Universality of charge transport in the interacting Haldane model

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Joint work with A. Giuliani and V. Mastropietro

Outline

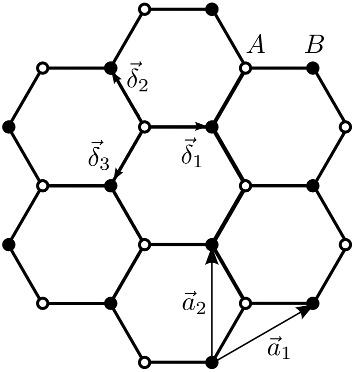
- Introduction: Graphene and the Haldane model.
- Integer Quantum Hall effect in the interacting Haldane model.
- Universality of graphene's optical conductivity
  (also related to Hall transitions in the Haldane model).
- Conclusions.

Introduction

Graphene and the Haldane model

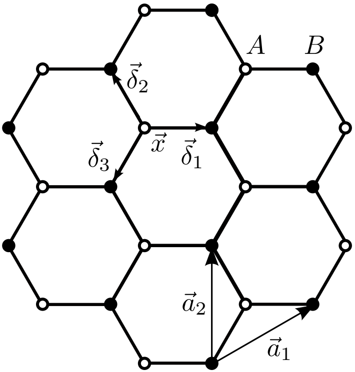
Graphene

Monoatomic layer of graphite. Isolated for the first time in 2004 (Geim & Novoselov, Nobel prize 2010). First realization of a "$2d$ crystal".



- Superposition of 2 triangular lattices Λ_A, Λ_B.
- Each site corresponds to a carbon atom.
- For neutral graphene, 1 electron per site (in average).

Noninteracting theory



In Fock space: $$\mathcal{H}^{(0)} = -t \sum_{\vec{x}\in\Lambda_A} \sum_{j=1,2,3} \sum_{\sigma=\uparrow\downarrow} a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_j,\sigma} + h.c.$$

$a^\pm$, $b^\pm$ = fermionic creation/annihilation operators.

Noninteracting theory

$$\mathcal{H}^{(0)} = -t \sum_{\vec{x}\in\Lambda_A} \sum_{j=1,2,3} \sum_{\sigma=\uparrow\downarrow} a^{+}_{\vec{x},\sigma} b^{-}_{\vec{x}+\vec{\delta}_j,\sigma} + h.c.$$

$$= \int_{\mathbb{T}^2} \sum_{\sigma=\uparrow\downarrow} \begin{pmatrix} \hat{a}^{+}_{\vec{k},\sigma} & \hat{b}^{+}_{\vec{k},\sigma} \end{pmatrix} \begin{pmatrix} 0 & -t\Omega(\vec{k})^* \\ -t\Omega(\vec{k}) & 0 \end{pmatrix} \begin{pmatrix} \hat{a}^{-}_{\vec{k},\sigma} \\ \hat{b}^{-}_{\vec{k},\sigma} \end{pmatrix}$$

$$\equiv \int_{\mathbb{T}^2} \sum_{\sigma=\uparrow\downarrow} \begin{pmatrix} \hat{a}^{+}_{\vec{k},\sigma} & \hat{b}^{+}_{\vec{k},\sigma} \end{pmatrix} H^{(0)}(\vec{k}) \begin{pmatrix} \hat{a}^{-}_{\vec{k},\sigma} \\ \hat{b}^{-}_{\vec{k},\sigma} \end{pmatrix}$$

$\Omega(\vec{k}) = \sum_{j=1,2,3} e^{i\vec{k}(\vec{\delta}_j - \vec{\delta}_1)}$.

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$\Omega(\vec{k}) = \sum_{j=1,2,3} e^{i\vec{k}(\vec{\delta}_j-\vec{\delta}_1)}$. Spectrum: $\sigma(H^{(0)}(\vec{k})) = \{-t|\Omega(\vec{k})|, t|\Omega(\vec{k})|\}$.

$$\Omega(\vec{k}) = 0 \iff \vec{k} = \vec{k}^\pm_F \,, \qquad \Omega(\vec{k}' + \vec{k}^\pm_F) \simeq (3/2)(ik'_1 \pm k'_2) + O(|\vec{k}'|^2)$$

