D^+_{\mathbf{x}_2,\mathbf{x}_1} \psi^-_{\mathbf{x}_3} D^-_{\mathbf{x}_4,\mathbf{x}_3}$$
$$\mathcal{R}_1 \mathcal{V}^{(0)}_6(\psi) = \int \prod_{i=1}^{6} d\mathbf{x}_i W^{(0)}_6(\underline{\mathbf{x}}) \psi^+_{\mathbf{x}_1} D^+_{\mathbf{x}_2,\mathbf{x}_1} D^+_{\mathbf{x}_3,\mathbf{x}_1} \psi^-_{\mathbf{x}_4} D^-_{\mathbf{x}_5,\mathbf{x}_4} D^-_{\mathbf{x}_6,\mathbf{x}_4}$$
where
$$D^{\varepsilon}_{\mathbf{x}_2,\mathbf{x}_1} = \psi^{\varepsilon}_{\mathbf{x}_2} - \psi^{\varepsilon}_{\mathbf{x}_1}$$

3. For $n = 1$

$$\mathcal{R}_1 \mathcal{V}_2^{(0)}(\psi) = \int d\mathbf{x}_1 d\mathbf{x}_2 W_2^{(0)}(\underline{\mathbf{x}}) \psi^+_{\mathbf{x}_1} H^-_{\mathbf{x}_1,\mathbf{x}_2}$$

where

$$H^-_{\mathbf{x}_1,\mathbf{x}_2} = \psi^-_{\mathbf{x}_2} - \psi^-_{\mathbf{x}_1} - (x_{0,1} - x_{0,2})\partial_0 \psi^-_{\mathbf{x}_1} - (x_1 - x_2)\tilde{\partial}_1 \psi^-_{\mathbf{x}_1} - \frac{1}{2}(x_1 - x_2)^2 \tilde{\Delta}_1 \psi$$

and

$$\tilde{\partial}_1 \psi^-_{\mathbf{x}} = \frac{1}{2}(\psi^-_{\mathbf{x}+(0,1)} - \psi^-_{\mathbf{x}-(0,1)}) = \int d\mathbf{k}\, i \sin k e^{i\mathbf{kx}} \hat{\psi}^-_{\mathbf{k}}$$

$$\tilde{\Delta}_1 \psi^-_{\mathbf{x}} = \psi^-_{\mathbf{x}+(0,1)} - 2\psi^-_{\mathbf{x}} + \psi^-_{\mathbf{x}-(0,1)} = 2\int d\mathbf{k}(\cos k - 1) e^{i\mathbf{kx}} \hat{\psi}^-_{\mathbf{k}}$$

As a consequence of the above definitions, calling

$$\hat{W}_2^{(0)}(\mathbf{k}) = \int d\mathbf{x} e^{i\mathbf{k}\mathbf{x}} W_2^{(0)}(\mathbf{x})$$

we get

$$\mathcal{L}_1 \mathcal{V}^{(0)} = \hat{W}_2^{(0)}(0) \int d\mathbf{x} \psi_{\mathbf{x}}^+ \psi_{\mathbf{x}}^- + \partial_0 \hat{W}_2^{(0)}(0) \int d\mathbf{x} \psi_{\mathbf{x}}^+ \partial_0 \psi_{\mathbf{x}}^- + \frac{1}{2} \partial_1^2 \hat{W}_2^{(0)}(0) \int d\mathbf{x} \psi_{\mathbf{x}}^+ \tilde{\Delta}_1 \psi_{\mathbf{x}}^-$$

where we have used that

i. $g^{(0)}(k_0, k) = g^{(0)}(k_0, -k)$, so that we get

$$\partial_1 \hat{W}_2^{(0)}(0) = 0$$

ii. There are no terms in $\mathcal{L}_1 \mathcal{V}^{(0)}$ with four or six fermionic fields, as

$$\psi_{\mathbf{x}_1}^{\varepsilon} D_{\mathbf{x}_2, \mathbf{x}_1}^{\varepsilon} = \psi_{\mathbf{x}_1}^{\varepsilon} \psi_{\mathbf{x}_2}^{\varepsilon}$$

and therefore

$$\mathcal{R}_1 \mathcal{V}_4^{(0)} = \mathcal{V}_4^{(0)} \qquad \mathcal{R}_1 \mathcal{V}_6^{(0)} = \mathcal{V}_6^{(0)}.$$

