VALIDITY OF THE SPIN-WAVE APPROXIMATION FOR THE HEISENBERG FERROMAGNET

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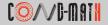
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11 June 2015 CRG2015: in memory of *Pierluigi Falce*

joint work with A. Giuliani (Roma 3) and R. Seiringer (IST Vienna)

M. Correggi (Roma 3) Ferromagnetic Heisenberg Model Frascati 11/06/2015 1 / 21

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In Motivations and mathematical setting:

- 3D quantum Heisenberg ferromagnet at low temperature.
- The spin-wave theory and the Holstein-Primakoff representation.

2 Main results:

- Validity of the spin-wave theory for the free energy at low temperature.
- Quasi long-range order.
- ③ Sketch of the proofs.

MAIN REFERENCES

- MC, A. GIULIANI, J. Stat. Phys. 149 (2012).
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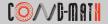
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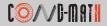
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- Very few rigorous results (mostly or almost exclusively based on *reflection positivity*).

- Absence of phase transitions in 1 or 2D [MERMIN, WAGNER '66].
- classical Heisenberg: proof of symmetry breaking [FRÖHLICH, SIMON, SPENCER '76]; spin-wave expansion [BALABAN '95-'98];
- plane rotator model: exactness of the spin-wave expansion to any order [BRICMONT, FONTAINE, LEBOWITZ, LIEB, SPENCER '81];
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FERROMAGNETIC QUANTUM HEISENBERG HAMILTONIAN

$$H = \sum_{\langle \mathbf{x}, \mathbf{y} \rangle \subset \Lambda} \left(S^2 - \hat{\mathbf{S}}_{\mathbf{x}} \cdot \hat{\mathbf{S}}_{\mathbf{y}} \right)$$

- $\Lambda \subset \mathbb{Z}^3$ is a 3D box of side length L with periodic boundary conditions.
- $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes nearest neighbors in Λ , i.e., $|\mathbf{x} \mathbf{y}| = 1$.
- $\hat{\mathbf{S}}$ quantum spin operator on \mathbb{C}^{2S+1} (2S integer), i.e., generator of a 2S + 1-dimensional representation of SU(2):

$$\left[\hat{S}_{\mathbf{x}}^{j}, \hat{S}_{\mathbf{y}}^{k}\right] = i\varepsilon_{jkl}\hat{S}_{\mathbf{x}}^{l}\delta_{\mathbf{x},\mathbf{y}} , \quad \hat{\mathbf{S}}_{\mathbf{x}}^{2} = (\hat{S}_{\mathbf{x}}^{1})^{2} + (\hat{S}_{\mathbf{x}}^{2})^{2} + (\hat{S}_{\mathbf{x}}^{3})^{2} = S(S+1).$$

- H acts on the Hilbert space $\mathscr{H}\simeq \mathbb{C}^{(2S+1)L^3}$
- H is normalized so that the ground state energy is 0.

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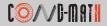
GROUND STATES

The ground states of ${\boldsymbol{H}}$ are the states with maximal total spin

 $\left|SL^3, S_T^3\right\rangle, \qquad \text{with } S_T^3 = -SL^3, \dots, SL^3.$

- Ground states are such that, for any *nearest neighbor* pair $\langle \mathbf{x}, \mathbf{y} \rangle$, $\hat{\mathbf{S}}_{\mathbf{x}} \cdot \hat{\mathbf{S}}_{\mathbf{y}}$ reaches its maximal value S^2 .
- Also the partial sums $(\hat{\mathbf{S}}_{\sigma(1)} + \cdots \cdot \hat{\mathbf{S}}_{\sigma(N-k)})^2$, with $k = 1, \dots, N-1$ and σ any perturbation, must be maximal on a ground state.
- The degeneracy $2SL^3 + 1$ is due to spherical symmetry of the model and could be removed by adding an external magnetic field.

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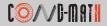
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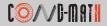
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EXCITED STATES: SPIN-WAVES



• Assume that the system is in the ground state $|SL^3, SL^3\rangle$ (e.g., because of a small h < 0) \implies one can think of producing an excited state by lowering *just one* spin: setting $\hat{S}^{\pm}_{\mathbf{x}} = \hat{S}^1_{\mathbf{x}} \pm i \hat{S}^2_{\mathbf{x}}$,

$$\left|\mathbf{x}\right\rangle = \frac{1}{\sqrt{2S}} \hat{S}_{\mathbf{x}}^{-} \left|SL^{3}, SL^{3}\right\rangle$$

 $\circ~|\mathbf{x}\rangle$ is not an eigenstate of H but a linear combination can be...

