Limit Theorems in Quantum Mechanics

—

"eth" in quantum theory

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Credits and Contents

Numerous useful discussions with: Ph. Blanchard, Bohmians (Dürr, Goldstein), colleagues at ETH (Graf, Hepp), P. Pickl and Chr. Schilling; challenge and stimulation coming from experiments by Haroche-Raimond group and from a paper by Bauer and Bernard; perspective from work by Maassen and Kümmerer.
This is the result of joint work primarily with my former PhD student Baptiste Schubnel; some of our efforts are joint work with M. Ballesteros, J. Faupin and M. Fraas.

1. Introduction – Some basic questions and claims
2. Direct (von Neumann) measurements
3. Indirect (Kraus) measurements
4. Conclusions – discussion

I apologize for not talking about (operator-theoretic) RG methods. But I think Alessandro can explain our approach to you better than I can. I thank him and his colleagues for this kind invitation!

In memoriam Pierluigi Falco

1. Introduction – Some basic questions and claims

In our courses, we tend to describe quantum-mechanical systems, S, as pairs of a Hilbert space, $\mathcal{H}$, and of a propagator, $(U(t,s))_{t,s\in\mathbb{R}}$, describing time evolution. Unfortunately, these data hardly encode any invariant information about S that would enable one to draw conclusions about its physical properties, and they give the erroneous impression that quantum theory might be a deterministic theory.

→Fundamental questions and problems:

- What do we have to add to the usual formalism of quantum theory to arrive at a mathematical structure that – through interpretation – can be given physical meaning, *without the intervention of "observers"*?
- Where does *intrinsic randomness* in quantum theory come from, given the deterministic character of the Schrödinger equation? Does it differ from classical randomness?

Introduction - ctd.

- What is the meaning of *states* and of *"observables"* in quantum mechanics? Do we understand the *time evolution* of states of quantum systems, and what does it have to do with solutions of the Schrödinger equation?
- What do we mean by an *isolated system* in quantum mechanics, and why may this be an important notion*? How may one prepare a system in a prescribed state?
  *Because *only for isolated systems* we know how to describe the *time evolution of "observables"*!

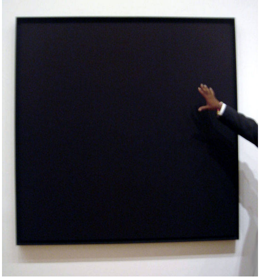
Basic claims:

- The quantum-mechanical state does *not* describe "what is". It is merely a mathematical device enabling one to make bets about what is likely to happen in the future.

Introduction – ctd.

- Fundamental *"Loss of Information"* and *Entanglement* with inaccessible ("lost") degrees of freedom enable us to understand why *pure states* may evolve into *mixed states* and to develop a rational theory of measurements, observations and experiments in quantum mechanics; ("Second Law of quantum measurement theory"!)
- *No information-* or *unitarity paradoxes* in quantum mechanics! Time evolution of states of quantum systems admitting observations and measurements is actually never unitary – it is "tree-like"; ("eth in QM"!).
- Operator algebras (including things like type III_1- von Neumann algebras), probability theory, stochastic differential equations, etc. have been invented to be *used* in Quantum Theory, rather than to be ignored!

Metaphor for the "mysterious holistic aspects" of Quantum Mechanics



QM is QM-as-QM and everything else is everything else

"The one thing to say about art is that it is one thing. Art is art-as-art and everything else is everything else." (Ad Reinhardt)

2. Direct (projective, or von Neumann) Measurements

Definition of qm systems – for pedestrians

An isolated quantum system, S, is characterized by the following data:

1. A pair
$$(\mathcal{H}, \{U(t,s)\}_{t,s\in\mathbb{R}})$$
of a Hilbert space $\mathcal{H}$ of pure state vectors and a propagator U;
2. a list,
$$\mathcal{O}_S = \{X_i\}_{i\in I_S},$$
of bounded operators on $\mathcal{H}$ representing *physical quantities/ potential properties* of S that can be measured or observed *directly*. For simplicity, we assume that $\mathcal{O}_S$ is abelian; then $\mathcal{O}_S$ can be chosen to be a *commutative algebra*.
(S must be an isolated system that includes appropriate instruments for the quantities represented by the operators $X_i, i \in I_S$, to be measurable directly.)

Direct measurements – ctd.