Since $\mathcal{L}_1\mathcal{V}^{(0)}$ is quadratic in the fields, we can include it in the free integration finding

$$e^{-\mathcal{W}(0)} = e^{-\beta L(F_0+e_0)} \int \tilde{P}(d\psi^{(\leq 0)}) e^{-\mathcal{R}_1\mathcal{V}^{(0)}(\psi^{(\leq 0)})}$$

where the propagator of $\tilde{P}(d\psi^{(\leq 0)})$ is now

$$\tilde{g}^{(\leq 0)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{kx}} \frac{\chi_{\leq 0}(\mathbf{k})}{D_{-1}(\mathbf{k})}$$

with

$$D_{-1}(\mathbf{k}) = -ik_0(1+z_{-1}) + (1+\alpha_{-1})(\cos k - 1) + r + \gamma^{-1}\mu_{-1}$$

and

$$z_{-1} = z_0 + \chi_{\leq 0}(\mathbf{k})\partial_0 \hat{W}_2^{(0)}(0) \qquad \alpha_{-1} = \alpha_0 + \chi_{\leq 0}(\mathbf{k})\partial_1^2 \hat{W}_2^{(0)}(0) \quad (4)$$

$$\gamma^{-1}\mu_{-1} = \mu_0 + \chi_{\leq 0}(\mathbf{k})\hat{W}_2^{(0)}(0) \quad (5)$$

where $z_0 = \alpha_0 = \mu_0 = 0$.

We can now write

$$\tilde{g}^{(\leq 0)}(\mathbf{x}) = g^{(\leq -1)}(\mathbf{x}) + \tilde{g}^{(0)}(\mathbf{x})$$

where

$$g^{(\leq -1)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{\leq -1}(\mathbf{k})}{D_{-1}(\mathbf{k})} \qquad \chi_{<-1}(\mathbf{k}) = \chi_0\left(\gamma |D_{-1}(\mathbf{k})|\right)$$

and

$$\tilde{g}^{(0)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{f_0(\mathbf{k})}{D_{-1}(\mathbf{k})} \qquad f_0(\mathbf{k}) = \chi_{\leq 0}(\mathbf{k}) - \chi_{\leq -1}(\mathbf{k}).$$

and define the new integrations

$$\begin{array}{ccc} \tilde{P}(d\psi^{(0)}) & \longleftrightarrow & \tilde{g}^{(0)}(\mathbf{x}) \\ P(d\psi^{(\leq -1)}) & \longleftrightarrow & \tilde{g}^{(\leq -1)}(\mathbf{x}) \end{array}$$

Using again additivity we get

$$e^{-\mathcal{W}(0)} = e^{-\beta L(F_0+e_0)} \int P(d\psi^{(\leq -1)}) \int \tilde{P}(d\psi^{(0)}) e^{-\mathcal{R}_1 \mathcal{V}^{(0)}(\psi^{(\leq 0)})} =$$
$$= e^{-\beta L F_{-1}} \int P(d\psi^{(\leq -1)}) e^{-\mathcal{V}^{(-1)}(\psi^{(\leq -1)})}$$

where

$$e^{-\beta L\tilde{e}_0 - \mathcal{V}^{(-1)}(\psi^{(\leq -1)})} = \int \tilde{P}(d\psi^{(0)}) e^{-\mathcal{R}_1 \mathcal{V}^{(0)}(\psi^{(\leq 0)})}$$

and $F_{-1} = F_0 + e_0 + \tilde{e}_0$.

Iteration.

