Validity of the Spin-Wave Approximation for the Heisenberg Ferromagnet

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joint work with A. Giuliani (Roma 3) and R. Seiringer (IST Vienna)

Outline

1. Motivations and mathematical setting:
   - 3D quantum Heisenberg ferromagnet at low temperature.
   - The spin-wave theory and the Holstein-Primakoff representation.
2. Main results:
   - Validity of the spin-wave theory for the free energy at low temperature.
   - Quasi long-range order.
3. Sketch of the proofs.

Main references

- MC, A. Giuliani, *J. Stat. Phys.* **149** (2012).
- MC, A. Giuliani, R. Seiringer, *Commun. Math. Phys.* to appear (full proofs);
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Motivations

- The Heisenberg model (HM) is a paradigmatic to study phase transitions in presence of a continuous symmetry.
- *Very few* rigorous results (mostly or almost exclusively based on *reflection positivity*).

Literature about HM

- *Absence* of phase transitions in 1 or 2D [Mermin, Wagner '66].
- classical Heisenberg: proof of symmetry breaking [Fröhlich, Simon, Spencer '76]; spin-wave expansion [Balaban '95–'98];
- plane rotator model: exactness of the spin-wave expansion to any order [Bricmont, Fontaine, Lebowitz, Lieb, Spencer '81];
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Ferromagnetic Heisenberg model

Ferromagnetic quantum Heisenberg Hamiltonian

$$H = \sum_{\langle \mathbf{x},\mathbf{y} \rangle \subset \Lambda} \left(S^2 - \hat{\mathbf{S}}_{\mathbf{x}} \cdot \hat{\mathbf{S}}_{\mathbf{y}} \right)$$

- $\Lambda \subset \mathbb{Z}^3$ is a 3D box of side length L with periodic boundary conditions.
- $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes nearest neighbors in Λ, i.e., $|\mathbf{x} - \mathbf{y}| = 1$.
- $\hat{\mathbf{S}}$ quantum spin operator on $\mathbb{C}^{2S+1}$ ($2S$ *integer*), i.e., generator of a $2S+1$-dimensional representation of $SU(2)$:

$$\left[\hat{S}^j_{\mathbf{x}}, \hat{S}^k_{\mathbf{y}} \right] = i\varepsilon_{jkl} \hat{S}^l_{\mathbf{x}} \delta_{\mathbf{x},\mathbf{y}} , \quad \hat{\mathbf{S}}^2_{\mathbf{x}} = (\hat{S}^1_{\mathbf{x}})^2 + (\hat{S}^2_{\mathbf{x}})^2 + (\hat{S}^3_{\mathbf{x}})^2 = S(S+1).$$

- H acts on the Hilbert space $\mathscr{H} \simeq \mathbb{C}^{(2S+1)L^3}$.
- H is normalized so that the ground state energy is 0.

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Ground states

- Introduce the total spin $\hat{\mathbf{S}}_T = \sum_{\mathbf{x}\in\Lambda} \hat{\mathbf{S}}_x$ and the eigenstates $|S_T, S_T^3\rangle$ of $\hat{\mathbf{S}}_T^2$, i.e., such that $\hat{\mathbf{S}}_T^2 |S_T, S_T^3\rangle = S_T(S_T+1) |S_T, S_T^3\rangle$.

Ground states

The ground states of H are the states with maximal total spin

$$|SL^3, S_T^3\rangle, \qquad \text{with } S_T^3 = -SL^3, \ldots, SL^3.$$

- Ground states are such that, for any *nearest neighbor* pair $\langle \mathbf{x}, \mathbf{y} \rangle$, $\hat{\mathbf{S}}_{\mathbf{x}} \cdot \hat{\mathbf{S}}_{\mathbf{y}}$ reaches its maximal value S^2.
- Also the *partial sums* $(\hat{\mathbf{S}}_{\sigma(1)} + \cdots \hat{\mathbf{S}}_{\sigma(N-k)})^2$, with $k = 1, \ldots, N-1$ and σ any perturbation, must be maximal on a ground state.
- The degeneracy $2SL^3 + 1$ is due to spherical symmetry of the model and could be removed by adding an external magnetic field.

