Renormalization group, Kondo effect and hierarchical models

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1-d lattice, fermions+impurity, "Kondo problem"

$$H_h = \sum_{\alpha=\pm} \Big(\sum_{x=-L/2}^{L/2-1} \psi^+_\alpha(x)\,(-\frac{1}{2}\Delta - 1)\,\psi^-_\alpha(x) + h\ \varphi^+\sigma^z\varphi^- \Big)$$

$$H_K = H_0 + \lambda \sum_{\substack{\alpha,\alpha'=\pm\\ \gamma,\gamma'=\pm}} \sum_{j=1}^{3} \psi^+_\alpha(0)\sigma^j_{\alpha,\alpha'}\psi^-_{\alpha'}(0)\ \varphi^+_\gamma\sigma^j_{\gamma,\gamma'}\varphi^-_{\gamma'} = H_h + V$$

(1) $\psi^\pm_\alpha(x), \varphi^\pm_\gamma$ C&A operators, σ^j, $j = 1, 2, 3$, Pauli matrices

(2) $x \in$ unit lattice, $-L/2$, $L/2$ identified (periodic b.c.)

(3) $\Delta f(x) = f(x + 1) - 2f(x) + f(x - 1)$ discrete Laplacian.

If $\lambda = 0$ impurity-electrons independent: classic or quantum

$$\chi(\beta, h) \propto \beta \xrightarrow[\beta\to\infty]{} \infty, \qquad \forall\, L \geq 1,\ \beta h < 1$$

Interaction (classic) elec.+imp.: field on both & $\lambda \neq 0$

$$\chi(\beta, h) = 4\beta \frac{(1 + e^{-2\lambda\beta} \cosh \beta h)}{(\cosh 2\beta h + e^{-2\lambda\beta})^2} \xrightarrow[\beta\to\pm\infty]{} \begin{matrix} 0 \text{ repulsive} \\ +\infty \text{ attractive} \end{matrix}$$

field on impurity only: $\chi(\beta, 0) = \beta \to \infty$
Reason: $\lambda < 0 \ \to$ rigidly antiparallel spins ????

Still true if $L < \infty$ classic&quantum or $L = \infty$ classic

XY model confirms (∞ both cases, exact)

Then Trivial? (0 repulsive, ∞ attractive ?)

BUT

If $L = \infty$ quantum chain: new phenomena

1) at $\lambda = 0 \Rightarrow$ Pauli paramagnetism (1926)

local or specific suscpt. $< \infty$ at $T \geq 0$:

$$\chi(\infty, 0) = \rho \frac{1}{k_B T_F} \frac{d}{2}, \qquad (Pauli)$$

2) at fixed $\lambda < 0 \Rightarrow$ Kondo effect:

susceptibility $\chi(\beta, h)$

smooth at $T = 0$ and $h \geq 0$

Kondo realized the problem (3^d-order P.T.) and gave arguments (1964) for $\chi < \infty$ (actually conductivity $< \infty$)

Anderson-Yuval-Hamann (1969,70) $\Rightarrow$ multiscale nature of the problem, relation with the 1D Coulomb gas & solved the $\lambda > 0$ case (no Kondo eff.), & stressed that lack of asymptotic freedom = obstacle for $\lambda < 0$

Wilson (1974-75) overcame asymptotic freedom by discussing a somewhat modified model and finding a recursion scheme, numerically implementable in an appropriately simplified model.

The method built a sequence of approximate Hamiltonians (with finitely many coefficients) more and more accurately representing the system on larger and larger scales, leading to the Kondo effect via a nontrivial fixed point.