At the h step (i.e. at scale h) we start with the integration

$$e^{-\mathcal{W}(0)} = e^{-\beta L F_h} \int P(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\psi^{(\leq h)})}$$

where

$$P(d\psi^{(0)}) \qquad \longleftrightarrow \qquad g^{(\leq h)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{\leq h}(\mathbf{k})}{D_h(\mathbf{k})}$$

with

$$D_h(\mathbf{k}) = -ik_0(1 + z_h) + (1 + \alpha_h)(\cos k - 1) + r + \gamma^h \mu_h$$

$$\chi_{\leq h}(\mathbf{k}) = \chi_0 \left(\gamma^{-h} a_0^{-1} |D_h(\mathbf{k})| \right) .$$

Finally

$$\mathcal{V}^{(h)}(\psi) = \sum_{n \geq 1} \int d\underline{\mathbf{x}} \int d\underline{\mathbf{y}} W_{2n}^{(h)}(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \prod_{i=1}^{n} \psi^+_{\mathbf{x}_i} \psi^-_{\mathbf{y}_i} = \sum_{n \geq 1} \mathcal{V}_{2n}^{(h)}(\psi)$$

Again we can wirte

$$\mathcal{V}^{(h)} = \mathcal{L}_1\mathcal{V}^{(h)} + \mathcal{R}_1\mathcal{V}^{(h)}.$$

where

$$\mathcal{L}_1\mathcal{V}^{(h)} = \hat{W}_2^{(h)}(0)\int d\mathbf{x}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} + \partial_0\hat{W}_2^{(h)}(0)\int d\mathbf{x}\psi^+_{\mathbf{x}}\partial_0\psi^-_{\mathbf{x}} + \frac{1}{2}\partial_1^2\hat{W}_2^{(h)}(0)\int d\mathbf{x}\psi^+_{\mathbf{x}}\tilde{\Delta}_1\psi^-_{\mathbf{x}}$$

Moving the relevant part of the effective potential into the integration we get

$$e^{-\mathcal{W}(0)} = e^{-\beta L(F_h+e_h)}\int \tilde{P}(d\psi^{(\le h)})e^{-\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h)})}$$

where the propagator of

$$\tilde{P}(d\psi^{(\le h)}) \qquad \longleftrightarrow \qquad \tilde{g}^{(\le h)}(\mathbf{x}) = \int d\mathbf{k}e^{i\mathbf{kx}}\frac{\chi_{\le h}(\mathbf{k})}{D_{h-1}(\mathbf{k})}$$

and the *running coupling constants* are defined recursively by

$$z_{h-1} = z_h + \chi_{\le h}(\mathbf{k})\partial_0\hat{W}_2^{(h)}(0) \qquad \alpha_{h-1} = \alpha_h + \chi_{\le h}(\mathbf{k})\partial_1^2\hat{W}_2^{(h)}(0)$$

$$\mu_{h-1} = \gamma\mu_h + \chi_{\le h}(\mathbf{k})\gamma^{-h}\hat{W}_2^{(h)}(0)$$

Finally we can write

$$e^{-\mathcal{W}(0)} = e^{-\beta L(F_h+e_h)} \int P(d\psi^{(\leq h-1)}) \int \tilde{P}(d\psi^{(h)}) e^{-\mathcal{R}_1 \mathcal{V}^{(h)}(\psi^{(\leq h)})}$$

where

$$\tilde{P}(d\psi^{(h)}) \qquad \longleftrightarrow \qquad \tilde{g}^{(h)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{kx}} \frac{f_h(\mathbf{k})}{D_{h-1}(\mathbf{k})}$$

and $f_h(\mathbf{k}) = \chi_{\leq h}(\mathbf{k}) - \chi_{\leq h-1}(\mathbf{k})$; thus

$$e^{-\beta L\bar{e}_h - \mathcal{V}^{h-1}} = \int \tilde{P}(d\psi^{(h)}) e^{-\mathcal{R}_1 \mathcal{V}^{(h)}(\psi^{(\leq h)})}.$$

To show that the above procedure is well defined we need a precise estimates on the propagator.

