Height fluctuations in interacting dimers

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Joint work with V. Mastropietro and F. Toninelli

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Outline

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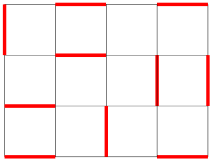
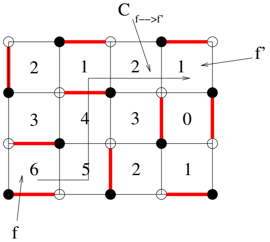
Perfect matchings of $\mathbb{Z}^2$ and height function





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$\sigma_b = \pm 1$ if b crossed with white on the right/left.

Note: white-to-black flux $(1_{b \in M} - 1/4)$ is divergence-free.

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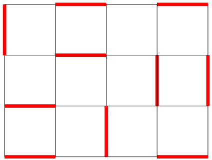

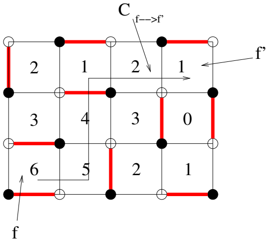

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Non-interacting dimers (uniform perfect matchings)

If Λ is a large domain, e.g. the $2L \times 2L$ square or torus, many ($\approx \exp(s|\Lambda|)$) perfect matchings exist.

Classical statmech/combinatorics problem: study the properties of the uniform measure $\langle \cdot \rangle_{\Lambda;0}$ on such perfect matchings.

Note: on the torus, the height profile is flat in average, i.e., $\langle h(f) - h(f') \rangle_{\Lambda;0} = 0$, because $\langle 1_{b \in M} \rangle_{\Lambda;0} = 1/4$ for every b.

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This “non-interacting” model is exactly solvable (Kasteleyn, Temperley-Fisher).

- The partition function is the Pfaffian of the complex adjacency matrix $K(x, y)$ (Kasteleyn matrix).

  The entropy per site in the thermodyn. limit is:

$$s = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\int_{-\pi}^{\pi}\frac{dk_2}{2\pi}\log(2\cos k_1+2i\cos k_2) = \frac{G}{\pi},$$

where G is Catalan's constant: $G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots$.

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Critical dimer correlations

- Dimer-dimer correlations are easy to compute in terms of a suitable Wick's rule. E.g.,

$$\langle 1_{(x,x+e_1)\in M} 1_{(y,y+e_1)\in M} \rangle_{\Lambda,0} =$$

$$= K^{-1}(x,x+e_1)K^{-1}(y,y+e_1) - K^{-1}(x,y+e_1)K^{-1}(y,x+e_1)$$

where K^{-1} is the inverse Kasteleyn matrix,

$$\lim_{\Lambda\to\mathbb{Z}^2} K^{-1}(x,y) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik(x-y)}}{-i\sin k_1 + \sin k_2}$$

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$$\lim_{\Lambda\to\mathbb{Z}^2} Var_{\Lambda,0}(h(f)-h(f')) \simeq \frac{1}{\pi^2}\log|f-f'|$$

as $|f-f'|\to\infty$ (Kenyon 2000, K-Okounkov-Sheffield 2006).

The computation is quite subtle:

$$Var_{\Lambda,0}(h(f)-h(f')) = \sum_{b,b'\in C_{f\to f'}} \sigma_b\sigma_{b'}\langle 1_{b\in M}; 1_{b'\in M}\rangle_{\Lambda,0}$$

If one replaces $\langle 1_{b\in M}; 1_{b'\in M}\rangle_{\Lambda,0}$ by its asymptotic behavior and the sums by integrals, one obtains an ambiguous (cutoff-dependent) integral.

Key ingredient: path-independence of the height.

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Height fluctuations II: higher order cumulants

- The height field is asymptotically Gaussian: for $m \geq 3$, the m^{th} cumulant of $h(f) - h(f')$ is

$$\langle h(f) - h(f'); m \rangle_{\Lambda,0} = o(Var_{\Lambda,0}(h(f) - h(f'))^{m/2}).$$

(recall: cumulants of X are zero for $m \geq 3$ iff X is Gaussian).

- Consequence: a coarse-grained version of $h(f)$ tends, in the scaling limit, to the 2D massless GFF (Kenyon 2001). This fact was heuristically known for this and similar interface models since the early 1980s (Nienhuis-Blöte-Hilhorst 1984).

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Height fluctuations III: conformal invariance and GFF

- More mathematical results: the microscopic fluctuations of $h(f)$ are asymptotically gaussian: the "electric correlator" behaves like

$$\lim_{\Lambda\to\mathbb{Z}^2} \langle e^{i\alpha(h(f)-h(f'))}\rangle_{0,\Lambda} \sim |f-f'|^{-\alpha^2/(2\pi^2)}$$

as $|f-f'|\to\infty$ (Pinson 2004, Dubedat 2011).