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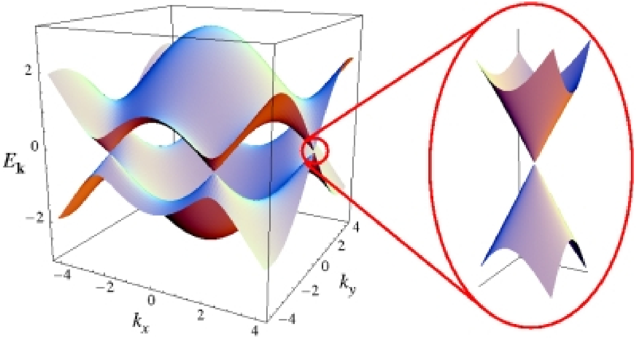
Close to $\vec{k}^\pm_F$, "relativistic" Hamiltonian:

$$H^{(0)}(\vec{k}) \simeq H^{(0)}_{\text{rel}}(\vec{k}) = \begin{pmatrix} 0 & -v(ik'_1 \pm k'_2) \\ -v(-ik'_1 \pm k'_2) & 0 \end{pmatrix}\,, \qquad v = \frac{3t}{2}$$

Emergent "relativistic" theory



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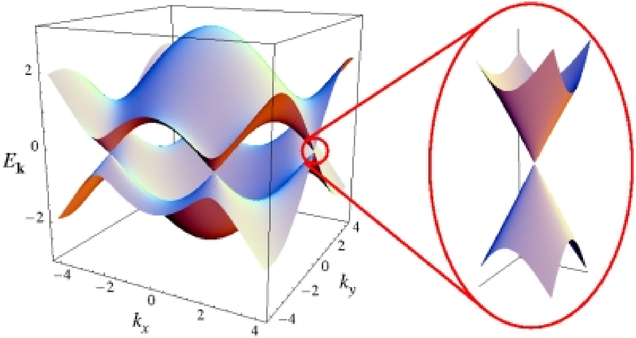

- Close to $\vec{k}_F^{\pm}$, the low-energy excitations are well described by $2+1$ dim. massless Dirac fermions ($v \simeq c/300$).

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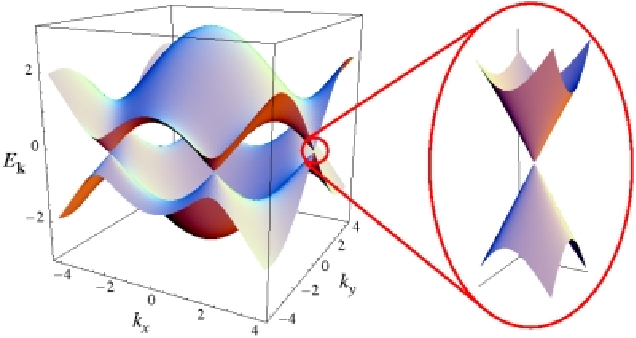

- Close to $\vec{k}_F^{\pm}$, the low-energy excitations are well described by $2+1$ dim. massless Dirac fermions ($v \simeq c/300$).
- The "relativistic" nature of the charge carriers in graphene gives rise to remarkable transport properties.

Example: Optical conductivity

- Linear response. Weak $\vec{E}(\omega)$,

$$J_i(\omega) \simeq \sum_{j=1,2} \sigma_{ij}(\omega) E_j(\omega)$$

- Optical regime. $\hbar\omega \gg k_{\mathrm{B}} T$.

- Prediction? Neglect lattice, disorder, interactions.
  Green-Kubo:

$$\sigma_{11} := \lim_{\omega \to 0^+} \lim_{T \to 0} \sigma_{11}(i\omega) = \frac{e^2}{h} \frac{\pi}{2}$$

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- How to measure it?