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Excited states: spin-waves

- Assume that the system is in the ground state $|SL^3, SL^3\rangle$ (e.g., because of a small $h < 0$) $\Longrightarrow$ one can think of producing an excited state by lowering *just one* spin: setting $\hat{S}^{\pm}_{\mathbf{x}} = \hat{S}^1_{\mathbf{x}} \pm i\hat{S}^2_{\mathbf{x}}$,

$$|\mathbf{x}\rangle = \frac{1}{\sqrt{2S}}\hat{S}^{-}_{\mathbf{x}} |SL^3, SL^3\rangle$$

- $|\mathbf{x}\rangle$ is not an eigenstate of H but a linear combination can be...

Spin waves

The spin waves are the orthonormal states (with $\mathbf{k} \in \Lambda^* = \frac{2\pi}{L}\mathbb{Z}^3$)

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Spin-wave approximation

- Neglecting the interaction, spin waves behave like free bosons, i.e., the mean number of excitations at $\beta \gg 1$ is given by the Bose statistics
$$\langle n_{\mathbf{k}} \rangle_\beta = \frac{1}{e^{S\beta\varepsilon(\mathbf{k})} - 1}$$
- If $\beta \gg 1$ the spin-wave approximation predicts:
  - free energy: $f_0(S,\beta) = -\lim_{L\to\infty} \frac{1}{\beta L^{3/2}} \log \operatorname{Tr}\left(e^{-\beta H_0}\right)$,
$$f_0(S,\beta) = \frac{1}{\beta} \int_{[-\pi,\pi]^3} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \log\left(1 - e^{-\beta S \varepsilon(\mathbf{k})}\right) = -\frac{\zeta(5/2)}{8(\pi S)^{3/2}} \frac{1}{\beta^{5/2}}$$
  - spontaneous magnetization $M_{\mathrm{sp}}(\beta) = \lim_{L\to\infty} \left[S - \frac{1}{L^3} \sum_{\mathbf{k}\in\Lambda^*} \langle n_{\mathbf{k}} \rangle \right]$,
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Holstein-Primakoff representation

Creation & annihilation operators

- For any $\mathbf{x} \in \Lambda$ one sets [Holstein, Primakoff '40]

$$a^\dagger_{\mathbf{x}} = \hat{S}^+_{\mathbf{x}} \frac{1}{\sqrt{S - \hat{S}^3_{\mathbf{x}}}}, \qquad a_{\mathbf{x}} = \frac{1}{\sqrt{S - \hat{S}^3_{\mathbf{x}}}} \hat{S}^-_{\mathbf{x}}, \qquad \hat{n}_{\mathbf{x}} = \hat{S}^3_{\mathbf{x}} + S,$$

- One has $[a_{\mathbf{x}}, a^\dagger_{\mathbf{x}}] = 1$, $\hat{n}_{\mathbf{x}} = a^\dagger_{\mathbf{x}} a_{\mathbf{x}}$ and $0 \leq \hat{n}_{\mathbf{x}} \leq 2S$.

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$$\mathcal{H} = \mathcal{H}_0 - \mathcal{K}, \qquad \mathcal{H}_0 = S \sum_{\langle \mathbf{x},\mathbf{y} \rangle \subset \Lambda} \left(a^\dagger_{\mathbf{x}} - a^\dagger_{\mathbf{y}} \right) \left(a_{\mathbf{x}} - a_{\mathbf{y}} \right),$$

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COMB-MATH

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$$\mathcal{H} = S \sum \left(a^\dagger_{\mathbf{x}} - a^\dagger_{\mathbf{y}} \right) \left(a_{\mathbf{x}} - a_{\mathbf{y}} \right) - \mathcal{K}$$

- The spin-wave approximation is given by dropping the spin-wave interaction:
  1. hard-core constraint $n_{\mathbf{x}} \leq 2S$;
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Physics of spin-waves

- $\mathcal{K}$ is formally of relative size S^{-1} with respect to $\mathcal{H}_0$.
- At least if $S \gg 1$, the spin-wave approximation should be *asymptotically* correct $\Longrightarrow$ to observe a non-trivial behavior one has to consider temperature scales of order S^{-1}.
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Known results