Evaluate $Z = \mathrm{Tr}\, e^{-\beta H_K}$ as a functional integral, (BG990).
The free fields $\psi^{\pm}(x), \varphi^{\pm}$

$$\psi^{\pm}(x) = \sum_m e^{\pm ikx} \psi^{\pm[m]}(x),\ \ \varphi^{\pm} = \sum_m \varphi^{\pm[m]}$$

can be decomposed into components of scale 2^{-m}, $m \in Z$

$$\psi^{\pm}(x) = \sum_{m=0}^{-\infty} \sum_{\omega=\pm} e^{\pm i\omega p_f x}\, 2^{\frac{1}{2}m}\, \psi_{\omega}^{\pm[m]}(2^m x), \quad \varphi^{\pm} = \sum_{m=0}^{-\infty} \varphi^{\pm[m]}$$

quasi particles, neglecting the UV (*i.e.* $m \le 0$). Then represent Z as a Grassmann integral.
Fields become Grassman variables.

But since the impurity is localized observ. localized at 0 depend on fields at 0, $\psi^{\pm}(0), \varphi^{\pm} \Rightarrow$ **1D problem** (AYH).

Key: response to field h acting on impurity site only depends on the propagators with $x = 0$.

By Wick $\Rightarrow$ only average values, over "time" of propagators at $x = 0$ needed. Propagators on scale m are $g^{[m]}(t - t')$

$$\delta_{m,m'} \sum_{\omega} \int \frac{dk_0 dk}{(2\pi)^2} \frac{e^{ik_0(t-t')}}{-ik_0 + \omega e(k)} \chi(2^{-2m}(k_0^2 + k^2)),$$

$$\delta_{m,m'} \int \frac{dk_0}{2\pi} \frac{e^{i\sigma k_0(t-t')}}{-i\sigma k_0} \chi(2^{-m} \frac{k_0}{2\pi})$$

singularity at $t - t' = 0$ (UV sing.) and at $t - t' = \infty$ (IR sing.) regularized via χ on scale 2^{-m}; $e(k) = -\cos k$.

Illustration of (AYH970) remark: 1D problem, (long range)

$$\text{Main operators :} \quad \vec{A}_x \stackrel{def}{=} \psi_x^+ \boldsymbol{\sigma} \psi_x^-, \vec{B}_x \stackrel{def}{=} \varphi^+ \boldsymbol{\sigma} \varphi^-$$

Interaction Ham. is constructed via the operators

$$O_0 = -\lambda^0 \vec{A} \cdot \vec{B},\ O_1 = \lambda^1 \vec{A}^2,\ O_2 = \lambda^2 \vec{B}^2,\ O_3 = \lambda^3 \vec{A}^2 \vec{B}^2$$

H_K on scale $m = 0$ is (with $\lambda^0 < 0$ and $\lambda^1 = \lambda^2 = \lambda^3 = 0$)

$$H_K = H_0 - \sum_x (\lambda^0 O_{x,0} + \lambda^1 O_{x,1} + \lambda^2 O_2 + \lambda^3 O_{x,3}) + \dots$$

Set RG analysis via (Grassmannian) as BG990 for $\mathrm{Tr} e^{-\beta H_K}$

Scaling O_0 = marginal, O_2 = relevant

Difficulty is immediate: multiscale PT at $h = 0$ generates a power series with at least the above 4 running costants $(\boldsymbol{\lambda}_n)$ $n \le 0$. Should be related by recurrence

$$\boldsymbol{\lambda}_n = \Lambda \boldsymbol{\lambda}_{n+1} + \mathcal{B}(\boldsymbol{\lambda}_{n+1}), \quad \lambda_0 = (-\lambda, 0, 0, 0)$$

with $\Lambda = (1, \frac{1}{2}, 2, \frac{1}{2})$ and $\mathcal{B}$ is a formal series.

Even forgetting convergence, PT of no use: marginal term grows (if $\lambda_0 < 0$) and generates relevant term!