Lemma

Assume that there exists a constant $K > 0$ *such that*

$$|z_h|, |\alpha_h|, |\mu_h| < K|\lambda|$$

for $h \geq h^*$. *Then for* $|x_0| \leq \beta/2$, *every N and* λ *small enough we have*

$$\left|\partial_0^{n_0} \tilde{\partial}_1^{n_1} \tilde{g}^{(h)}(\mathbf{x})\right| \leq C_N \frac{\gamma^{\frac{h}{2}}}{1 + [\gamma^h |x_0| + \gamma^{\frac{h}{2}} |x|]^N} \gamma^{h(n_0 + n_1/2)}$$

with C_N *independent from* K.

Lemma

There exists a constants $\lambda_0 > 0$, independent of β, L and r, such that the kernels $W_l^{(h)}$ in the domain $|\lambda| \le \lambda_0$, are analytic function of λ and satisfy for $h \ge h^*$

$$\frac{1}{\beta L}\int d\underline{\mathbf{x}}d\underline{\mathbf{y}}|W_{2l}^{(h)}(\underline{\mathbf{x}},\underline{\mathbf{y}})| \le \gamma^{h\left(\frac{3}{2}-\frac{l}{2}\right)}\gamma^{\vartheta h}(C|\lambda|)^{max(1,l-1)}$$

with $\vartheta = \frac{1}{4}$.

Observe that for $h = h^*$ we get

$$\frac{1}{\beta L}\int d\underline{\mathbf{x}}d\underline{\mathbf{y}}|W_{2l}^{(h^*)}(\underline{\mathbf{x}},\underline{\mathbf{y}})| \le (C|\lambda|)^{max(1,l-1)}r^{\frac{3}{2}-\frac{l}{2}}$$

that is a generalization of the condition for the effective coupling since we get

$$\frac{1}{\beta L}\int d\underline{\mathbf{x}}d\underline{\mathbf{y}}|W_{4}^{(h^*)}(\underline{\mathbf{x}},\underline{\mathbf{y}})| \le C\lambda r^{\frac{1}{2}}$$

The second region: Anomalous scaling

We have now to consider the integration of the scales with $h < h^*$, that is

$$e^{-\mathcal{W}(0)} = e^{-\beta L F_{h^*}} \int P(d\psi^{(\leq h^*)}) e^{-\mathcal{V}^{(h^*)}(\psi^{(\leq h^*)})}$$

where

$$P(d\psi^{(\leq h^*)}) \qquad \longleftrightarrow \qquad g^{(\leq h^*)}(\mathbf{x})$$

with

$$g^{(\leq h^*)}(\mathbf{x}) = \int d\mathbf{k} \frac{\chi_{\leq h^*}(\mathbf{k})}{-ik_0(1+z_{h^*}) + (1+\alpha_{h^*})(\cos k - 1) + r + \gamma^{h^*}\mu_{h^*}}$$

We first extract a counterterm to fix the fermi surface

$$\int P(d\psi^{(\leq h^*)}) e^{-\mathcal{V}^{(h^*)}(\psi^{(\leq h^*)})} =$$

$$\int \tilde{P}(d\psi^{(\leq h^*)}) e^{-\mathcal{V}^{(h^*)}(\psi^{(\leq h^*)}) - \gamma^{h^*} \nu_{h^*} \int d\mathbf{x} \psi_{\mathbf{x}}^{(\leq h^*)+} \psi_{\mathbf{x}}^{(\leq h^*)-}}$$

where

$$\tilde{P}(d\psi^{(\leq h^*)}) \qquad \longleftrightarrow \qquad g^{(\leq h^*)}(\mathbf{x})$$

with

$$g^{(\leq h^*)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{\leq h^*}(\mathbf{k})}{-ik_0(1+z_{h^*}) + (1+\alpha_{h^*})(\cos k - \cos p_F)}$$

and

$$(1+\alpha_{h^*}) \cos p_F = (1+\alpha_{h^*}) - r - \gamma^{h^*} \mu_{h^*} + \gamma^{h^*} \nu_{h^*}$$

The strategy of the analysis is the following:

1. we will perform a multiscale analysis. In this analysis we will have to chose $\nu_{h^*} = O(\lambda)$ as function of p_F and λ to obtain a convergent expansion.
2. at the end of the above construction we will use the above relation between p_F and ν_{h^*} to obtain the Fermi momentum p_F as function of λ and r.