Choose fiducial time t_0 at which physical quantities are identified with operators in $\mathcal{O}_S$ and define (Heisenberg picture!)

$$X(t) := U(t_0, t) X U(t, t_0), \qquad X \in \mathcal{O}_S,$$

to be the operator representing the quantity corresponding to $X \in \mathcal{O}_S$ at some time $t \quad \to \mathcal{O}_S(t)$. Potential properties/phys. quantities observable at times $s \geq t$ generate a W*- algebra, $\mathcal{E}_{\geq t}$:

$$\mathcal{E}_{\geq t} := \langle \prod_i X_i(t_i) | X_i \in \mathcal{O}_S, t_i \geq t \rangle^{-}, \qquad \text{with } \mathcal{E} := \mathcal{E}_{>-\infty} \quad (1)$$

Then

$$B(\mathcal{H}) \supseteq \mathcal{E} \supseteq \mathcal{E}_{\geq t} \supseteq \mathcal{E}_{\geq s} \supseteq \mathcal{O}_S(s), \qquad s \geq t \quad (2)$$

$$\neq \leftarrow I.L.!$$

($I.L.$ = “Information Loss”!)

Direct measurements – ctd.

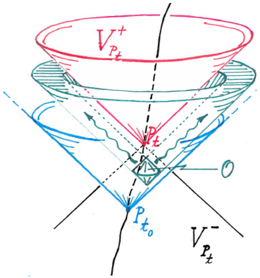
"Information Loss" can be interpreted as saying that time translations, τ_t, (with $\tau_t(X(s)) = X(s+t),\ X \in \mathcal{O}_S$), are **endomorphisms*

$$\tau_t : \mathcal{E}_{\geq s} \to \mathcal{E}_{\geq s+t},$$

rather than *automorphisms – (at least if, between time s and time $s+t$, a direct measurement happens).
It is not difficult to construct examples of (generally *non-autonomous*) quantum systems exhibiting Information Loss, in the above sense: "Small systems" (e.g., an n-level atom) temporarily interacting with a quantized wave medium, such as the quantized e.m. field, or phonons, etc.
More fundamentally, Information Loss always occurs in QFT's with massless particles, such as QED, as pointed out by Buchholz and Roberts:

Information Loss and Huyghens Principle

In QED, (2) ($\Rightarrow$ Information Loss and Entanglement with inaccessible degrees of freedom) is a consequence of *Huyghens' Principle* for the electromagnetic field:



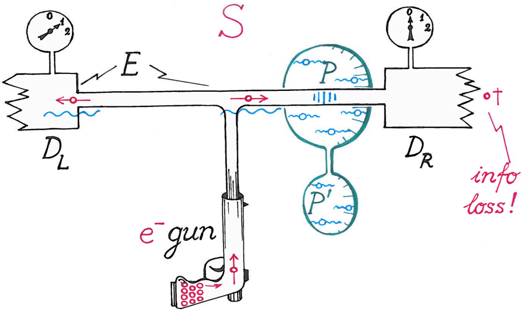
world line of J. F. $\nearrow$

All operators in $\mathcal{E}_{\geq t}$ are located in $V^+_{P_t}$, $P_t = (t, \vec{x})$. The Figure shows that $\mathcal{E}_{\geq t_0}$ *properly contains* $\mathcal{E}_{\geq t}$, for $t > t_0$, and

$$(\mathcal{E}_{\geq t})' \cap \mathcal{E}_{\geq t_0} \supset \mathcal{A}^{\text{out}}_{\mathcal{O}}$$

Example of a mesoscopic qm system:

conducting "T-channel" ending in detectors D_L and D_R and in an electron gun, quantum dot, $P \vee P'$, with P binding up to N electrons.



Example – ctd.

E contains all measuring devices, including the two e^-- detectors, D_L and D_R. The *only* directly observable quantity in this system is the *click* of either D_L or D_R: A detector clicks iff an e^- is entering it. Mathematically, this quantity is represented by the linear operator

$$X = \mathbf{1}_{\overline{P}} \otimes \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_E , \qquad \text{with } \overline{P} = P \vee P', \tag{3}$$

which has the (infinitely degenerate) eigenvalues $\xi = \pm 1$, with

$$\xi = +1 \leftrightarrow D_L \text{ clicks}, \qquad \xi = -1 \leftrightarrow D_R \text{ clicks}.$$

$\mathcal{O}_S$ consists of all bounded functions of X. We want to understand what it means that the quantity represented by the operator X is directly (projectively) measurable, and (in the next section) what information can be gained from long sequences of successive projective measurements of X. (It will turn out that they reveal the number of e^- in P.)