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The spin waves are the orthonormal states (with $\mathbf{k} \in \Lambda^* = \frac{2\pi}{L}\mathbb{Z}^3$)

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 $H |\mathbf{k}\rangle = S\varepsilon(\mathbf{k}) |\mathbf{k}\rangle, \qquad \varepsilon(\mathbf{k}) = 2\sum (1 - \cos k_i).$

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• Neglecting the interaction, spin waves behave like free bosons, i.e., the mean number of excitations at $\beta \gg 1$ is given by the Bose statistics

 $\langle n_{\bf k}\rangle_\beta = \frac{1}{e^{S\beta\varepsilon({\bf k})}-1}$ • If $\beta\gg 1$ the spin-wave approximation predicts:

• free energy: $f_0(S,\beta) = -\lim_{L \to \infty} \frac{1}{\beta L^{3/2}} \log \operatorname{Tr} \left(e^{-\beta H_0} \right)$,

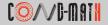
$$f_0(S,\beta) = \frac{1}{\beta} \int_{[-\pi,\pi]^3} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \log\left(1 - e^{-\beta S\varepsilon(\mathbf{k})}\right) = -\frac{\zeta(5/2)}{8(\pi S)^{3/2}} \frac{1}{\beta^{5/2}}$$

• spontaneous magnetization $M_{\rm sp}(\beta) = \lim_{L \to \infty} \left[S - \frac{1}{L^3} \sum_{\mathbf{k} \in \Lambda^*} \langle n_{\mathbf{k}} \rangle \right]$,

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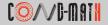
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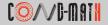
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• free energy: $f_0(S,\beta) = -\lim_{L\to\infty} \frac{1}{\beta L^{3/2}} \log \operatorname{Tr} \left(e^{-\beta H_0} \right)$,

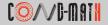
$$f_0(S,\beta) = \frac{1}{\beta} \int_{[-\pi,\pi]^3} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \log\left(1 - e^{-\beta S\varepsilon(\mathbf{k})}\right) = -\frac{\zeta(5/2)}{8(\pi S)^{3/2}} \frac{1}{\beta^{5/2}}$$

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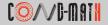
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CREATION & ANNIHILATION OPERATORS

• For any $\mathbf{x} \in \Lambda$ one sets [HOLSTEIN, PRIMAKOFF '40]

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VALIDITY OF SPIN-WAVE APPROXIMATION



$$\mathcal{H} = S \sum \left(a_{\mathbf{x}}^{\dagger} - a_{\mathbf{y}}^{\dagger} \right) \left(a_{\mathbf{x}} - a_{\mathbf{y}} \right) - \mathcal{K}$$

- The spin-wave approximation is given by dropping the spin-wave interaction:
 - 1) hard-core constraint $n_{\mathbf{x}} \leq 2S$;
 - 2 attractive interaction $\mathcal{K}.$

Physics of spin-waves

- ${\cal K}$ is formally of relative size S^{-1} with respect to ${\cal H}_0.$
- At least if S ≫ 1, the spin-wave approximation should be asymptotically correct ⇒ to observe a non-trivial behavior one has to consider temperature scales of order S⁻¹.
- What if S is fixed and $\beta \to \infty$? spin-wave approximation is still expected to be correct! [DYSON '56; ZITTARTZ '65].

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(MATH) LITERATURE



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- In the regime $\beta \to \infty$ with S fixed, there was only an upper bound to the free energy (obtained through probabilistic methods) [CONLON, SOLOVEJ '91; TOTH '93]

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Theorem (free energy [MC, Giuliani, Seiringer '13]) For any $S \geq \frac{1}{2}$,

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Remarks

- The result is uniform in S for any finite S.
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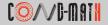
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MAIN RESULTS

QUASI LONG-RANGE ORDER



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• A first consequence of the main result is that the energy per site $e(S,\beta) = \partial_{\beta}(\beta f(S,\beta))$ is as $\beta \to \infty$

$$e(S,\beta) \simeq -CS^{-3/2}\beta^{-5/2}, \qquad C = \frac{3\zeta(5/2)}{16\pi^{3/2}}$$

• A more relevant by-product of the result together with a crucial estimate on the excitation spectrum of H is that

$$\left\langle S^2 - \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}} \right\rangle_{\beta} \leq \frac{27}{8} |\mathbf{x} - \mathbf{y}|^2 e(S, \beta) \simeq C \beta^{-5/2} |\mathbf{x} - \mathbf{y}|^2.$$

which yields

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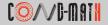
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M. Correggi (Roma 3) Ferromagnetic Heisenberg Model Frascati 11/06/2015 12 / 21