To understand a simpler problem turn to hierarchical model

The propagators $g^{[m]}(t-t')$ are $\tilde{c}$onstant for $t > t'$ on scale m, *i.e.* $t, t' \in I_m = [n2^{-m}, (n+1)2^{-m}]$, antisymmetric in t, t' and fast decay on scale 2^{-m}

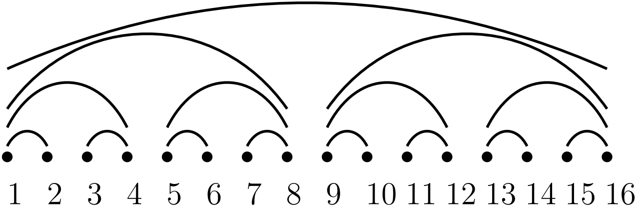
Hierarchical fields will be defined by assigning to each I_m two Grassmannians $2^{\frac{1}{2}m} z^{[m]}(t), \zeta^{[m]}(t)$

1) exactly constant in each half of I_m

2) propagator 1 for $t \in I_m^-, t' \in I_m^+$, -1 for $t \in I_m^+, t' \in I_m^-$

3) independent for $t \in I_m$, $t' \in I_{m'} \neq I_m$

$$\psi_\alpha^{[\le m]\pm}(t) = 2^{\frac{m}{2}} \Big(z_\alpha^{[m]\pm}(t) + \frac{1}{\sqrt{2}} Z_\alpha^{[m-1]\pm} \Big),$$
$$\varphi_\beta^{[\le m]\pm}(t) = \zeta_\beta^{[m]\pm}(t) + \Xi_\beta^{[m-1]\pm}$$



Hierarchy of lattice sites $[1, \ldots, 2^N]$: i intervals on scale 0

$$\psi_\alpha^{[\leq m]\pm}(t) = 2^{\frac{m}{2}} \left(z_\alpha^{[m]\pm}(t) + \frac{1}{\sqrt{2}} Z_\alpha^{[m-1]\pm} \right),$$
$$\varphi_\beta^{[\leq m]\pm}(t) = \zeta_\beta^{[m]\pm}(t) + \Xi_\beta^{[m-1]\pm}$$

where z, ζ are fields of scale m while Z e Ξ are constant on scale m (not $m-1$).

A second key is that the integral on a given scale can be exactly computed: no new operators arise

$$V(I) = \frac{1}{2}\Big(\lambda_0\,(\mathbf{A_1}\cdot\mathbf{B_1} + \mathbf{A_2}\cdot\mathbf{B_2}) + \lambda_{\mathbf{1}}\,(\mathbf{A_1^2} + \mathbf{A_2^2}) + \lambda_2\,(\mathbf{B_1^2} + \mathbf{B_2^2}) + \lambda_{\mathbf{3}}\,(\mathbf{A_1^2}\mathbf{B_1^2} + \mathbf{A_2^2}\varphi\mathbf{B_2^2})\Big)$$

with $\mathbf{A}_1 \stackrel{def}{=} \psi^+(t)\boldsymbol{\sigma}\psi^-(t), \mathbf{B}_1(t) \stackrel{def}{=} \varphi^+(t)\boldsymbol{\sigma}\varphi^-(t)\ t \in I^-$.

$$e^{V'(I)} = C\exp\frac{1}{2}\Big(\lambda'_0\,\mathbf{a}\cdot\mathbf{b} + \lambda'_{\mathbf{1}}\,\mathbf{a^2} + \lambda'_{\mathbf{2}}\,\mathbf{b^2} + \lambda'_{\mathbf{3}}\,\mathbf{a^2}\mathbf{b^2}\Big) = \int e^{V(I)}\,P(dz)P(d\zeta)$$

defines exact recursion

The running couplings $\boldsymbol{\lambda}_n$ can be explicitly computed in closed form in terms of the $\boldsymbol{\lambda}_{n+1}$.