Meaning of direct measurements

Clarify what it means that a quantity rep. by operator $X \in \mathcal{O}_S$ is measured *directly* (or projectively) around some time t, and what the roles of Information Loss and Entanglement with inaccessible degrees of freedom are. We assume, for simplicity, that the spectrum of X consists of finitely many (typically infinitely degenerate) eigenvalues $\xi_1, ..., \xi_N$. Then

$$X(t) = \sum_{j=1}^{N} \xi_j \Pi_j(t), \tag{4}$$

where $\Pi_j(t)$ is the spectral proj. corresp. to ξ_j.
It is compatible with the "Copenhagen mumbo-jumbo" to assume that measuring X at time t implies that the *state* of S,

$$\rho(\cdot) = \mathrm{Tr}(P(\cdot)), \qquad \text{where } P \text{ is a density matrix on } B(\mathcal{H}),$$

Direct measurements

restricted to the algebra $\mathcal{E}_{\geq t}$, is an incoherent superposition of eigenstates of the operator $X(t)$. This means that

$$\rho_t(A) = \sum_{j=1}^{N} \rho_t(\Pi_j(t)\ A\ \Pi_j(t)), \qquad \forall A \in \mathcal{E}_{\geq t}, \tag{5}$$

where $\rho_t := \rho|_{\mathcal{E}_{\geq t}}$, (up to possibly a tiny error). Information Loss & Entanglement with inaccessible degrees of freedom $\Rightarrow$ ρ_t is, in general, a *mixed state* on $\mathcal{E}_{\geq t}$, *even* if ρ is a *pure* state on $B(\mathcal{H})$. Eq. (5) $\Leftrightarrow$ $X(t)$ belongs to the *"stabilizer"*, $\mathcal{C}_{\rho_t}$, of the state ρ_t, i.e.,

$$ad_{X(t)}(\rho_t) = 0.$$

By $\mathcal{Z}_{\rho_t}$ we denote the center of $\mathcal{C}_{\rho_t}$.

Fundamental axiom of the quantum-mechanical measurement process

1. The quantity represented by $X \in \mathcal{O}_S$ is measured/observed around time t iff

$$\mathrm{dist}(X(t), \mathcal{Z}_{\rho_t}) := \|\epsilon_{\rho_t}(X(t)) - X(t)\| \approx 0, \tag{6}$$

   where ϵ_{ρ_t} = conditional expectation of $\mathcal{E}_{\rho_t}$ given $\mathcal{Z}_{\rho_t}$.
   Clearly, Eq. (6) $\Rightarrow$ Eq. (5)!
   In simple prose, Eq. (6) means that if ρ_t can be represented by a density matrix, P_t, on $\mathcal{E}_{\geq t}$ then, on the range of P_t, $X(t)$ "equals" a function of P_t, multiplied by a central element, z.

2. If X is observed around time t then X has a *value* $\in \{\xi_1, ..., \xi_N\}$ around time t.

3. Randomness in quantum mechanics
   The probability of observing ξ_j is given by

$$p_j^X(t) = \rho(\Pi_j(t)), \tag{7}$$

Born's Rule

Fundamental axiom – ctd.

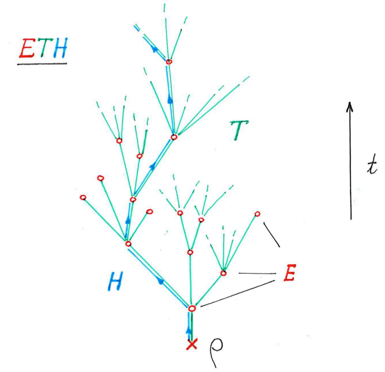
If ξ_j has been observed then the state

$$\rho_j^X(\cdot) := p_j^X(t)^{-1} \; \rho_t(\Pi_j(t)(\cdot)\Pi_j(t)) \tag{8}$$

must be used for improved predictions of future at times $\geq t$.

Remark: Apparently, if it is known that the system S has been prepared in a state ρ before some direct measurement/observation is made then the quantum-mechanical dynamics of S *predicts which quantity* X will first be observed and, approximately, at what time t; but the *measured value* ξ_j is *not* predictable with certainty!