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Striking agreement with experiments:

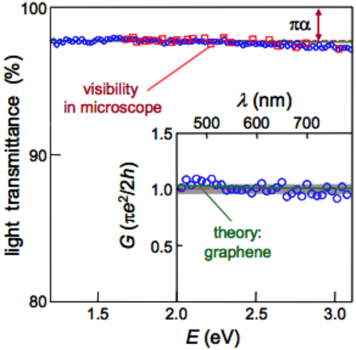

Figure: Nair et al., Science '08

Integer Quantum Hall effect

- Graphene is an ideal compound for integer quantum Hall effect.
- In general, in $d = 2$ at $T = 0$, if the Fermi energy lies in a gap and the system is exposed to a transverse magnetic field:

$$\sigma_{11} = 0 \,, \qquad \sigma_{12} = \frac{e^2}{h} \cdot \nu \,, \qquad \nu \in \mathbb{Z} \,.$$

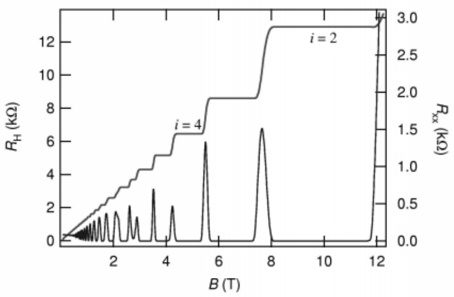

Figure: The resistivity matrix $R = \sigma^{-1}$ in a Quantum Hall system.

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For noninteracting systems, the quantization of σ_{12} has a deep topological explanation (Avron-Seiler-Simon '83, Bellissard-Van Elst-Schulz Baldes '94...)

$$\sigma_{12} = i \mathrm{Tr}\, P_{\leq\mu} \big[[P_{\leq\mu}, x_1],\, [P_{\leq\mu}, x_2] \big] \in \mathbb{Z}$$

where $P_{\leq\mu}$ = projector over energy levels $\leq \mu$.

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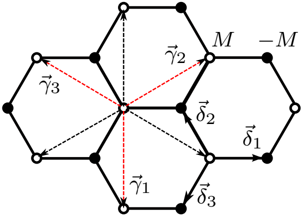
- Today. Simple graphene-like model for IQHE, the Haldane model. For free particles, the "topological phase diagram" can be explicitly computed.
- Goal. Stability of IQHE in presence of weak many-body interactions.

IQHE and the Haldane model

- Haldane, '88. Graphene (spinless) + staggered potential + nnn hopping.

$$\mathcal{H}_{\mathsf{H}}^{(0)} = -t_1 \sum_{j=1,2,3} \sum_{\vec{x}\in\Lambda_A} a^+_{\vec{x}} b^-_{\vec{x}+\vec{\delta}_j} + b^+_{\vec{x}+\vec{\delta}_j} a^-_{\vec{x}}$$
$$+ M\Big[\sum_{\vec{x}\in\Lambda_A} a^+_{\vec{x}} a^-_{\vec{x}} - \sum_{\vec{x}\in\Lambda_B} b^+_{\vec{x}} b^-_{\vec{x}} \Big]$$
$$- t_2 \sum_{\substack{\alpha=\pm \\ j=1,2,3}} \Big[\sum_{\vec{x}\in\Lambda_A} e^{i\alpha\phi} a^+_{\vec{x}} a^-_{\vec{x}+\alpha\vec{\gamma}_j} + \sum_{\vec{x}\in\Lambda_B} e^{-i\alpha\phi} b^+_{\vec{x}} b^-_{\vec{x}+\alpha\vec{\gamma}_j} \Big]$$

- Black: $t_2 e^{i\phi}$. Red: $t_2 e^{-i\phi}$
- Zero net flux.



IQHE and the Haldane model

- M breaks inversion, ϕ breaks time-reversal.
- Graphene's cones split. Gaps: $\Delta_{\pm} = |m_{\pm}|$, with $m_{\pm} = M \pm 3\sqrt{3}t_2 \sin\phi$

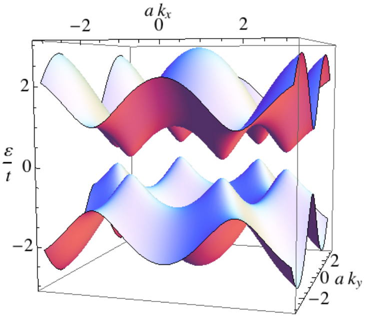


Figure: Gapped spectrum.

Phase diagram of the Haldane model

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- IQHE without net external flux: $\sigma_{12} = \frac{e^2}{2h}\big[\text{sgn}(m_+) - \text{sgn}(m_-)\big]$.