We can now write

$$\chi_{\leq h^*}(\mathbf{k}) = \chi_{\leq h^*,1}(\mathbf{k}) + \chi_{\leq h^*,-1}(\mathbf{k})$$

where

$$\chi_{\leq h^*,\omega}(\mathbf{k}) = \tilde{\vartheta}\left(\omega\frac{k}{p_F}\right)\chi_{\leq h^*}(\mathbf{k})$$

where $\omega = \pm 1$, $\tilde{\vartheta}$ is a smooth function such that $\tilde{\vartheta}(k) = 1$ for $k > \frac{1}{2}$ and $\tilde{\vartheta}(k) = 0$ for $k < -\frac{1}{2}$ and

$$\tilde{\vartheta}(k) + \tilde{\vartheta}(-k) = 1$$

for every k.
Thus we can write

$$g^{(\leq h^*)}(\mathbf{x}) = \sum_{\omega=\pm 1} e^{i\omega p_F x} g_\omega^{(\leq h^*)}(\mathbf{x})$$

where

$$g_\omega^{(\leq h^*)}(\mathbf{x}) = \int d\mathbf{k} e^{i(\mathbf{k}-\omega\mathbf{p}_F)\mathbf{x}} \frac{\chi_{\leq h^*,\omega}(\mathbf{k})}{-ik_0(1+z_{h^*}) + (1+\alpha_{h^*})(\cos k - \cos p_F)}$$

with $\mathbf{p}_F = (0, p_F)$.

A new localization operation is defined in the following way

$$\mathcal{L}_2 \int d\underline{\mathbf{x}} W_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \prod_{i=1}^{4} \psi^{\varepsilon_i}_{\mathbf{x}_i,\omega_i} = \hat{W}_4(0) \int d\mathbf{x} \psi^+_{\mathbf{x},1} \psi^-_{\mathbf{x},1} \psi^+_{\mathbf{x},-1} \psi^-_{\mathbf{x},-1}$$

$$\mathcal{L}_2 \int d\underline{\mathbf{x}} W_2(\mathbf{x}_1, \mathbf{x}_2) \psi^+_{\mathbf{x}_1,\omega} \psi^-_{\mathbf{x}_2,\omega} = \hat{W}_2(0) \int \psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} d\mathbf{x} +$$

$$\partial_1 \hat{W}_2(0) \int \bar{\psi}^+_{\mathbf{x},\omega} \Delta_1 \psi^-_{\mathbf{x},\omega} d\mathbf{x} + \partial_0 \hat{W}_2(0) \int \psi^+_{\mathbf{x},\omega} \partial_0 \psi^-_{\mathbf{x},\omega} d\mathbf{x}$$

where

$$\bar{\Delta}_1 f(\mathbf{x}) = 2 \int d\mathbf{k} (\cos k - \cos p_F) e^{i\mathbf{k}\mathbf{x}} \hat{f}(\mathbf{k}) \quad \text{if} \quad f(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \hat{f}(\mathbf{k})$$

After the integration of the scale $\psi^{(h^*)}, ..\psi^{(h)}$ we get

$$e^{-\mathcal{W}(0)} = e^{-\beta L F_h} \int P_{Z_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})}$$

where

$$P_{Z_h}(d\psi^{(\leq h)}) \qquad \longleftrightarrow \qquad \frac{g_\omega^{(\leq h)}}{Z_h}$$

with

$$g_\omega^{(\leq h)}(\mathbf{x}) = \int d\mathbf{k} e^{i(\mathbf{k}-\omega\mathbf{p}_F)} \frac{\chi_{\leq h,\omega}(\mathbf{k})}{-(1+z_{h^*})ik_0 + (1+\alpha_{h^*})(\cos k - \cos p_F)}$$

where

$$\chi_{\leq h,\omega}(\mathbf{k}) = \widetilde{\vartheta}\left(\omega \frac{k}{p_F}\right) \chi_0(a_0\gamma^{-h}|(1+z_{h^*})ik_0 - (1+\alpha_{h^*})(\cos k - \cos p_F)|).$$