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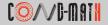
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M. Correggi (Roma 3) Ferromagnetic Heisenberg Model Frascati 11/06/2015

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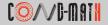
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• A more relevant by-product of the result together with a crucial estimate on the excitation spectrum of H is that

$$\left\langle S^2 - \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}} \right\rangle_{\beta} \leq \frac{27}{8} |\mathbf{x} - \mathbf{y}|^2 e(S, \beta) \simeq C \beta^{-5/2} |\mathbf{x} - \mathbf{y}|^2.$$
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which yields

$$\left< \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}} \right>_{\beta} \ge S^2 - C\beta^{-5/2} |\mathbf{x} - \mathbf{y}|^2 = S^2 + o(1),$$

as long as $|\mathbf{x} - \mathbf{y}| \ll \beta^{5/4}$.

• Hence we get a proof that long-range order persists up to length scales of order $\beta^{5/4}$, although one would actually expect *infinite* long-range order...

UPPER BOUND

$$f(\beta) \le C_0 \left(\frac{1}{2}\right)^{-3/2} \beta^{-5/2} \left(1 + \mathcal{O}(\beta^{-3/8})\right)$$

1) Localization into boxes of side length $\ell \gg \sqrt{eta}$ with Dirichlet b.c.;

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$$\Gamma = \frac{P e^{-\beta \mathcal{H}_0} P}{\mathrm{Tr} P e^{-\beta \mathcal{H}_0}}$$

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SKETCH OF THE PROOF $(S = \frac{1}{2})$

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$$f(\beta) \ge C_0 \left(\frac{1}{2}\right)^{-3/2} \beta^{-5/2} \left(1 + \mathcal{O}(\beta^{-\kappa})\right), \qquad \kappa < \frac{1}{40}$$

- Localization into Neumann boxes;
- ② Sharp lower bound on $H \Longrightarrow$ preliminary lower bound on $f(\beta)$ off the mark by $\log \beta \Longrightarrow$ restriction of the trace to states with small energy;
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By dropping the positive interaction among different subcells, one has

$$f(\beta) \ge f(\beta, \Lambda_{\ell}) = -\frac{1}{\ell^{3}\beta} \log \operatorname{Tr} e^{-\beta H},$$



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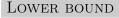
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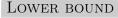
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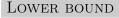
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PROPOSITION (OPERATOR INEQUALITY) In the subspace \mathscr{H}_{S_T} with total spin $S_T(S_T + 1)$

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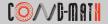
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$$\operatorname{Tr}_{H \le E_0} e^{-\beta H} = \sum_{S_T = S_0}^{\frac{1}{2}\ell^3} (2S_T + 1) \operatorname{Tr}_{S_T^3 = -S_T} e^{-\beta H}$$

• $H \leq E_0 \Longrightarrow S_T \geq S_0$ with $S_0 = \frac{1}{2}\ell^3 - \ell^2 E_0$.

- In the HP representation the number of bosons N is bounded: $N = \frac{1}{2}\ell^3 + S_T^3 = \frac{1}{2}\ell^3 - S_T \le \ell^2 E_0 \le \ell^5 (\log \beta/\beta)^{5/2}$
- For $\ell \gtrsim \beta^{1/2+\varepsilon}$ the number of bosons is small $N \sim \beta^{5\varepsilon}$ and their density is very small $\rho \sim \beta^{2\varepsilon-3/2}$.

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$$Z \leq (\ell^3 + 2) \operatorname{Tr}_{\mathcal{H} \leq E_0} e^{-\beta \mathcal{H}}$$
$$\mathcal{H} = \frac{1}{2} P \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \left[\left(a_{\mathbf{x}}^{\dagger} - a_{\mathbf{y}}^{\dagger} \right) \left(a_{\mathbf{x}} - a_{\mathbf{y}} \right) - \hat{n}_{\mathbf{x}} \hat{n}_{\mathbf{y}} \right] P$$

8 Estimate of the interaction

- Peierls-Bogoliubov inequality $\operatorname{Tr} e^{A+B}/\operatorname{Tr} e^A \leq \exp{\{\operatorname{Tr} (Be^A)/\operatorname{Tr} e^A)\}}.$
- To conclude the proof we thus have to estimate the expectation value

$$\left\langle E \left| \mathcal{K} \left| E \right\rangle = \sum_{\left\langle \mathbf{x}, \mathbf{y}
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angle \subset \Lambda}
ho(\mathbf{x}, \mathbf{y}) \le C \ell^3 \left\| \rho \right\|_{\infty}$$

of the bosonic interaction \mathcal{K} over eigenstates $|E\rangle$ of \mathcal{H} with two-particle density $\rho(\mathbf{x}, \mathbf{y}) = \langle E | a_{\mathbf{x}}^{\dagger} a_{\mathbf{y}}^{\dagger} a_{\mathbf{x}} a_{\mathbf{y}} | E \rangle$.