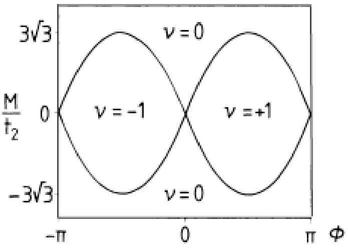

Figure: $\nu = \frac{1}{2}[\text{sgn}(m_+) - \text{sgn}(m_-)]$ (Haldane '88).

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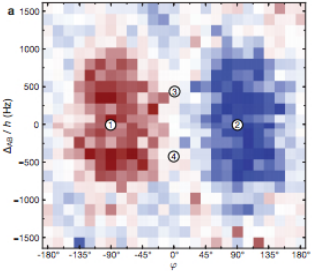

Figure: Experimental realization, Esslinger group, ETHZ (Nature '14)

The interacting Haldane model
Quantization of the Hall conductivity

The interacting Haldane model

- Let $\Lambda = \Lambda_A \cup \Lambda_B$. We consider:

$$\mathcal{H}_{\mathsf{H}} = \mathcal{H}_{\mathsf{H}}^{(0)} + U\mathcal{V}$$
$$\mathcal{V} = \frac{1}{2}\sum_{\vec{x},\vec{y}\in\Lambda} n_{\vec{x}}\, n_{\vec{y}}\, v(\vec{x}-\vec{y})\ .$$

with:

$$n_{\vec{x}} = \begin{cases} a^+_{\vec{x}} a^-_{\vec{x}} & \vec{x}\in\Lambda_A \\ b^+_{\vec{x}} b^-_{\vec{x}} & \vec{x}\in\Lambda_B \end{cases}$$

- $v(\vec{x})$ is assumed to be short-ranged, with Fourier transform $\hat{v}(\vec{p})$ in C^∞.
- The grandcanonical Gibbs state is:

$$\langle\cdot\rangle_{\beta,\Lambda} = \frac{\mathrm{Tr}\, e^{-\beta(\mathcal{H}_{\mathsf{H}}-\mu\mathcal{N})}\,\cdot}{\mathrm{Tr}\, e^{-\beta(\mathcal{H}_{\mathsf{H}}-\mu\mathcal{N})}}$$

with $\mathcal{N} = \sum_{\vec{x}\in\Lambda} n_{\vec{x}}$ = number operator.

Current operator and Green-Kubo formula

- The current operator is defined as:

$$\vec{J} := i\big[\mathcal{H}_{\mathsf{H}}\,, d\Gamma_\Lambda(\vec{x})\big] = i\big[\mathcal{H}^{(0)}_{\mathsf{H}}\,, d\Gamma_\Lambda(\vec{x})\big]$$

$$d\Gamma_\Lambda(\vec{x}) := \sum_{\vec{x}\in\Lambda_A} \vec{x}\, a^+_{\vec{x}} a^-_{\vec{x}} + \sum_{\vec{x}\in\Lambda_B} \vec{x}\, b^+_{\vec{x}} b^-_{\vec{x}}\ .$$

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- The conductivity matrix is defined starting from Green-Kubo formula:

$$\sigma_{ij} := -\frac{1}{A}\lim_{p_0\to 0}\frac{\partial}{\partial p_0}\widehat{K}_{ij}(p_0)$$

with $A = 3\sqrt{3}/2 =$ area of the hexagonal cell and

$$\widehat{K}_{ij}(p_0) = \lim_{\beta,\,|\Lambda|\to\infty}\frac{1}{\beta|\Lambda|}\int_0^\beta dx_0\int_0^\beta dy_0\, e^{ip_0(x_0-y_0)}\big\langle \mathbf{T}\{J_i(x_0)\,;J_j(y_0)\}\big\rangle_{\beta,\Lambda}$$

- $\vec{J}(x_0) = e^{(\mathcal{H}_{\mathsf{H}}-\mu\mathcal{N})x_0}\vec{J}e^{-(\mathcal{H}_{\mathsf{H}}-\mu\mathcal{N})x_0}$, $\mathbf{T} =$ fermionic time ordering.

QHE for interacting systems

- Hastings-Michalakis '15. Rigorous proof of quantization of σ_{12} for general interacting systems, under the assumptions:

  1. The interacting ground state is nondegenerate;
  2. $\exists$ gap separating interacting ground state and first excited state.