We can now write

$$\int P_{Z_h}(d\psi^{(\leq h)})e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} = \int \tilde{P}_{Z_{h-1}}(d\psi^{(\leq h)})e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})}$$

where

$$\begin{aligned}\mathcal{L}_2\tilde{\mathcal{V}}^{(h)}(\psi) =& l_h \int d\mathbf{x}\psi^+_{\mathbf{x},1}\psi^-_{1,\mathbf{x}}\psi^+_{\mathbf{x},-1}\psi^-_{-1,\mathbf{x}} + (a_h - z_h)\sum_\omega \int d\mathbf{x}\psi^+_{\omega,\mathbf{x}}\Delta_1\psi^-_{\omega,\mathbf{x}} + \\ & n_h \int d\mathbf{x}\psi^+_{\omega,\mathbf{x}}\psi^-_{\omega,\mathbf{x}}\end{aligned} \tag{6}$$

while $\tilde{P}_h(d\psi^{(\leq h)})$ is the integration with propagator identical to $g^{(\leq h)}_\omega(\mathbf{x})$ but with $\chi_{\leq h,\omega}(\mathbf{k})$ replaced by $\frac{\chi_{\leq h,\omega}(\mathbf{k})}{\tilde{Z}_{h-1}(\mathbf{k})}$ with

$$\tilde{Z}_{h-1}(\mathbf{k}) = Z_h + \chi_{\leq h,\omega}(\mathbf{k})Z_h z_h$$

Setting $Z_{h-1} = \tilde{Z}_{h-1}(0)$, we can finally write

$$\int P_{Z_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} =$$

$$e^{-\beta L e_h} \int P_{Z_{h-1}}(d\psi^{(\leq h-1)}) \int \tilde{P}_{Z_{h-1}}(d\psi^{(h)}) e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})}$$

where

$$\tilde{P}_{Z_{h-1}} \qquad \longleftrightarrow \qquad \frac{\tilde{g}_\omega^{(h)}}{Z_{h-1}}$$

with

$$\tilde{g}_\omega^{(h)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{x}(\mathbf{k}-\omega \mathbf{p}_F)} \frac{\tilde{f}_{h,\omega}(\mathbf{k})}{-(1+z_{h^*})ik_0 + (1+\alpha_{h^*})(\cos k - \cos p_F)}$$

and

$$\tilde{f}_{h,\omega}(\mathbf{k}) = Z_{h-1}\left[\frac{\chi_{\leq h,\omega}(\mathbf{k})}{\tilde{Z}_{h-1}(\mathbf{k})} - \frac{\chi_{\leq h-1,\omega}(\mathbf{k})}{Z_{h-1}}\right].$$

Finally we have

$$\bar{\mathcal{V}}^{(h)}(\psi^{(\leq h)}) = \tilde{\mathcal{V}}^{(h)}\left(\sqrt{\frac{Z_h}{Z_{h-1}}}\psi^{(\leq h)}\right)$$

so that

$$\mathcal{L}_2\tilde{\mathcal{V}}^h = \lambda_h \int d\mathbf{x}\psi^+_{\mathbf{x},1}\psi^-_{1,\mathbf{x}}\psi^+_{\mathbf{x},-1}\psi^-_{-1,\mathbf{x}} + \delta_h \sum_\omega \int d\mathbf{x}\psi^+_{\omega,\mathbf{x}}\Delta_1\psi_{\omega,\mathbf{x}} + \gamma^h\nu_h \sum_\omega \int d\mathbf{x}\psi^+_{\omega,\mathbf{x}}\psi_{\omega,\mathbf{x}}$$

with

$$\gamma^h\nu_h = \frac{Z_h}{Z_{h-1}} n_h \qquad \delta_h = \frac{Z_h}{Z_{h-1}}(a_h - z_h) \qquad \lambda_h = \left(\frac{Z_h}{Z_{h-1}}\right)^2 l_h$$

We can now integrate the field $\psi^{(h)}$

$$\int \tilde{P}_{Z_{h-1}}(d\psi^{(h)})e^{-\bar{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})} = e^{-\beta L\tilde{e}_h - \mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})}$$

so that the procedure can be iterated.