- Exactness of the spin-wave theory for the computation of the free energy, when $S \to \infty$ with $\beta \propto S^{-1}$ and a magnetic field $h \propto S$ [Conlon, Solovej '90].
- In the regime $\beta \to \infty$ with S fixed, there was only an upper bound to the free energy (obtained through probabilistic methods) [Conlon, Solovej '91; Toth '93]

$$f_0(1/2, \beta) \leq \frac{C_1}{\beta^{5/2}}, \qquad \text{with } C_1 > -\frac{\zeta(5/2)}{(2\pi)^{3/2}}$$

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Theorem (free energy [MC, Giuliani, Seiringer '13])

For any $S \geq \frac{1}{2}$,

$$\lim_{\beta\to\infty} S^{3/2}\beta^{5/2} f(S,\beta) = \int_{\mathbb{R}^3} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \log\left(1 - e^{-k^2}\right) = -\frac{\zeta(5/2)}{8\pi^{3/2}}$$

Remarks

- The result is uniform in S for any finite S.
- In fact S needs not to be fixed but it is necessary that $\beta S \to \infty$, under the additional constraint $\beta S \gg S^{\alpha}$.
- The upper bound proven in [Toth '93] was

$$\left(\tfrac{1}{2}\right)^{3/2} \beta^{5/2} f(1/2,\beta) \leq -\frac{\zeta(5/2)}{8\pi^{3/2}} \log 2 + o(1)$$

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Quasi long-range order

- A first consequence of the main result is that the energy per site $e(S,\beta) = \partial_\beta(\beta f(S,\beta))$ is as $\beta \to \infty$

$$e(S,\beta) \simeq -C S^{-3/2}\beta^{-5/2}, \qquad C = \frac{3\zeta(5/2)}{16\pi^{3/2}}$$

- A more relevant by-product of the result together with a crucial estimate on the excitation spectrum of H is that

$$\left\langle S^2 - \mathbf{S_x}\cdot\mathbf{S_y}\right\rangle_\beta \leq \tfrac{27}{8}|\mathbf{x}-\mathbf{y}|^2 e(S,\beta) \simeq C\beta^{-5/2}|\mathbf{x}-\mathbf{y}|^2.$$

which yields

$$\boxed{\left\langle \mathbf{S_x}\cdot\mathbf{S_y}\right\rangle_\beta \geq S^2 - C\beta^{-5/2}|\mathbf{x}-\mathbf{y}|^2 = S^2 + o(1),}$$

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Sketch of the proof

Upper bound

$$f(\beta) \leq C_0 \left(\tfrac{1}{2}\right)^{-3/2} \beta^{-5/2} \left(1 + \mathcal{O}(\beta^{-3/8})\right)$$

1. Localization into boxes of side length $\ell \gg \sqrt{\beta}$ with *Dirichlet* b.c.;
2. Gibbs variational principle $f(\beta) \leq \frac{1}{\ell^3} \mathrm{Tr}(\mathcal{H}\Gamma + \frac{1}{\beta}\Gamma \log \Gamma)$ + trial state
$$\Gamma = \frac{P e^{-\beta \mathcal{H}_0} P}{\mathrm{Tr} P e^{-\beta \mathcal{H}_0}}$$
with $P = \prod P_{\mathbf{x}}$ projecting onto *hard-core* states with $n_{\mathbf{x}} \leq 1$.
3. Key estimate $1 - P \leq \frac{1}{2} \sum \hat{n}_{\mathbf{x}} (\hat{n}_{\mathbf{x}} - 1)$ + Wick's theorem and $\langle \hat{n}_{\mathbf{x}} \rangle_\beta = \mathcal{O}(\beta^{-3/2})$:
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$$Z_0^{-1} \mathrm{Tr}(1 - P) e^{-\beta \mathcal{H}_0} \leq C \ell^3 \beta^{-3}$$
4. Optimization with respect to ℓ ($\sim \beta^{7/8}$).

