Constructive Renormalization Group :

A conference in memory of **Pierluigi Falco**

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Anomaly cancellation to all orders in chiral abelian gauge theories
A perturbative analysis based on (Bi)flow equations

Christoph Kopper and Benjamin Lévêque

Benjamin Lévêque, 1986-2014

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I. Introduction

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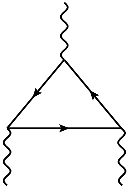
In (perturbative) gauge theories physical consistency is linked to local gauge symmetry. If quantum contributions violate gauge symmetry, the theory generally becomes inconsistent. Such contributions are called anomalies.
In the standard model anomalies are potentially present due to the existence of chiral fermions whose couplings involve the Dirac matrix γ_5 . They potentially give rise to the Adler- or ABJ-anomaly linked to the Feynman amplitude of the triangle diagram

- absent in QED
- linearly UV divergent by power counting in $d = 4$
- does not have a relevant local part



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Way out in practice :
for a given fermion species of charge q (w.r.t. the external gauge field considered) the triangle contribution is proportional to q^3 $\Rightarrow$

If $\sum_{Quarks} (q_1 q_2 q_3 + q_1 q_2 q_3 + q_1 q_2 q_3) + \sum_{Leptons} q_1 q_2 q_3 = 0$ there is

"Anomaly cancellation"

What happens in higher orders ?

Adler 1968, Adler-Bardeen 1969

Zee 1972, Lowenstein-Schroer 1972, Costa-Marinucci-Tonin 1976, Bandelloni-Becchi-Blasi-Collina 1980

Fujikawa 1979

....

Anselmi 2014

"Non-renormalization of the anomaly"

"no anomaly in 1-loop order $\Rightarrow$ no anomaly at all"

II. Chiral abelian gauge theory and the anomaly

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ACTION OF THE MODEL :

$$\mathcal{S}_{cl} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_j \bar{\psi}_j (i \not\partial + q_j \gamma_5 \not A_\mu) \psi_j \right)$$

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$$\mathcal{S}_{c.t.} = \int d^4x \Big[\sum_j (z_{1j} \, \bar{\psi}_j q_j \, \gamma_5 \not{A}_\mu \psi_j + z_{2j} \, \bar{\psi}_j i \not{\partial} \psi_j) + z_3 \frac{\mu^2}{2} A^2$$

$$+ z_4 \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + z_5 \frac{1}{2\alpha} (\partial \cdot A)^2 + z_6 \frac{1}{4!} (A^2)^2 \Big] .$$

The classical theory is invariant w.r.t chiral local gauge transformations :

$$A_\mu(x) \;\rightarrow\; A_\mu(x) + \partial_\mu u(x) \;, \qquad u \in \mathcal{S}(\mathbb{R}^4)$$

$$\psi_j(x) \;\rightarrow\; e^{iq_j\gamma_5 u(x)}\,\psi_j(x) \;, \quad \bar{\psi}_j(x) \;\rightarrow\; \bar{\psi}_j(x)\,e^{iq_j\gamma_5 u(x)}$$

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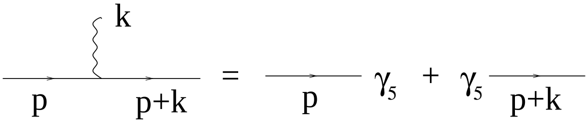

"Ward's Ward identity"

$$\frac{1}{\not p}\,\gamma_5\,\not k\,\frac{1}{\not p + \not k} \quad = \quad \frac{1}{\not p}\,\gamma_5 \quad + \quad \gamma_5\,\frac{1}{\not p + \not k}$$

We will study the theory with the aid of the flow equations with

UV-cutoff Λ_0 and IR- (= Wilson-) cutoff Λ

on the propagators. We choose

$$\sigma_{\Lambda,\Lambda_0}(p) = \exp(-\frac{p^2}{\Lambda_0^2}) - \exp(-\frac{p^2}{\Lambda^2})$$

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Choosing the one-particle irreducible correlation functions

$$\Gamma^{\Lambda,\Lambda_0}_{m,n,L}$$

m : # external bosons, n : # external fermions, L : loop order

to describe the theory we obtain the following

General form of the violated Ward-Identities :

$$\underbrace{\Omega^{\Lambda,\Lambda_0}_{m+1,n}(p_{[m+n]})}_{=0 \text{ if WI's hold}} = p_1 \cdot \Gamma^{\Lambda,\Lambda_0}_{m+1,n}(p_{[m+n]})$$

$$+ \sum_{k\in[n/2]} q_{j(k)}\, \underset{\uparrow}{\gamma_5}\, \Gamma^{\Lambda,\Lambda_0}_{m,n}(p_2, \ldots, \underset{\uparrow}{p_{m+k+1}} + p_1, \ldots)$$

$$+ \sum_{k\in[n/2]} q_{j(k)}\, \Gamma^{\Lambda,\Lambda_0}_{m,n}(p_2, \ldots, \underset{\uparrow}{p_{m+n/2+k+1}} + p_1, \ldots)\, \underset{\uparrow}{\gamma_5}$$