Effective time evolution of states



E: "events" (proj. measnts.), *T*: "trees" (of states),
H: "histories" ; probs. of "histories" are det. by QM

3. Indirect (Kraus) measurements

Assume, $\mathcal{O}_S$ is a finite-dimensional, commutative algebra with spectrum $\mathcal{X}_S$ = a finite set of points, $\{1, ..., N\}$. In the concrete model considered in Section 2, $\mathcal{O}_S$ consists of all functions of the operator

$$X = \mathbf{1}_{\overline{P}} \otimes \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_E ,$$

hence $\mathcal{X}_S = \{-1, +1\}$.

With each point $\xi \in \mathcal{X}_S$ we associate a spectral projection $\pi_\xi \in \mathcal{O}_S$, and all operators in $\mathcal{O}_S$ are linear combinations of the π_ξ's. We suppose that successive projective measurements/ observations of quantities corresponding to operators in $\mathcal{O}_S$ at times $\approx t_1, ..., \approx t_k$ have yielded a sequence of measurement results,

$$\underline{\xi}^{(k)} := \{\xi_1, ..., \xi_k\} \in \mathcal{X}_S^{\times k}. \qquad (9)$$

Kraus measurements – ctd.

We assume that S has been prepared in the state ρ before measnts./observations of quantities corresponding to ops. in $\mathcal{O}_S$ have started. QM predicts that the probability (frequency) of a measurement protocol $\underline{\xi}^{(k)}$ is given by

$$\mu_\rho(\xi_1, ..., \xi_k) = \mathrm{Tr}(\pi_{\xi_k}(t_k) \cdots \pi_{\xi_1}(t_1)\ P\ \pi_{\xi_1}(t_1) \cdots \pi_{\xi_k}(t_k)), \quad (10)$$

where P is the density matrix coresponding to the state ρ; (*LSW - formula*).Obviously,

$$\sum_{\xi_k \in \mathcal{X}_S} \mu_\rho(\underline{\xi}^{(k-1)}, \xi_k) = \mu_\rho(\underline{\xi}^{(k-1)}), \qquad \mu_\rho(\emptyset) = 1. \qquad (11)$$

It follows that μ_ρ extends to a probability measure on the space, Ξ, of infinitely long measurement protocols; (equipped with the σ-algebra of cylinder sets).

Kraus measurements – ctd.

Let $\mathcal{O}_S[\rho, \infty]$ be the algebra of functions in $L^\infty(\Xi, [\mu_\rho])$ that do not depend on any finite set of measurement outcomes: *Observables at infinity*. (Observables at infinity can sometimes be related to the commutant of $\mathcal{E}$; see Eq. (1). But the general picture is not entirely clear, yet.)
Let $\Xi[\rho, \infty]$ be the spectrum of $\mathcal{O}_S[\rho, \infty]$. Then the measure μ_ρ can be decomposed into a convex combination of *"extremal measures"*:

$$\mu_\rho(\cdot) = \int_{\Xi[\rho,\infty]} \mathrm{d}P(\nu)\mu_\rho(\cdot|\nu). \tag{12}$$

For different points ν, the measures $\mu_\rho(\cdot|\nu)$ are *mutually singular*. Thus, a very long measurement protocol $\underline{\xi}^{(k)}$ determines a point $\nu \in \Xi[\rho, \infty]$ (called a "fact") with an error likelihood that tends to 0, as $k \to \infty$, and ν then determines the values of all "observables at infinity".

Exchangeable measures

If the order in which the measurement results $\xi_1, ..., \xi_k$ are obtained does not matter, for any k, (i.e., if successive measurements commute with each other) then $\mu_\rho(\xi_{\sigma(1)}, ..., \xi_{\sigma(k)})$ is *independent* of the permutation σ, $\forall\sigma$ and all k. Then Eq. (12) follows from *de Finetti's theorem*, which also says that the measures $\mu_\rho(\cdot|\nu)$ are product measures:

$$\mu_\rho(\xi_1, ..., \xi_k|\nu) = \prod_{j=1}^{k} p(\xi_j|\nu), \tag{13}$$

with $p(\xi|\nu) \geq 0$ and $\sum_\xi p(\xi|\nu) = 1$.
A simple example of this situation is a model of the system described in Section 2, for which $\mathcal{O}_S = \langle X \rangle$ and $\mathcal{X}_S = \{-1, +1\}$. (Assuming that the electrons moving through the "T-channel" are entirely independent of each other and that the detectors D_L and D_R return to the same state after each measurement, and before the next electron travels through the "T-channel", one concludes that the measures μ_ρ are *exchangeable*.)