The propagator.

Again we need precise estimates on the propagator.

Lemma

For $h \leq h^*$, every N and λ small enough we have

$$|\partial_0^{n_0}\partial_1^{n_1}\tilde{g}_\omega^{(h)}(\mathbf{x})| \leq C_N \frac{v_F^{-1}\gamma^h}{1+[\gamma^h|x_0|+v_F^{-1}\gamma^h|x|]^N}\gamma^{h(n_0+n_1)}v_F^{-n_1}$$

with $v_F = \sin(p_F) = O(r^{\frac{1}{2}})$.

Assume now that, given $h < h^*$ we have

$$|\lambda_k|, |\delta_k| \leq C v_F |\lambda| \quad |\nu_k| \leq C|\lambda| \qquad h^* \geq k > h$$

then we have

Lemma

There exists a constants $\lambda_0 > 0$, independent of β, L and r, such that the kernels $W_{2l}^{(h)}$ are analytic functions of λ for $|\lambda| \leq \lambda_0$. Moreover they satisfy

$$\frac{1}{\beta L}\int d\underline{\mathbf{x}}d\underline{\mathbf{y}}|W_{2l}^{(h)}(\underline{\mathbf{x}},\underline{\mathbf{y}})| \leq \gamma^{h(2-l)}v_F^{l-1}\left(C|\lambda|\right)^{max(1,l-1)} .$$

We need to prove by induction that, for $h \leq h^*$ and $\vartheta = \frac{1}{4}$ we have

$$|\lambda_h| \leq C|\lambda| r^{\frac{1}{2}+\vartheta}, \quad |\delta_h| \leq C|\lambda| r^{\frac{1}{2}+\vartheta} \quad |\nu_h| \leq C|\lambda| \gamma^{\vartheta h}$$

It is not hard to check that this is true for $h = h^*$.

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It is not hard to check that this is true for $h = h^*$.

The main observation is that we can decompose the propagator as

$$\tilde{g}^{(h)}_{\omega}(\mathbf{x}) = g^{(h)}_{\omega,L}(\mathbf{x}) + r^{(h)}_{\omega}(\mathbf{x})$$

where

$$g^{(h)}_{\omega,L}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\tilde{f}_h(\mathbf{k})}{-ik_0 + \omega v_F k}$$

and

$$|r^{(h)}_{\omega}(\mathbf{x})| \le \left(\frac{\gamma^h}{v_F}\right)^3 \frac{C_N}{1 + \gamma^h(|x_0| + v_F^{-1}|x|)^N}.$$

The flow of ν_h is given by

$$\nu_{h-1} = \gamma \nu_h + \beta_\nu^{(h)}(\vec{v}_h, ..., \vec{v}_{h^*})$$

where $\vec{v}_h = (\lambda_h, \delta_h, \nu_h)$. Using the decomposition of the propagator we can write

$$\beta_\nu^{(h)} = \bar{\beta}_\nu^{(h)} + \beta_{\nu,R}^{(h)}$$

where

$$\bar{\beta}_\nu^{(h)} = 0 \qquad\qquad \beta_{\nu,R}^{(h)} = O(\lambda \gamma^{\vartheta h})$$

Thus by iteration

$$\nu_{h-1} = \gamma^{-h+h^*}[\nu_{h^*} + \sum_{k=h}^{h^*} \gamma^{k-h^*} \beta_\nu^{(k)}].$$

and we can choose ν_{h^*} so that

$$\nu_{h^*} = -\sum_{k=-\infty}^{h^*} \gamma^{k-h^*} \beta_\nu^{(k)}$$

This implies that

$$\nu_{h-1} = \gamma^{-h+h^*}[-\sum_{k=-\infty}^{h} \gamma^{k-h^*} \beta_\nu^{(k)}]$$

and $|\nu_h| \leq C|\lambda|\gamma^{\vartheta h}$.