- Scaling limit is conformally invariant (Kenyon 2001): if the model is defined on a (discretization Λ of) $\mathcal{D}\subset\mathbb{C}$, the limiting moments, such as

$$g_{\mathcal{D}}(x,y)=\lim_{mesh\to 0}\langle (h_x-\langle h_x\rangle_{\Lambda,0})(h_y-\langle h_y\rangle_{\Lambda,0})\rangle_{\Lambda,0}$$

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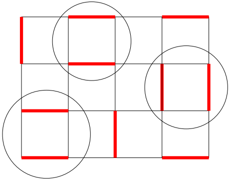
Interacting dimers

Associate an energy $\lambda \in \mathbb{R}$ to adjacent dimers:



Interacting measure:

$$\langle \cdot \rangle_{\Lambda,\lambda} = \frac{\sum_M e^{\lambda N(M)} \cdot}{Z_{\Lambda,\lambda}},$$

with $N(M) = \#$ adjacent pairs of dimers in M.

Quantum Dimer Models at the RK point

If $\lambda \neq 0$, the model is *not exactly solvable*: the exact Pfaffian structure breaks down.

At close packing, it is expected to remain critical even if $\lambda \neq 0$.

The model arises naturally in the context of simplified models for short-ranged RVB states (Quantum Dimer Models, Rokhsar-Kivelson 1988, ...).

For certain fine-tuned choices of the hopping vs interaction parameters, the ground state of the QDM is known exactly and its correlations coincide with those of the interacting classical dimer model.

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Phase diagram of the interacting dimer model

The phase diagram of this system has been analyzed extensively, by using MonteCarlo simulations and an effective field theory description that extends the non-interacting one.

Known: at large λ the system is an ordered "columnar" phase (Heilmann and Praestgaard 1977; existence of the columnar phase in the quantum model: Giuliani-Lieb 2015)

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The "liquid phase" (small λ)

We shall focus on the case of small λ.
Known features (Falco 2013):

- no long range order
- anomalous correlations.

E.g., if $b = (x, x + e_1)$ and $b' = (y, y + e_1)$

$$\lim_{\Lambda \to \mathbb{Z}^2} \langle 1_b ; 1_{b'} \rangle_{\Lambda,\lambda} = \frac{(-1)^{x-y}}{2\pi^2} A(\lambda) \frac{(x_1 - y_1)^2 - (x_2 - y_2)^2}{|x - y|^4} + \frac{(-1)^{x_1 - y_1}}{2\pi^2} B(\lambda) \frac{1}{|x - y|^{2+\eta(\lambda)}} + R(x - y) ,$$

where $A(\cdot), B(\cdot), \eta(\cdot)$ are analytic, $A(0) = B(0) = 1$ and $\eta(0) = 0$;
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Main results

Theorem [G., Mastropietro, Toninelli 2014]
If $|\lambda| \leq \lambda_0$ then:

- Height fluctuations still grow logarithmically:

$$\lim_{\Lambda \to \mathbb{Z}^2} Var_{\Lambda,\lambda}(h(f) - h(f')) \simeq \frac{K(\lambda)}{\pi^2} \log|f - f'|$$

with $K(\cdot)$ analytic and $K(0) = 1$;

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- convergence to the GFF: if $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \varphi(x)dx = 0$ then, as $\epsilon \to 0$,

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Corollary (Coarse-grained electric correlator).
Let $\chi_x : \mathbb{R}^2 \to \mathbb{R}$ be a smooth, positive, compactly supported function, centered at $x \in \mathbb{R}^2$ and s.t. $\int_{\mathbb{R}^2} \chi_x = 1$, then

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That is, a coarse-grained version of the "electric correlator" $\langle e^{i\alpha(h(f)-h(f'))}\rangle_{\mathbb{Z}^2,\lambda}$ decays at infinity with an anomalous critical exponents. The problem of controlling the electric correlator directly is beyond the current state-of-art (at $\lambda = 0$: Pinson 2004, Dubedat 2011).

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  Key features: robust under a large class of local non-integrable perturbations; the proof uses (some simple instances of) the underlying discrete holomorphicity of the model.