  These assumptions might be very hard to prove in translation invariant, interacting systems.

- Earlier (nonrigorous) approaches, based on effective field theory description:

  Wen '90, Fröhlich et al. '91, Zhang '92, ... Fröhlich-Werner '14.

  (Powerful ideas based on gauge invariance and Ward identities, extensions to fractional Quantum Hall effect and topological insulators).

- Our approach is close in spirit to these last ones, and in particular to:

  Coleman-Hill '85: no corrections beyond 1-loop to the "topological mass" in QED_{2+1}.

Stability of IQHE in the interacting Haldane model

Theorem (Giuliani - Mastropietro - P., '15.)

Let M/t_2, ϕ be away from the critical lines. Let μ be in a spectral gap of $\mathcal{H}_H^{(0)}$. Then, there exists $U_0 > 0$ such that:

1) $\widehat{K}_{ij}(\mathbf{p})$ is analytic in $|U| \leq U_0$, and C^∞ in $\mathbf{p} \in \mathbb{R} \times \mathbb{T}^2$;

2) The conductivity matrix $(\sigma_{ij})_{i,j=1,2}$ is given by (restoring e and h):

$$\sigma_{11} = \sigma_{22} = 0\,, \qquad \sigma_{12} = -\sigma_{21} = \frac{e^2}{2h}\big[\mathit{sgn}(m_+) - \mathit{sgn}(m_-)\big]\,,$$

with $m_\pm = M \pm 3\sqrt{3}t_2 \sin\phi$.

Remarks

- Our result provides the first rigorous proof of quantization for the Hall conductivity for an interacting system, without extra assumptions.
- Point 1) is a standard application of fermionic cluster expansion and multiscale analysis. Methods developed in the last 30 years by:

  Brydges-Battle-Federbush, Gawedzki-Kupiainen, Lesniewski, Benfatto-Gallavotti, Feldman-Knörrer-Trubowitz, Magnen-Rivasseau-Sénéor, Benfatto-Mastropietro...

  Recall: away from the critical line, the theory is massive.
- Point 2), that is the universality of σ_{ij}, is based on Ward identities.

Sketch of the proof

- Let $\mathbf{p} = (p_0, \vec{p}) \in \mathbb{R} \times \mathbb{T}^2$. Starting point:

$$\widehat{K}_{ij}(\mathbf{p}) - \widehat{K}^{(0)}_{ij}(\mathbf{p}) = \int_0^U dU' \frac{\partial}{\partial U'} \widehat{K}_{ij}(\mathbf{p}) = \sum_{k \geq 1} \frac{U^k}{k!} \widehat{K}^{(k)}_{ij}(\mathbf{p})$$

- Evaluating the derivative we get, for $k \geq 1$:

$$\begin{aligned}\widehat{K}^{(k)}_{i,j}(\mathbf{p}) = \frac{1}{2} \int \frac{d\mathbf{q}}{(2\pi)^3} \hat{v}(\vec{q}) \widehat{K}^{(k-1)}_{i,j,0,0}(\mathbf{p}, -\mathbf{p}, \mathbf{q}) \\ + \sum_{m=0}^{k-1} \binom{k-1}{m} \hat{v}(\vec{0}) \widehat{K}^{(m)}_{i,j,0}(\mathbf{p}, -\mathbf{p}) \widehat{K}^{(k-1-m)}_{0}(\mathbf{0}) \\ + \sum_{m=0}^{k-1} \binom{k-1}{m} \hat{v}(\vec{p}) \widehat{K}^{(m)}_{i,0}(\mathbf{p}) \widehat{K}^{(k-1-m)}_{j,0}(-\mathbf{p})\end{aligned}$$

with:

$$\widehat{K}_{i,j,0,0}(\mathbf{p}) = \left\langle \mathbf{T}\{\hat{J}_{i,\mathbf{p}}\,;\, \hat{J}_{j,-\mathbf{p}}\,;\, \hat{\rho}_{\mathbf{q}}\,;\, \hat{\rho}_{-\mathbf{q}}\} \right\rangle , \qquad \hat{\rho}_{\mathbf{q}} = \hat{n}^A_{\mathbf{q}} + e^{i\mathbf{q}\cdot\delta_1} \hat{n}^B_{\mathbf{q}} .$$