The flow equations for λ_h and δ_h with $h < h^*$ are

$$\lambda_{h-1} = \lambda_h + \beta_\lambda^{(h)}(\vec{v}_h, ..., \vec{v}_{h^*})$$
$$\delta_{h-1} = \delta_h + \beta_\delta^{(h)}(\vec{v}_h, ..., \vec{v}_{h^*})$$

Again we can use decomposition of the propagator and decompose the beta function for $\alpha = \lambda, \delta$ as

$$\beta_\alpha^{(h)}(\vec{v}_h, ..., \vec{v}_{h^*}) = \bar{\beta}_\alpha^{(h)}(\lambda_h, \delta_h, ..., \lambda_0, \delta_0) + \beta_{\alpha,R}^{(h)}(\vec{v}_h, ..., \vec{v}_{h^*})$$

where $\bar{\beta}_\alpha^{(h)}$ contains only propagators $g_{\omega,L}^{(h)}(\mathbf{x})$ and end-points to which is associated λ_k, δ_k.
The dimensional gains give us:

$$|\beta_{\alpha,R}^{(h)}| \leq C v_F \lambda^2 \gamma^{\vartheta h}$$

It is easy to see that

$$\bar{\beta}_\lambda^{(h)}(\lambda_h, \delta_h, ..., \lambda_{h^*}, \delta_{h^*}) = v_F \hat{\beta}_\lambda^{(h)} \left(\frac{\lambda_h}{v_F}, \frac{\delta_h}{v_F}, ..., \frac{\lambda_{h^*}}{v_F}, \frac{\delta_{h^*}}{v_F} \right) \tag{7}$$
$$\bar{\beta}_\delta^{(h)}(\lambda_h, \delta_h, ..., \lambda_{h^*}, \delta_{h^*}) = v_F \hat{\beta}_\delta^{(h)} \left(\frac{\lambda_h}{v_F}, \frac{\delta_h}{v_F}, .., \frac{\lambda_{h^*}}{v_F}, \frac{\delta_{h^*}}{v_F} \right) \tag{8}$$

where $\hat{\beta}_\lambda^{(h)}(\lambda_d, \delta_h, .., \lambda, \delta_0)$ is the beta function of a Luttinger model with $v_F = 1$.

The following crucial result, called *asymptotic vanishing of the beta function*,

$$|\hat{\beta}^{(h)}_{\lambda}(\lambda_d, \delta_h, .., \lambda_{h^*}, \delta_{h^*})| \leq C[\max(|\lambda_k|, |\delta_k|)]^2 \gamma^{\vartheta(h-h^*)}$$

Assuming by induction that $|\lambda_k|, |\delta_k| \leq 2|\lambda| r^{\frac{1}{2}+\vartheta}$ for $k \geq h$ we get

$$|\bar{\beta}^{(h)}_{\alpha}(\lambda_h, \delta_h, ..., \lambda_{h^*}, \delta_{h^*})| \leq 4Cv_F\lambda^2 r^{1+2\vartheta}\gamma^{\vartheta h} v_F^{-2} r^{-\vartheta} \leq 4Cv_F\lambda^2\gamma^{\vartheta h} r^{\vartheta}$$

Thus

$$|\lambda_{h-1}| \leq |\lambda_{h^*}| + \sum_{k=h}^{h^*} 4Cv_F\lambda^2\gamma^{\vartheta h} r^{\vartheta} \leq 2|\lambda| r^{\frac{1}{2}+2\vartheta}$$

and the same is true for δ_h.

Moreover we have

$$\frac{Z_{h-1}}{Z_h} = 1 + \beta_z^{(h)}$$

so that

$$\gamma^\eta = 1 + \beta^{-\infty}\left(\frac{\lambda_{-\infty}}{v_F}\right)$$

where $\beta^{-\infty}$ is the beta function with $v_F = 1$; therefore

$$Z_h = \gamma^{-\eta(h-h^*)}(1 + A(\lambda))$$

with $|A(\lambda)| \leq C|\lambda|$. Observe that $\eta = O(\lambda^2 r^{4\vartheta})$, hence is vanishing as $r \to 0$ as $O(\lambda^2 r)$.

Thank You.