$$\begin{aligned}
\lambda'_0 =& \frac{2}{C}\Big(\frac{1}{2}\lambda_0 + 9\lambda_0\lambda_3 + \frac{3}{2}\lambda_0\lambda_2 + \frac{3}{2}\lambda_0\lambda_1 - \lambda_0^2\Big)\\
\lambda'_1 =& \frac{2}{C}\Big(\frac{1}{4}\lambda_1 + \frac{9}{2}\lambda_2\lambda_3 + \frac{1}{8}\lambda_0^2\Big),\\
\lambda'_2 =& \frac{2}{C}\Big(\lambda_2 + 18\lambda_1\lambda_3 + \frac{1}{2}\lambda_0^2\Big)\\
\lambda'_3 =& \frac{2}{C}\Big(\frac{1}{4}\lambda_3 + \frac{1}{8}\lambda_1\lambda_2 + \frac{1}{48}\lambda_0^2\Big)\\
C =& 1 + 3\lambda_0^2 + 9\lambda_1^2 + 9\lambda_2^2 + 324\lambda_3^2
\end{aligned}$$

In other words model exactly reducible to a 4 dim. map

Non perturbative as for $n \to -\infty$ (IR limit) $\boldsymbol{\lambda}_n$ converge to non trivial fixed point if $h = 0$, exactly computable.

If only marginal and relevant retained 4 fixed points

$$\lambda_0' = \frac{2}{C}\left(\frac{1}{2}\lambda_0 + \frac{3}{2}\lambda_0\lambda_2 - \lambda_0^2\right)$$
$$\lambda_2' = \frac{2}{C}\left(\lambda_2 + \frac{1}{2}\lambda_0^2\right)$$
$$C = 1 + 3\lambda_0^2 + 9\lambda_2^2$$

f0=$(0,0)$ marginal, repelling $\lambda_0 < 0$
f1=$(0,\frac{1}{3})$ unstable with marginal repelling $\lambda_0 < 0$,
f2=$(0,-\frac{1}{3})$ stable, fixed point for $\lambda_0 > 0$,
f*=$(-\frac{2}{3},\frac{1}{3})$ stable

Starting from $\lambda_0 < 0$ quickly close to f1 then slowly to f*
Kondo temp. $\beta_K = 2^{n_0(\lambda_0)}$ is β the temperature at which the non-trivial fixed point is reached by all components
For small λ, $j = 0, 1, 3$
$n_j(\lambda_0) = c_0|\lambda_0|^{-1}, \qquad c_0 \simeq 0.5$
$n_2(\lambda_0) = c_2|\lambda_0|^{-1}, \qquad c_2 \simeq 2.$

Susceptibility: new operators needed to close beta

$$O_4 = \vec{A}\cdot\vec{h},\ O_5 = \vec{B}\cdot\vec{h},\ O_6 = \vec{A}\cdot\vec{h}\,\vec{B}\cdot\vec{h},$$
$$O_7 = \vec{A}^2\vec{B}\cdot\vec{h},\ O_8 = \vec{B}^2\vec{A}\cdot\vec{h}$$

O_0, O_4, O_6, O_8 marginal and O_2, O_5 relevant + irrelevant

Calculating beta function: ~ 20 Feynman graphs at $h = 0$ but (person dependent) 52100 graphs for $h \neq 0$; after simplifications beta function with 81 coeff.

$$\widetilde{\lambda}_0 = \frac{1}{2}\lambda_0 - \lambda_0\lambda_6 + 9\lambda_0\lambda_3 + \frac{3}{2}\lambda_0\lambda_2 + \frac{3}{2}\lambda_0\lambda_1 - \lambda_0^2$$
$$\widetilde{\lambda}_1 = \frac{1}{4}\lambda_1 + \frac{9}{2}\lambda_2\lambda_3 + \frac{3}{4}\lambda_8^2 + \frac{1}{24}\lambda_6^2 + \frac{1}{2}\lambda_5\lambda_7 + \frac{1}{48}\lambda_4^2 + \frac{1}{2}\lambda_0\lambda_6 + \frac{1}{8}\lambda_0^2$$
$$\widetilde{\lambda}_2 = \lambda_2 + 18\lambda_1\lambda_3 + \frac{1}{2}\lambda_0^2 + 3\lambda_7^2 + \frac{1}{6}\lambda_6^2 + \frac{1}{12}\lambda_5^2 + \lambda_4\lambda_8 + \frac{1}{3}\lambda_0\lambda_6$$
$$\widetilde{\lambda}_3 = \frac{1}{4}\lambda_3 + \frac{1}{8}\lambda_1\lambda_2 + \frac{1}{24}\lambda_0^2 + \frac{1}{72}\lambda_0\lambda_6\frac{1}{144}\lambda_6^2 + 6\lambda_5\lambda_7 + \frac{1}{24}\lambda_4\lambda_8$$