Sketch of the proof ($S = \frac{1}{2}$)

Lower bound

$$f(\beta) \geq C_0 \left(\tfrac{1}{2}\right)^{-3/2} \beta^{-5/2} \left(1 + \mathcal{O}(\beta^{-\kappa})\right), \qquad \kappa < \tfrac{1}{40}$$

1. Localization into *Neumann* boxes;
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By dropping the *positive* interaction among different subcells, one has

$$f(\beta) \geq f(\beta, \Lambda_\ell) = -\frac{1}{\ell^3 \beta} \log \mathrm{Tr} e^{-\beta H},$$

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In the subspace $\mathscr{H}_{S_T}$ *with total spin* $S_T(S_T+1)$

$$H\big|_{\mathscr{H}_{S_T}} \geq \frac{C}{\ell^2}\left(\tfrac{1}{2}\ell^3 - S_T\right)$$

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For $\ell \geq \beta^{1/2}$ *and* $\beta \gg 1$*, one has* $f(\beta,\Lambda_\ell) \geq -C\left(\log\beta/\beta\right)^{5/2}$.

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- H commutes with $\hat{\mathbf{S}}_T \Longrightarrow$ in the trace we can fix $S_T^3 = -S_T$

$$\mathrm{Tr} e^{-\beta H} = \sum_{S_T=0}^{\frac{1}{2}\ell^3} (2S_T + 1) \mathrm{Tr}_{S_T^3 = -S_T} e^{-\beta H}$$

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$$\mathcal{H} = \tfrac{1}{2} P \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \left[\left(a^\dagger_{\mathbf{x}} - a^\dagger_{\mathbf{y}} \right) (a_{\mathbf{x}} - a_{\mathbf{y}}) - \hat{n}_{\mathbf{x}} \hat{n}_{\mathbf{y}} \right] P$$

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- Peierls-Bogoliubov inequality $\mathrm{Tr} e^{A+B} / \mathrm{Tr} e^{A} \leq \exp\{\mathrm{Tr}(B e^{A}) / \mathrm{Tr} e^{A})\}$.
- To conclude the proof we thus have to estimate the expectation value

  $$\langle E | \mathcal{K} | E \rangle = \sum_{\langle \mathbf{x}, \mathbf{y} \rangle \subset \Lambda} \rho(\mathbf{x}, \mathbf{y}) \leq C \ell^3 \| \rho \|_\infty$$

  of the bosonic interaction $\mathcal{K}$ over eigenstates $|E\rangle$ of $\mathcal{H}$ with two-particle density $\rho(\mathbf{x}, \mathbf{y}) = \langle E | a^\dagger_{\mathbf{x}} a^\dagger_{\mathbf{y}} a_{\mathbf{x}} a_{\mathbf{y}} | E \rangle$.
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If $|E\rangle$ *is an eigenfunction of* $\mathcal{H}$ *and* $\rho(\mathbf{x}, \mathbf{y}) = \langle E | a^\dagger_{\mathbf{x}} a^\dagger_{\mathbf{y}} a_{\mathbf{x}} a_{\mathbf{y}} | E \rangle$ *(*$E > 0$*),*

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- Since $\|\rho\|_1 \leq C N^2$, for $\ell \gtrsim \beta^{1/2+\varepsilon}$ and $E \leq E_0$,

$$\langle E | \mathcal{K} | E \rangle \leq C \ell^3 \|\rho\|_\infty \leq C \beta^{-3/2} \beta^{\varepsilon'}$$

- The expectation value of the interaction $\langle E | \mathcal{K} | E \rangle = \mathcal{O}(\beta^{-3/2+\varepsilon'})$ is much smaller that the kinetic term $\langle E | \mathcal{H}_0 | E \rangle = C \ell^{-2} = \mathcal{O}(\beta^{-1})$:

$$e^{\beta \langle \mathcal{K} \rangle_\beta} \leq \exp\{\beta^{-1/2+\varepsilon'}\} \leq 1 + o(1)$$