• The gas is very *dilute*: its density is $\sim \beta^{-3/2}$ and therefore $\langle \mathcal{K} \rangle_{\beta} \sim \beta^{-3/2}$ as well.

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6 Estimate of the interaction

• Since $\|\rho\|_1 \leq CN^2$, for $\ell \gtrsim \beta^{1/2+\varepsilon}$ and $E \leq E_0$, $\langle E|\mathcal{K}|E \rangle \leq C\ell^3 \|\rho\|_{\infty} \leq C\beta^{-3/2}\beta^{\varepsilon}$

• The expectation value of the interaction $\langle E | \mathcal{K} | E \rangle = \mathcal{O}(\beta^{-3/2+\varepsilon'})$ is much smaller that the kinetic term $\langle E | \mathcal{H}_0 | E \rangle = C\ell^{-2} = \mathcal{O}(\beta^{-1})$: $e^{\beta \langle \mathcal{K} \rangle_{\beta}} \le \exp\{\beta^{-1/2+\varepsilon'}\} \le 1 + o(1)$

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$$f(\beta) \geq \frac{1}{\beta \ell^3} \sum_{\mathbf{k} \in \Lambda_{\ell}^*, \mathbf{k} \neq 0} \log \left(1 - e^{-\frac{1}{2}\beta \varepsilon(\mathbf{k})} \right) (1 + o_{\ell,\beta}(1))$$

3 Leading term

- Ignoring the constraint $N \leq \ell^2 E_0$ for all states but $\mathbf{k} = 0$, $\operatorname{Tr}_{N \leq \ell^2 E_0} e^{-\beta \mathcal{H}_0} \leq \left(\ell^2 E_0 + 1\right) \prod_{\mathbf{k} \in \Lambda_{\ell}^*, \mathbf{k} \neq 0} \frac{1}{1 - e^{-\frac{1}{2}\beta \varepsilon(\mathbf{k})}}$
- Riemann sum approximation

$$\frac{1}{\ell^3} \sum \log\left(1 - e^{-\frac{1}{2}\beta\varepsilon(\mathbf{k})}\right) \simeq \frac{1}{(2\pi)^3} \int_{[\pi,\pi]^3} d\mathbf{k} \,\log\left(1 - e^{-\frac{1}{2}\beta\varepsilon(\mathbf{k})}\right)$$

- Dispersion relation $\varepsilon(\mathbf{k}) = 2\sum (1 \cos k_i) \simeq k^2$ for $k \ll 1$.
- Optimization w.r.t. $\ell \sim \beta^{21/40}$

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$$Z \le (\ell^3 + 2) \left(\ell^2 E_0 + 1\right) \prod_{\mathbf{k} \in \Lambda_{\ell}^*, \mathbf{k} \ne 0} \frac{1}{1 - e^{-\frac{1}{2}\beta\varepsilon(\mathbf{k})}}$$
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• Reduce the N-body eigenvalue equation $H\left|\Psi\right\rangle=E\left|\Psi\right\rangle$ to the differential inequality

$$\begin{split} -\Delta\rho(\mathbf{x},\mathbf{y}) &\leq 4E\rho(\mathbf{x},\mathbf{y})\\ \text{where } \tilde{\Delta} \text{ is the discrete Laplacian on } \Lambda_\ell \times \Lambda_\ell \setminus \{(\mathbf{x},\mathbf{x}),\mathbf{x} \in \Lambda_\ell\}, \text{ i.}\\ &-\tilde{\Delta}\rho(\mathbf{x},\mathbf{y}) = (-\Delta_\mathbf{x} - \Delta_\mathbf{y})\,\rho(\mathbf{x},\mathbf{y}) + 2\rho(\mathbf{x},\mathbf{y})\mathbbm{1}_{\{|\mathbf{x}-\mathbf{y}|=1\}}\\ &-\Delta\rho(\mathbf{x}) = \sum_{\mathbf{y} \in \Lambda_\ell, |\mathbf{y}-\mathbf{x}|=1} \left(\rho(\mathbf{x}) - \rho(\mathbf{y})\right) \end{split}$$

• Extend the inequality to the whole of \mathbb{Z}^6 via reflections: $-\Delta\rho(\mathbf{z}) \leq 4E\rho(\mathbf{z}) + 2\rho(\mathbf{z})\chi(\mathbf{z})$

here P_n is the probability of a random walk on \mathbb{Z}^d .