$$\widetilde{\lambda}_4 = \frac{1}{2}\lambda_4 + 3\lambda_6\lambda_7 + \frac{1}{2}\lambda_5\lambda_6 + 54\lambda_3\lambda_8 + 9\lambda_2\lambda_8 + \frac{3}{2}\lambda_1\lambda_4 + 3\lambda_0\lambda_7 + \frac{1}{2}\lambda_0\lambda_5$$

$$\widetilde{\lambda}_5 = \lambda_5 + 6\lambda_6\lambda_8 + \lambda_4\lambda_6 + 108\lambda_3\lambda_7 + 3\lambda_2\lambda_5 + 18\lambda_1\lambda_7 + 6\lambda_0\lambda_8 + \lambda_0\lambda_4$$

$$\widetilde{\lambda}_6 = \frac{1}{2}\lambda_6 + 9\lambda_7\lambda_8 + \frac{3}{2}\lambda_5\lambda_8 + \frac{3}{2}\lambda_4\lambda_7 + \frac{1}{4}\lambda_4\lambda_5 + 9\lambda_3\lambda_6 + \frac{3}{2}\lambda_2\lambda_6 + \frac{3}{2}\lambda_1\lambda_6 + \lambda_0\lambda_6$$

$$\widetilde{\lambda}_7 = \frac{1}{4}\lambda_7 + \frac{1}{4}\lambda_6\lambda_8 + \frac{1}{24}\lambda_4\lambda_6 + \frac{3}{4}\lambda_3\lambda_5 + \frac{3}{4}\lambda_2\lambda_7 + \frac{1}{8}\lambda_1\lambda_5 + \frac{1}{4}\lambda_0\lambda_8 + \frac{1}{24}\lambda_0\lambda_4$$

$$\widetilde{\lambda}_8 = \frac{1}{2}\lambda_8 + \frac{1}{2}\lambda_6\lambda_7 + \frac{1}{12}\lambda_5\lambda_6 + \frac{3}{2}\lambda_3\lambda_4 + \frac{1}{4}\lambda_2\lambda_4 + \frac{3}{2}\lambda_1\lambda_8 + \frac{1}{2}\lambda_0\lambda_7 + \frac{1}{12}\lambda_0\lambda_5$$

$$C = 1 + 2\lambda_0^2 + (\lambda_0 + \lambda_6)^2 + 9\lambda_1^2 + 9\lambda_2^2 + 324\lambda_3^2 + \frac{1}{2}\lambda_4^2 + \frac{1}{2}\lambda_5^2 + 18\lambda_7^2 + 18\lambda_8^2$$

Flow of running const: relevant,marginal,irrelevant $h = 0$

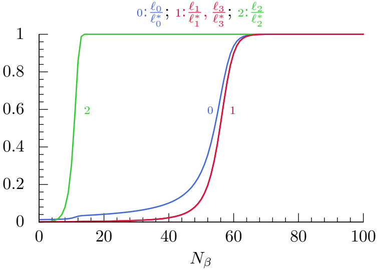


Fig.1:plot of $\frac{\ell}{\ell^*}$ as a function of N_β, $\lambda_0 \equiv \alpha_0 = -0.01$.