Exchangeable measures – ctd.

Let $\nu \in \{1, ..., N\}$ be the number of e^- in the quantum dot P. Let us assume, for the time being, that ν is *time-independent*, i.e., we consider a *non-demolition measurement* of ν. Because μ_ρ is exchangeable, we have that

$$\mu_\rho(\underline{\xi}^{(k)}) = \sum_{\nu=1}^{N} P_\rho(\nu)\mu(\underline{\xi}^{(k)}|\nu), \tag{14}$$

with

$$\mu_\rho(\underline{\xi}^{(k)}|\nu) = \prod_{j=1}^{k} p(\xi_j|\nu),$$

where:
$P_\rho(\nu)$: Born probability for ν e^- bound by P;
$p(\xi|\nu)$: QM probability for an e^- in the "T-channel" to be scattered into $D_\xi, \xi = -1(R), +1(L)$, ($\nearrow$ a QM-I calculation!).

Frequencies of "events"

An example of an "observable at infinity" that is usually well defined is the *"asymptotic frequency"*, $p(\xi|\cdot)$, of an event $\xi \in \mathcal{X}_S$. We define

$$f_{\xi}^{(l,l+k)}(\underline{\xi}) := \frac{1}{k}\left(\sum_{j=l+1}^{l+k} \delta_{\xi,\xi_j}\right), \quad \text{with} \ \sum_{\xi} f_{\xi}^{(l,l+k)}(\underline{\xi}) = 1. \tag{15}$$

One expects that, for "most" states ρ, the
(1) *Law of Large Numbers*

$$\lim_{k\to\infty} f_{\xi}^{(l,l+k)}(\underline{\xi}) =: p(\xi|\nu), \tag{16}$$

for some point (or "fact") $\nu \in \Xi[\rho, \infty]$, holds. This is indeed the case for the simple model described above.

Hypothesis: We continue to assume that $\mathcal{O}_S$ is finite-dimensional and that $\mathrm{card}(\Xi[\rho, \infty]) < \infty$, with

$$\min_{\nu_1 \neq \nu_2} |p(\xi|\nu_1) - p(\xi|\nu_2)| \geq \kappa > 0, \quad \text{for some } \xi \in \mathcal{X}_S \tag{17}$$

“q-hypothesis testing”

With each $\nu \in \Xi[\rho, \infty]$ we associate a subset

$$\Xi_\nu(l, k; \underline{\varepsilon}) := \{\underline{\xi} | \; |f_{\underline{\xi}}^{(l,l+k)}(\underline{\xi}) - p(\underline{\xi}|\nu)| < \epsilon_k\}, \qquad (18)$$

where

$$\epsilon_k \to 0, \sqrt{k}\; \epsilon_k \to \infty, \quad \text{as } k \to \infty$$

Main Results:

(2) It follows from Hyp. (17) and definition (18) that, for k so large that $\epsilon_k < \kappa/2$,

$$\Xi_{\nu_1}(l, k; \underline{\varepsilon}) \cap \Xi_{\nu_2}(l, k; \underline{\varepsilon}) = \emptyset, \quad \nu_1 \neq \nu_2$$

(3) *Central Limit Theorem:* $\Rightarrow$ Under suitable hypotheses on the states ρ,

$$\mu_\rho \left(\bigcup_\nu \Xi_\nu(l, k; \underline{\varepsilon}) \right) \to 1, \quad k \to \infty$$

hypothesis testing – ctd.

(4) *Theorem of Boltzmann-Sanov* $\Rightarrow$ If the measures μ_ρ are exchangeable one has that

$$\mu\left(\Xi_{\nu_1}(I, k; \underline{\varepsilon})|\nu_2\right) \leq C\ e^{-k\sigma(\nu_1\|\nu_2)}$$

where σ is a relative entropy.

(5) *Theorem of Maassen and Kümmerer* $\Rightarrow$ In the simple model described above, the state of S, restricted to $B(\mathcal{H}_P)$ approaches a state with a *fixed number of electrons* in the quantum dot P; ("purification").

The theory of indirect measurements outlined here only concerns measurements of time-independent "facts", which correspond to points in $\Xi[\rho, \infty]$ (*non-demolition measurements*). However, most interesting "facts" depend on time! Thus, one must ask how one can acquire information concerning *time-dependent* facts indirectly, through repeated, successive direct measnts. of quantities corresponding to operators in $\mathcal{O}_S$.