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$$\partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0}_{m,n,L} = \ldots \int_k \dot{C}^\Lambda_F(k)\ \mathcal{L}^{\Lambda,\Lambda_0}_{m,n+2,L-1} + \ldots \int_k \dot{C}^\Lambda_B(k)\ \mathcal{L}^{\Lambda,\Lambda_0}_{m+2,n,L-1} +$$

$$\sum \ldots \mathcal{L}^{\Lambda,\Lambda_0}_{m',n',L'}\ \dot{C}^\Lambda_F\ \mathcal{L}^{\Lambda,\Lambda_0}_{m'',n'',L''} + \sum \ldots \mathcal{L}^{\Lambda,\Lambda_0}_{m',n',L'}\ \dot{C}^\Lambda_B\ \mathcal{L}^{\Lambda,\Lambda_0}_{m'',n'',L''}\Big|_{\textit{sum rules}}$$

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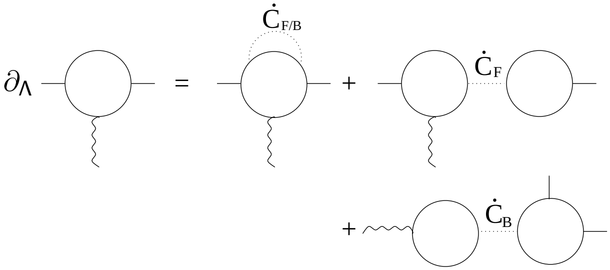
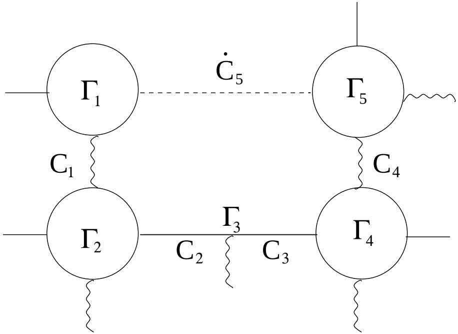


Example: contributions to the flow of $\mathcal{L}_{1,2}$

rewritten for the one-particle irreducible functions

$$\partial_\Lambda \Gamma^{\Lambda,\Lambda_0}_{m,n,L} = \sum_k \sum_{m_i,n_i,L_i} \ldots \prod_{i<k} (\Gamma^{\Lambda,\Lambda_0}_{m_i,n_i.L_i} C_i^{\Lambda,\Lambda_0})\, \Gamma^{\Lambda,\Lambda_0}_{m_k,n_k,L_k}\ \dot{C}_k^\Lambda \Big|_{sum\ rules}$$



Example: contribution to the flow of $\Gamma_{4,4}$; Γ_3 *is of loop order* 0

From the FE we deduce inductively in the loop order L the following properties of the one-particle irreducible Schwinger functions $\Gamma^{\Lambda,\Lambda_0}_{m,n,L}$ of chiral abelian gauge theory (for suitable renormalization conditions) :

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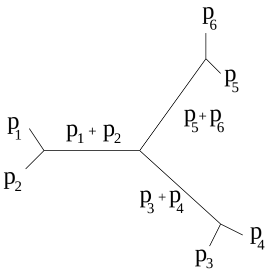
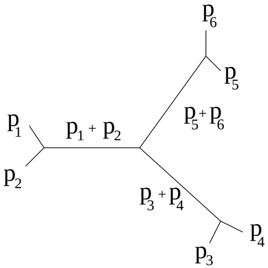
This proves *weak renormalizability* of the theory, independently of the presence of the triangle anomaly



Example : tree for a six-point function : $m + n = 6$.



Example : tree for a six-point function : $m + n = 6$.

The corresponding bound is of the form

$$\frac{1}{\sup(|p_1 + p_2|, \Lambda)^{\theta_1}} \ \frac{1}{\sup(|p_3 + p_4|, \Lambda)^{\theta_2}} \ \frac{1}{\sup(|p_5 + p_6|, \Lambda)^{\theta_3}}$$

$$\times \ \mathcal{P}_{2L}\Big(\log_+ \frac{\Lambda}{\mu}, \log_+ \frac{\sup |p_i|}{\sup\{\Lambda, \inf\{\eta(p), \mu\}\}}\Big)$$

$\sum \theta_i = 4 - m - 3/2\, n$; μ *is the renormalization scale* ; $\eta(p)$ *is the smallest strict subsum of external momenta*