Projection of the flow on plane λ_0, λ_2

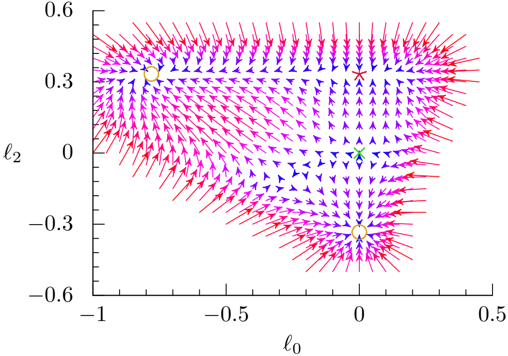


Fig.2: phase diagram projected on the (ℓ_0, ℓ_2) plane, with initial conditions chosen in the plane that contains all four fixed points: f* (linearly stable, yellow, nontrivial fixed pt.), f0 (trivial fixed pt. 1 unstable dir. and 1 marginal, green cross), f1 (1 linearly stable direction and one marginal, red star), and f2 (linearly stable, yellow).

The flow to "high T fixed pt" at scale $\propto 1/|\log h|$

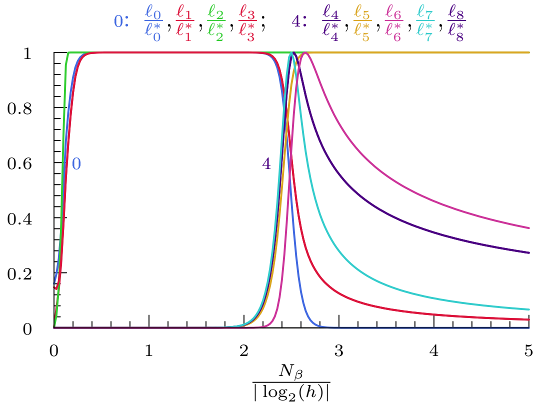


Fig.3: $\frac{\ell}{\ell^*}$ versus $N_\beta = \log_2 \beta$, at $\lambda_0 = -0.01$, $h = 2^{-40}$. ℓ_0^* – ℓ_3^* are components of on-trivial fix. pt., ℓ_4^* through ℓ_8^* are the values reached with largest absolute value by $\ell_4 - \ell_8$. Flow is on $h = 0$-path until ℓ_4–ℓ_8 large: $N_\beta \sim r(h)$. Then $\to 0$, **except relevant**: $r(h) = c_r \log_2 \beta / |\log_2 h|$, $c_r = 2.6$.

The equation of state

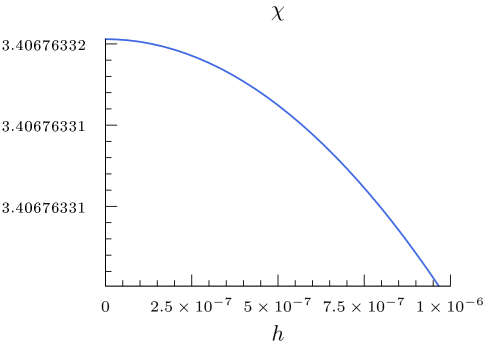


Fig.4: plot of $\chi(\beta, h)$ for $h \leq 10^{-6}$ at $\lambda_0 = -0.28$ and $\beta = 2^{20}$ (so that the largest value for βh is ~ 1)

Attempt at reproducing Wilson's last graph

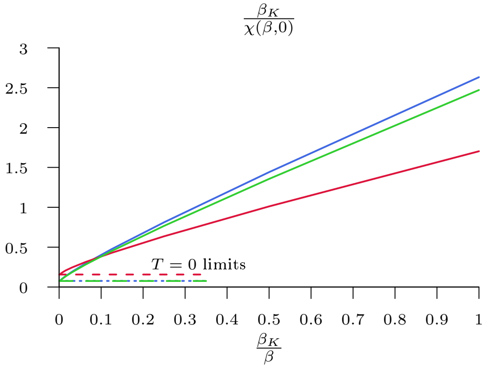


Fig.5: plot of $\frac{\beta_K}{\chi(\beta,0)}$ as a function of $\frac{\beta_K}{\beta}$ for various values of λ_0: $\lambda_0 = -0.024$ (blue), $\lambda_0 = -0.02412$ (green),$\lambda_0 = -0.05$ (red). In [7], $\lambda_0 = -0.024$ and -0.02412. Note that the abscissa of the data points are 2^{-n} for $n \geq 0$, so that there are only 3 points in the range $[0.25, 1]$. The lines are drawn for visual aid.