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If $|E\rangle$ *is an eigenfunction of* $\mathcal{H}$ *and* $\rho(\mathbf{x}, \mathbf{y}) = \langle E | a^\dagger_{\mathbf{x}} a^\dagger_{\mathbf{y}} a_{\mathbf{x}} a_{\mathbf{y}} | E \rangle$ *(*$E > 0$*),*

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❸ Leading term

- Ignoring the constraint $N \leq \ell^2 E_0$ for all states but $\mathbf{k} = 0$,

$$\mathrm{Tr}_{N \leq \ell^2 E_0} e^{-\beta \mathcal{H}_0} \leq \left(\ell^2 E_0 + 1\right) \prod_{\mathbf{k} \in \Lambda_\ell^*, \mathbf{k} \neq 0} \frac{1}{1 - e^{-\frac{1}{2}\beta\varepsilon(\mathbf{k})}}$$

- Riemann sum approximation

$$\frac{1}{\ell^3} \sum \log\left(1 - e^{-\frac{1}{2}\beta\varepsilon(\mathbf{k})}\right) \simeq \frac{1}{(2\pi)^3} \int_{[\pi,\pi]^3} \mathrm{d}\mathbf{k}\, \log\left(1 - e^{-\frac{1}{2}\beta\varepsilon(\mathbf{k})}\right)$$

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Proof of Theorem ($\star$)

- Reduce the N-body eigenvalue equation $H\left|\Psi\right\rangle = E\left|\Psi\right\rangle$ to the differential inequality

$$-\tilde{\Delta}\rho(\mathbf{x},\mathbf{y}) \leq 4E\rho(\mathbf{x},\mathbf{y})$$

where $\tilde{\Delta}$ is the discrete Laplacian on $\Lambda_\ell \times \Lambda_\ell \setminus \{(\mathbf{x},\mathbf{x}), \mathbf{x} \in \Lambda_\ell\}$, i.e.,

$$-\tilde{\Delta}\rho(\mathbf{x},\mathbf{y}) = \left(-\Delta_{\mathbf{x}} - \Delta_{\mathbf{y}}\right)\rho(\mathbf{x},\mathbf{y}) + 2\rho(\mathbf{x},\mathbf{y})\mathbb{1}_{\{|\mathbf{x}-\mathbf{y}|=1\}}$$

$$-\Delta\rho(\mathbf{x}) = \sum_{\mathbf{y}\in\Lambda_\ell, |\mathbf{y}-\mathbf{x}|=1} \left(\rho(\mathbf{x}) - \rho(\mathbf{y})\right)$$

- Extend the inequality to the whole of $\mathbb{Z}^6$ via reflections:

$$-\Delta\rho(\mathbf{z}) \leq 4E\rho(\mathbf{z}) + 2\rho(\mathbf{z})\chi(\mathbf{z})$$

where P_n is the probability of a random walk on $\mathbb{Z}^d$.

- Pick $n \simeq E^{-1} \gg 1$: $P_n(\mathbf{z},\mathbf{z}') \propto n^{-3} e^{-3|\mathbf{z}-\mathbf{z}'|/n}$ so that for some $c < 1$

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Perspectives

(✔) Upper bound for $f(S, \beta)$ in the regime $S \to \infty$ and $\beta = \tilde{\beta} S^{-1}$ to the first order in S^{-1} [with N. Benedikter]:

$$\frac{f(S,\beta)}{S} \leq \frac{1}{(2\pi)^3 \tilde{\beta}} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \log\left(1 - e^{-\tilde{\beta}\varepsilon(\mathbf{k})}\right) + \frac{C_1}{S} + \mathcal{O}(1/S^2),$$

where C_1 is computed via the spin-wave approximation.

(✗) Upper bound for $f(\beta)$ as $\beta \to \infty$ up to the first non-trivial contribution:

$$f(\beta) \leq \beta^{-5/2}\left(C_0 + C_1' \beta^{-1} + C_2' \beta^{-2} + C_3' \beta^{-5/2} + o(\beta^{-5/2})\right),$$

where C_3' is *still* the spin-wave prediction.

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(✗) Spontaneous magnetization and breaking of the rotational symmetry.

Thank you for the attention!

Perspectives

COMB-MATH

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