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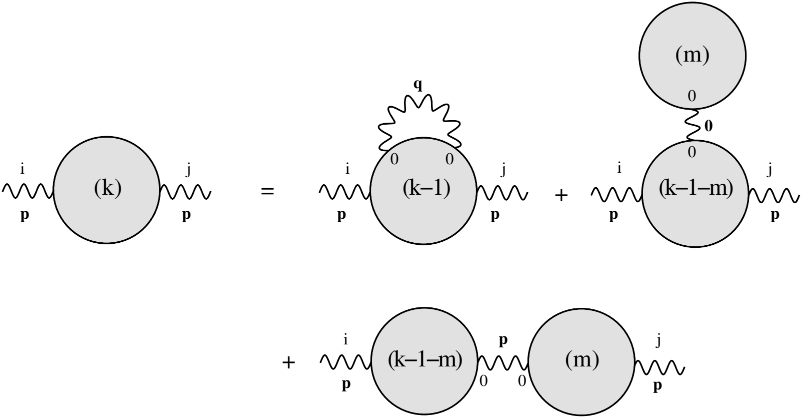

Ward identities

- Being σ_{ij} analytic for $|U| \leq U_0$, to prove universality it is sufficient to show:

$$\sigma_{ij}^{(k)} = -\frac{2}{3\sqrt{3}} \lim_{p_0 \to 0} \frac{\partial}{\partial p_0} \widehat{K}_{i,j}^{(k)}(p_0, \vec{0}) = 0 \, , \qquad \forall k \geq 1 \, .$$

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- Continuity equation:

$$\partial_{x_0} \hat{\rho}_{\vec{p}}(x_0) = \left[\mathcal{H}_{\mathsf{H}},\, \hat{\rho}_{\vec{p}}(x_0)\right] = \sum_{i=1,2} p_i \cdot \hat{J}_{i,\vec{p}}(x_0) \,.$$

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- It implies the following Ward identity, for all $n \geq 0$:

$$\sum_{\mu=0,1,2} p_\mu \widehat{K}_{\mu,0}^{(n)}(\mathbf{p}) = 0 \Rightarrow \widehat{K}_{i,0}^{(n)}(\mathbf{p}) = -\sum_{\mu=0,1,2} p_\mu \frac{\partial}{\partial p_i} \widehat{K}_{\mu,0}^{(n)}(\mathbf{p}) \ .$$

Similarly: $\widehat{K}_{i,j,0,0}^{(n)}(\mathbf{p}, -\mathbf{p}, \mathbf{q}) = O(\mathbf{p}^2)$, $\widehat{K}_{i,j,0}^{(n)}(\mathbf{p}, -\mathbf{p}) = O(\mathbf{p}^2)$.

Sketch of the proof

$$\widehat{K}^{(n)}_{i,0}(\mathbf{p}) = O(\mathbf{p})\,, \qquad \widehat{K}^{(n)}_{i,j,0}(\mathbf{p},-\mathbf{p}) = O(\mathbf{p}^2)\,, \qquad \widehat{K}^{(n)}_{i,j,0,0}(\mathbf{p},-\mathbf{p},\mathbf{q}) = O(\mathbf{p}^2)\,.$$

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$$\widehat{K}^{(n)}_{i,0}(\mathbf{p}) = O(\mathbf{p}) \,, \qquad \widehat{K}^{(n)}_{i,j,0}(\mathbf{p},-\mathbf{p}) = O(\mathbf{p}^2) \,, \qquad \widehat{K}^{(n)}_{i,j,0,0}(\mathbf{p},-\mathbf{p},\mathbf{q}) = O(\mathbf{p}^2) \,.$$

This implies:

$$\begin{aligned}\widehat{K}^{(k)}_{i,j}(\mathbf{p}) &= \frac{1}{2}\int \frac{d\mathbf{q}}{(2\pi)^3}\,\hat{v}(\vec{q})\widehat{K}^{(k-1)}_{i,j,0,0}(\mathbf{p},-\mathbf{p},\mathbf{q}) \\ &\quad + \sum_{m=0}^{k-1}\binom{k-1}{m}\hat{v}(\vec{0})\widehat{K}^{(m)}_{i,j,0}(\mathbf{p},-\mathbf{p})\widehat{K}^{(k-1-m)}_{0}(\mathbf{0}) \\ &\quad + \sum_{m=0}^{k-1}\binom{k-1}{m}\hat{v}(\vec{p})\widehat{K}^{(m)}_{i,0}(\mathbf{p})\widehat{K}^{(k-1-m)}_{j,0}(-\mathbf{p}) \\ &= O(\mathbf{p}^2) \,.\end{aligned}$$