Sharp saturation of the marginal and irrelevant contants at $\beta_K \simeq \frac{1}{2}\frac{1}{|\lambda_0|}$

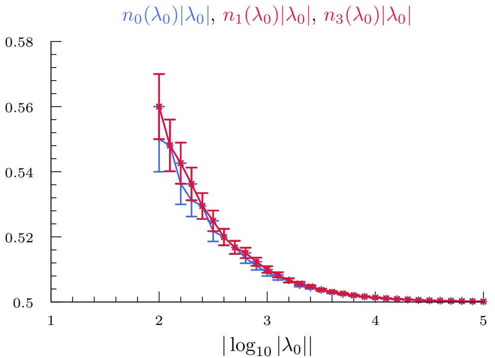


Fig.6: plot of $n_j(\lambda_0)|\lambda_0|$ for $j = 0$ (blue) and $j = 1, 3$ (red) as a function of $|\log_{10}|\lambda_0||$.

Sharp saturation of the relevant contants at $\beta_K \simeq 2\frac{1}{|\lambda_0|}$

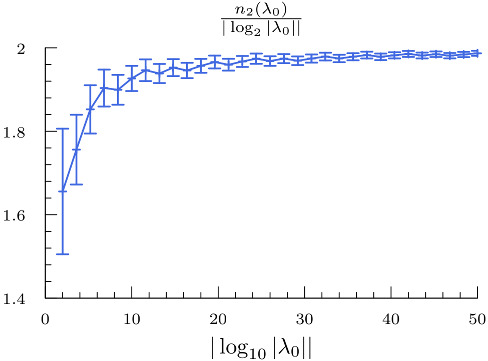


Fig.6: plot of $n_2(\lambda_0)|\log_2|\lambda_0|||^{-1}$ as a function of $|\log_{10}|\lambda_0||$.

The relevant constants approach the fixed point 4 times more slowly than the marginal and irrelevant (unlike their faster approach to the temporary fixed point f1)

$h > 0$: sharp decay as $h \to 0$ to the "high temperature" fixed point $r_j(h) \simeq 2.6 \frac{1}{-\log_2 h}$

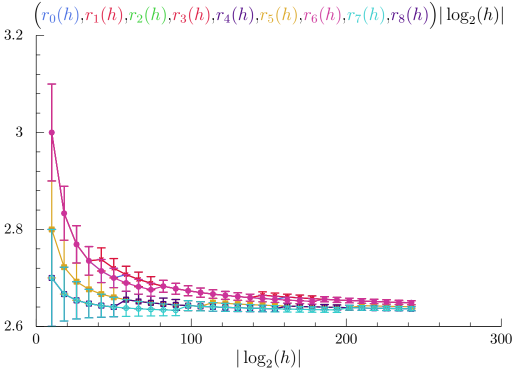


Fig.6: plot of $r_j(h)|\log_2(h)||$ as a function of $|\log_2(h)|$.

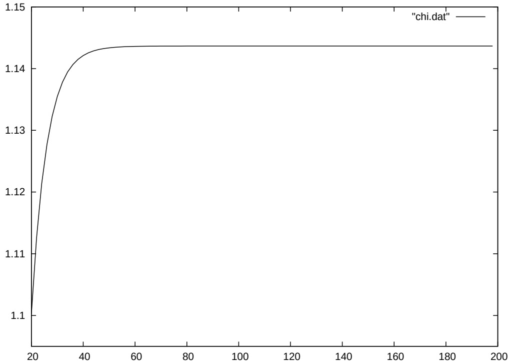


Fig.5a: Plot of $\chi(\beta, 0)$ as a function of $\log_2 \beta$ for $\ell_0 = -0.28$ [1, 6, 2, 3, 4, 5]

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