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- Related results in lattice interface models:
  - log fluctuations and the roughening transition in 6V, SOS models and anharmonic crystals (Falco, Brascamp-Lieb--Lebowitz, Fröhlich-Spencer, Ioffe-Shlosman-Velenik, Milos-Peled)
  - convergence of Ginzburg-Landau type gradient models to the GFF (Conlon-Naddaf-Spencer, Giacomin-Olla-Spohn, Miller),
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- Rigorous foundation of the effective field theory description of QDM at the RK point.
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Outline

Fermionic representation

Algebraic identity: Pfaffian can be written as Grassmann Gaussian integrals:

$$Pf(K) = \int \prod_{u \in \Lambda} d\psi_u e^{-\frac{1}{2}(\psi, K\psi)}$$

where $\{\psi_x\}_{x \in \Lambda}$ are Grassmann variables. Similarly,

$$K^{-1}(x,y) = \frac{1}{Pf(K)} \int \prod_{u \in \Lambda} d\psi_u e^{-\frac{1}{2}(\psi, K\psi)} \psi_x \psi_y .$$

Interacting dimers as interacting fermions

Similarly, the partition function of the interacting model is written as

$$\frac{Z_{\Lambda,\lambda}}{Z_{\Lambda,0}} = \frac{1}{Pf(K)} \int \prod_{x\in\Lambda} d\psi_x e^{-\frac{1}{2}(\psi,K\psi)+V(\psi)} \equiv \left\langle e^{V(\psi)} \right\rangle_{\Lambda,0}$$

with

$$V(\psi) = V_4(\psi) + V_6(\psi) + \dots,$$

and

$$V_4(\psi) = 2\lambda \sum_x \psi_x \psi_{x+e_1} \psi_{x+e_2} \psi_{x+e_1+e_2}.$$

NB: for finite Λ, these are just exact identities, V is a polynomial (finite degree).

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Constructive Renormalization Group

Analysis of the interacting fermionic theory by constructive field theory methods, due to:

- Gawedski-Kupiainen, Battle-Brydges-Federbush, Lesniewski, Benfatto-Gallavotti, Feldman-Magnen-Rivasseau-Trubowitz, ...

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Dimer-dimer correlations, interacting case

If λ is small, the constructive RG analysis provides "explicit" formulas for all the dimer correlations, e.g.,

$$\sigma_b \sigma_{b'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{b \in M}; 1_{b' \in M} \rangle_{\Lambda,\lambda} = A_{b,b'} + B_{b,b'} + C_{b,b'}$$

$$= -\frac{K(\lambda)}{2\pi^2} \mathrm{Re} \left[\Delta z_b \Delta z_{b'} \frac{1}{(z_b - z_{b'})^2} \right]$$

$$+ Osc(z_b, z_{b'}) \frac{1}{|z_b - z_{b'}|^{2+\eta(\lambda)}} + O(|z_b - z_{b'}|^{-3+O(\lambda)}).$$

with $K(\cdot)$, $\eta(\cdot)$ analytic and $K(0) = 1, \eta(0) = 0$.

Height variance, interacting case

Note:

- the behavior of the dimer-dimer correlation is non-universal: an anomalous exponent emerges in the $B_{b,b'}$ term.
- Due to the oscillating factor in front of $B_{b,b'}$, the dominant contribution to $\langle (h(f) - h(f'))^2 \rangle$ is

$$\sum_{\substack{b \in C_{f \to f'}, \\ b' \in C'_{f \to f'}}} A_{b,b'} \simeq -\frac{K(\lambda)}{2\pi^2} \mathrm{Re} \int_f^{f'} \int_{\tilde{f}}^{\tilde{f}'} \frac{dz dz'}{(z - z')^2} \simeq \frac{K(\lambda)}{\pi^2} \log |f - f'|$$

Ward Identities and path-independence

- The asymptotic computation of the correlations, the emergence of $\eta(\lambda)$, and the proof that $A_{b,b'}$ has no anomalous critical exponent requires the implementation of hidden Ward Identities in the RG flow, as well as the rigorous control of the associated anomalies.
- In order to exhibit the necessary cancellations, a suitable deformation of the paths along which the factors in $(h(f) - h(f'))^m$ are computed is required (idea borrowed from Kenyon, Kenyon-Okounkov--Sheffield, Dubedat, Laslier-Toninelli).

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Conclusions

- Proof of Gaussian behavior for the height function of non-integrable dimer models.
- Novelties:
  - match between constructive QFT methods (huge literature) and some (simple) discrete complex analysis ideas
  - control of a non-local fermionic observable (height field) in a non-integrable case
- While critical exponent of dimer-dimer correlations is not universal, logarithmic growth of variance is.
- To be done (major difficulties):
  - get rid of periodic b.c., work with general domains (necessary to study conformal invariance).
  - control the exponential of the height function

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