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Therefore:

$$\lim_{p_0\to 0}\frac{\partial}{\partial p_0}\widehat{K}^{(k)}_{i,j}(p_0,\vec{0}) = 0 \qquad \forall k \geq 1\ . \qquad \blacksquare$$

Universality of conductivity in interacting graphene

(Related to Hall transitions in the Haldane model)

Critical line

- So far, we discussed the universality of $(\sigma_{ij})_{i,j=1,2}$ away from the boundary.

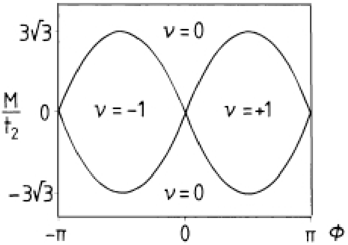

Figure: The spectral gap closes at the critical line.

If naively estimated, the radius of convergence U_0 goes to zero as $m_\pm \to 0$.

- To study the boundary, one needs renormalization group.
- Open problem. Transport properties on the (renormalized) boundary?

Graphene as a gapless limit of the Haldane model

- Consider the points $M = 0$, $\phi = 0, \pi$. In this situation, the Hamiltonian is:

$$\widetilde{\mathcal{H}}_{\mathsf{H}} = -t_1 \sum_{j=1,2,3} \sum_{\vec{x}\in\Lambda_A} a^+_{\vec{x}} b^-_{\vec{x}+\vec{\delta}_j} + b^+_{\vec{x}+\vec{\delta}_j} a^-_{\vec{x}}$$
$$- t_2 \sum_{\substack{\alpha=\pm \\ j=1,2,3}} \left[\sum_{\vec{x}\in\Lambda_A} a^+_{\vec{x}} a^-_{\vec{x}+\alpha\vec{\gamma}_j} + \sum_{\vec{x}\in\Lambda_B} b^+_{\vec{x}} b^-_{\vec{x}+\alpha\vec{\gamma}_j} \right] + U\mathcal{V}$$

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- The standard graphene model can be obtained as

$$\mathcal{H}_{\mathsf{graphene}} = \lim_{t_2\to 0} \widetilde{\mathcal{H}}_{\mathsf{H}} \; .$$

- In this limit, particle-hole symmetry: $a^\pm \to a^\mp$, $b^\pm \to -b^\mp$. It implies half-filling, which is the relevant choice for neutral graphene.

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- Giuliani-Mastropietro '10. Analyticity of graphene's ground state correlations, for U small.

Universality of conductivity in interacting graphene

Theorem (Giuliani-Mastropietro-P., 2012)

Let $\phi = 0, \pi$, $M = t_2 = 0$, assume half-filling. The conductivity matrix $(\sigma_{ij})_{i,j=1,2}$ is analytic for U small. Moreover, it is given by (restoring e and h):

$$\sigma_{ij} := -\frac{2}{3\sqrt{3}} \lim_{p_0 \to 0^+} \frac{1}{p_0} \left[\widehat{K}_{ij}(p_0) - \widehat{K}_{ij}(0) \right] = \frac{e^2}{h} \frac{\pi}{4} \delta_{ij} \; .$$

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- First proof of universality of conductivity for a $2d$ interacting system. Analogous universality results in $1d$: Benfatto-Falco-Mastropietro '10-'15.

Conclusions

- We discussed the universality properties of transport coefficients of interacting graphene-like models. We proved:
  1. the stability of IQHE for the interacting Haldane model against weak many-body interactions.
  2. the universality of the conductivity matrix in interacting graphene.
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     Ludwig et al. '94: Random $2d$ massless Dirac fermions $\equiv$ marginal SUSY QFT.
     Mastropietro '13: perturbative renormalization of marginal disorder (random $U(1)$ gauge field)

Thank you!