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$$\partial_\Lambda \Gamma^{\Lambda,\Lambda_0}_{\mathcal{O}(x),m,n,L} \;=$$

$$\sum_{k_1}' \sum_{m_i,n_i,L_i}' \cdots \prod_{i\leq k_1} (\Gamma^{\Lambda,\Lambda_0}_{m_i,n_i,L_i} C^{\Lambda,\Lambda_0}_i) \left[\sum_{m_O,n_O,L_O}' \Gamma^{\Lambda,\Lambda_0}_{\mathcal{O}(x),m_O,n_O,L_O} \right]$$

$$\sum_{k_2}' \sum_{m_j,n_j,L_j}' \cdots \prod_{j\leq k_2} (C^{\Lambda,\Lambda_0}_j \; \Gamma^{\Lambda,\Lambda_0}_{m_j,n'_j,L_j}) \; \dot{C}^{\Lambda}_{F/B} \Big|_{sum\ rules}$$

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This system is *linear with respect to the inserted functional* $\Gamma^{\Lambda,\Lambda_0}_{\mathcal{O}(x)}$

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- It can be shown that *the violations of the Ward identities for* $\Lambda = 0$ *,* $\Omega_{m.n}^{0,\Lambda_0}$ *, are the moments of an operator insertion functional* of dimension 5. This functional corresponds to the violation of BRST-invariance in the regularized functional integral of the theory considered.

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- *If the relevant part of* Ω^{0,Λ_0} (in the sense of the renormalization group) *can be made vanish* - by choosing appropriate renormalization conditions for the $\Gamma^{0,\Lambda_0}_{m,n,L}$ - we can show with the aid of the operator inserted Flow equations

$$\lim_{\Lambda_0 \to \infty} \Omega^{0,\Lambda_0}_{m,n,L} = 0 \quad (\mathit{Restitution\,Theorem})$$

$\Rightarrow$ *the Ward identities are restored for* $\Lambda_0 \to \infty$

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The one-loop triangle $\Gamma^{0,\Lambda_0}_{3,0}(p_{[3]})$ *has no relevant local content :*
By general symmetry considerations the lowest order local term of order three in A_μ is in fact of dimension six : $\partial\cdot A \;\; \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$.
Correspondingly one verifies explicitly that $\Gamma^{0,\Lambda_0}_{3,0,L=1}(p_{[3]})$ is of order three in the external momenta.

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Correspondingly one verifies explicitly that $\Gamma^{0,\Lambda_0}_{3,0,L=1}(p_{[3]})$ is of order three in the external momenta.

$\Omega^{0,\Lambda_0}_{3,0}(p_{[3]})$ *can however have a local relevant contribution*

We find for a single fermion species of charge q :

$$\lim_{\Lambda_0\to\infty}\Omega^{0,\Lambda_0}_{3,0,L=1}(p_{[3]})_{\mu_2,\mu_3} = \frac{q^3}{2\pi^2}\,\varepsilon_{\mu_2\mu_3\rho\sigma}\,p_2^\rho\,p_3^\sigma$$

We thus have

$$\Omega_{3,0}^{0,\Lambda_0}(p_{[3]})\Big|_{rel} \equiv \; p_1 \cdot \Gamma_{3,0}^{0,\Lambda_0}(p_{[3]})\Big|_{rel} \neq 0$$

and *there does not exist a local counter* term for $\Gamma_{3,0}^{0,\Lambda_0}(p_{[3]})$ to annihilate the relevant part of $\Omega_{3,0}^{0,\Lambda_0}(p_{[3]})|_{rel}$

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$$\Rightarrow$$

If $\Gamma_{3,0}^{0,\Lambda_0} \neq 0$ *at one-loop order, the Ward identities cannot be restored and gauge invariance is lost.*

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We are interested in the situation where $\Lambda_F < \Lambda_B$. In this case we obtain similar bounds as before, replacing $\Lambda \to \Lambda_B$ in terms growing with Λ and replacing $\Lambda \to \Lambda_F$ in terms growing with $1/\Lambda$.

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For the tree of the previous example :

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We are in particular interested in the limit $\Lambda_F \to 0\,,\ \Lambda_{0F} \to \infty$ since we can show that the Ward Identities can already be restored in this limit without taking $\Lambda_B \to 0\,,\ \Lambda_{0B} \to \infty\,.$

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To be safe on the infrared side we consider the theory on a torus of side length L in position space, with antiperiodic boundary conditions for the fermions.