We now consider the simple model introduced above. We assume that electrons can enter into, or tunnel out of the component P of the quantum dot $\overline{P}$, i.e., the number of electrons, ν, in P may slowly vary in time. We define

$$\Xi_{\nu_1,\ldots,\nu_r}(k;\underline{\varepsilon}) := \{\underline{\xi} \mid |f_L^{(ik-k,ik)}(\underline{\xi}) - p(L|\nu_i)| < \varepsilon_k, \forall i = 1, \ldots, r\}$$

and

$$\mathcal{P}_\rho(\nu_1, \ldots, \nu_r) := \mu_\rho(\Xi_{\nu_1,\ldots,\nu_r}(k;\underline{\varepsilon}))$$

(6) *Theorem on quantum jumps:* For each $r < \infty$,

$$\sum_{\nu_1,\ldots,\nu_r} \mathcal{P}_\rho(\nu_1, \ldots, \nu_r) \to 1,$$

in the limit where the temporal variation of the number of electrons in P tends to 0 and $k \to \infty$.

Remark. In suitable limiting regimes, $\mathcal{P}_\rho(\nu_1, \ldots, \nu_r)$ is the path-space measure of a *Markov chain* with state space $= \{1, \ldots, N\}$.

4. Conclusions – discussion

"In all my films, I have been faithful to these suspension points in the conclusions. Besides, I have never written the word 'END' on the screen."
(Federico Fellini)

"Everyone wants to understand art (physics)*. Why don't we try to understand the song of a bird? Why do we love the night, the flowers, everything around us, without trying to understand them? But in the case of a painting* (result in physics)*, people think they have to understand." (Pablo Picasso)*

Vi ringrazio per l'attenzione!

References to some of our papers

1. J. Fröhlich and B. Schubnel, "Do we understand quantum mechanics – finally?", in: Wolfgang Reiter et al. (eds.), *Erwin Schrödinger – 50 years after*, Zurich: European Math. Soc. Publ. 2013, pages 37 - 84.

2. J. Fröhlich and B. Schubnel, "Quantum Probability Theory and the Foundations of Quantum Mechanics", in: Philippe Blanchard and Jürg Fröhlich (eds.), *The Message of Quantum Science – Attempts Towards a Synthesis*, Lecture Notes in Physics vol. **899**, Berlin-Heidelberg: Springer-Verlag 2015, pages 131 - 193

3. M. Ballesteros, M. Fraas, J. Fröhlich and B. Schubnel, "Indirect retrieval of information and the emergence of facts in quantum mechanics", arXiv:1506.01213; and refs. given there.

4. J. Faupin, J. Fröhlich and B. Schubnel, "On the probabilistic nature of quantum mechanics and the notion of closed systems", to appear in Ann. Henri Poincar, 2015.

Please, take a look at some of the excellent papers by numerous colleagues that we have quoted in the works listed above.

My Manifesto

I propose that, at all decent institutions of higher education, *one or two days per semester* will be declared to be

Days of Reflection and of Protest!

During these days, we will not teach or attend committee meetings, and there won't be any exercise classes. Instead, we will discuss some of the serious problems threatening our civilization, draft declarations and reach out to the media, with the aim to make it clear to all circles wielding power that we no longer accept – (to mention some examples among others):

My Manifesto, ctd.

- That internal tensions and conflicts in countries, such as the *Ukraine*, are “solved” by armed conflicts rather than by political dialogue and compromise;
- that innocent people are slaughtered in ugly civil wars and by terrorist activities, such as those in Syria and Iraq;
- that countries threaten other countries with warfare;
- that weapons are sold to (clans) in countries plagued by civil war or other forms of unrest and conflict;
- that religions are abused for purposes of power and suppression of people;
- that the dignity and the rights of women are abused and offended in the name of religion;

My Manifesto, ctd.

- that people are harassed or killed because of their race or faith;
- that nothing is done against the perversions of 21st Century Capitalism;
- that the resources of Planet Earth continue to be looted shamelessly.

These are *but some examples of numerous problems* threatening the survival of humankind in peace and dignity. –

Where is the *"Peace Movement"*, where are movements such as *"Occupy Wall Street"*, *"Survivre et Vivre"*? What is the *"Club of Rome"* doing? Why are the media silent about the activities of these and other groups? Why do academics not have a strong voice in political debates, anymore?

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Students and Academics raise your voices, arise!