We proceed inductively in the loop order L

- Show that for appropriate renormalization conditions

$$\Omega_{m,n,L}^{\Lambda_{0B},\Lambda_{0B},0,\infty} = 0$$

For $\Lambda_B = \Lambda_{0B}$ the bosonic propagator vanishes. So one has to show that diagrams with one fermionic loop *including the coupling constant and fermionic wave function counter terms* $z_1(\Lambda_{0B}, \infty)$ *and* $z_2(\Lambda_{0B}, \infty)$, vanish when contracted with one external momentum p_1 .

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The proof is based on "Ward's Ward identity" once it has been shown inductively that *the theory can be renormalized such that*

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Remember that the contracted (generalized) triangle is finite for $\Lambda_{0F} \to \infty$ *, and the sum of the triangle contributions vanishes if* $\sum q_j^3 = 0$ *.*

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Proof :
Write the *bosonic flow equations* for the contributions to

$$\Omega_{m,n}^{\Lambda_B,\Lambda_{0B},0,\Lambda_{0F}} \quad .$$

Sum over these contributions to obtain

$$\partial_{\Lambda_B}\Omega_{m,n,L}^{\Lambda_B,\Lambda_{0B},0,\infty} \;=\; \dots$$

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a) Terms having an additional external boson carrying momentum p_1, which then is attached to some $\Gamma^{\Lambda_B,\Lambda_{0B},0,\infty}_{m',n',L'}$. This new channel is then contracted with the 4-momentum p_1. So the result is

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b) Terms where the momentum of one external fermion -each of them appearing in some $\Gamma^{\Lambda_B,\Lambda_{0B},0,\infty}_{m',n',L'}$- is increased by p_1. The corresponding spinor index is contracted with γ_5 and multiplied by the corresponding charge q' :

$$\Gamma^{\Lambda_B,\Lambda_{0B},0,\infty}_{m',n',L'}(\ldots,p',\ldots) \;\to\; q'\; \Gamma^{\Lambda_B,\Lambda_{0B},0,\infty}_{m'+1,n',L'}(\ldots,p'+p_1,\ldots)\,\gamma_5$$

c) Terms where the additional external boson is attached directly to a fermionic line producing a new vertex of loop order 0 with the corresponding charge

$$\frac{1}{\not{p}} \;\rightarrow\; \frac{1}{\not{p}}\, q\, \gamma_5\, \not{p}_1\, \frac{1}{\not{p} + \not{p}_1}$$

c) Terms where the additional external boson is attached directly to a fermionic line producing a new vertex of loop order 0 with the corresponding charge

$$\frac{1}{\not p} \;\to\; \frac{1}{\not p}\, q\, \gamma_5\, \not p_1\, \frac{1}{\not p + \not p_1}$$

We realize that the sum of a) and b) reproduces $\Omega^{\Lambda_B,\Lambda_{0B},0,\infty}_{m',n',L'}$ - unless $\Omega^{\Lambda_B,\Lambda_{0B},0,\infty}_{m',n',L'}$ is linked to the chain via a fermionic propagator. In this case the term where the momentum of **this** propagator is shifted, is lacking.

In this last case consider contribution c) and use Ward's Ward identity :

$$\Gamma(\ldots,-p)\ \frac{1}{\not p}\ \gamma_5\ \not p_1\ \frac{1}{\not p+\not p_1}\ \ \Gamma(p+p_1,\ldots)\ =$$

$$\Gamma(\ldots,-p-p_1+p_1)\,\gamma_5\ \frac{1}{\not p+\not p_1}\Gamma(p+p_1,\ldots)+\Gamma(\ldots,-p)\ \frac{1}{\not p}\gamma_5\ \Gamma(p+p_1,\ldots)$$

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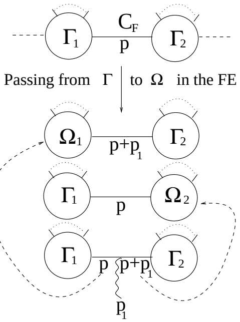

Regrouping contributions to the flow of Ω

The two terms from Ward's Ward identity are the lacking terms from the previous sum of a) and b) $\Rightarrow$ $\Omega^{\Lambda_B,\Lambda_{0B},0,\infty}$ is reproduced on the r.h.s. of the FE

$\Rightarrow$ *The bosonic FE for Ω is linear in Ω*

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Boundary conditions : $\Omega^{\Lambda_{0B},\Lambda_{0B},0,\infty}_{m,n,L} = 0$ (see above)

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Inductive assumption : $\Omega^{\Lambda_B,\Lambda_{0B},0,\infty}_{m',n',L'} = 0$ for $L' < L$

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$$\Rightarrow \quad \boxed{\Omega^{\Lambda_B,\Lambda_{0B},0,\Lambda_{0F}}_{m,n,L} = 0 \quad \text{at loop order } L}$$

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To resume :

Chiral abelian gauge theory is a renormalizable gauge theory to all orders of perturbation theory if $\sum_j q_j^3 = 0$.