• Pick $n \simeq E^{-1} \gg 1$: $P_n(\mathbf{z}, \mathbf{z}') \propto n^{-3} e^{-3|\mathbf{z}-\mathbf{z}'|/n}$ so that for some c < 1 $\rho(\mathbf{z}) \le (1 + o(1)) \left(CE^3 \|\rho\|_1 + c \|\rho\|_{\infty} \right)$

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 $\,\circ\,$ Reduce the N-body eigenvalue equation $H\,|\Psi\rangle=E\,|\Psi\rangle$ to the differential inequality

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where $\tilde{\Delta}$ is the discrete Laplacian on $\Lambda_{\ell} \times \Lambda_{\ell} \setminus \{(\mathbf{x}, \mathbf{x}), \mathbf{x} \in \Lambda_{\ell}\}$, i.e., $-\tilde{\Delta}\rho(\mathbf{x}, \mathbf{y}) = (-\Delta_{\mathbf{x}} - \Delta_{\mathbf{y}})\rho(\mathbf{x}, \mathbf{y}) + 2\rho(\mathbf{x}, \mathbf{y})\mathbb{1}_{\{|\mathbf{x}-\mathbf{y}|=1\}}$ $-\Delta\rho(\mathbf{x}) = \sum_{\mathbf{y}\in\Lambda_{\ell}, |\mathbf{y}-\mathbf{x}|=1} (\rho(\mathbf{x}) - \rho(\mathbf{y}))$

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• Extend the inequality to the whole of \mathbb{Z}^6 via reflections:

$$\rho(\mathbf{z}) \le (1 - E/3)^{-1} \left(\langle \rho \rangle(\mathbf{z}) + \|\rho\|_{\infty} \chi(\mathbf{z}) \right)$$

where P_n is the probability of a random walk on \mathbb{Z}^d .

• Pick $n \simeq E^{-1} \gg 1$: $P_n(\mathbf{z}, \mathbf{z}') \propto n^{-3} e^{-3|\mathbf{z}-\mathbf{z}'|/n}$ so that for some c < 1 $\rho(\mathbf{z}) \le (1 + o(1)) \left(CE^3 \|\rho\|_1 + c \|\rho\|_{\infty} \right)$

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• Reduce the N-body eigenvalue equation $H\left|\Psi\right>=E\left|\Psi\right>$ to the differential inequality

$$-\tilde{\Delta}\rho(\mathbf{x},\mathbf{y}) \le 4E\rho(\mathbf{x},\mathbf{y})$$

where $\tilde{\Delta}$ is the discrete Laplacian on $\Lambda_{\ell} \times \Lambda_{\ell} \setminus \{(\mathbf{x}, \mathbf{x}), \mathbf{x} \in \Lambda_{\ell}\}$, i.e., $-\tilde{\Delta}\rho(\mathbf{x}, \mathbf{y}) = (-\Delta_{\mathbf{x}} - \Delta_{\mathbf{y}})\rho(\mathbf{x}, \mathbf{y}) + 2\rho(\mathbf{x}, \mathbf{y})\mathbb{1}_{\{|\mathbf{x}-\mathbf{y}|=1\}}$ $-\Delta\rho(\mathbf{x}) = \sum_{\mathbf{y}\in\Lambda_{\ell}, |\mathbf{y}-\mathbf{x}|=1} (\rho(\mathbf{x}) - \rho(\mathbf{y}))$

 $\, \bullet \,$ Extend the inequality to the whole of \mathbb{Z}^6 via reflections:

 $\rho(\mathbf{z}) \le (1 - E/3)^{-n} \left((P_n * \rho)(\mathbf{z}) + \frac{1}{6} \|\rho\|_{\infty} \sum_{j=0}^{n-1} P_j * \chi(\mathbf{z}) \right)$

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M. Correggi (Roma 3) Feb

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(\checkmark) Upper bound for $f(S,\beta)$ in the regime $S \to \infty$ and $\beta = \tilde{\beta}S^{-1}$ to the first order in S^{-1} [with N. BENEDIKTER]:

$$\frac{f(S,\beta)}{S} \le \frac{1}{(2\pi)^3 \tilde{\beta}} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \log\left(1 - e^{-\tilde{\beta}\varepsilon(\mathbf{k})}\right) + \frac{C_1}{S} + \mathcal{O}(1/S^2),$$

where C_1 is computed via the spin-wave approximation.

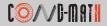
(\checkmark) Upper bound for $f(\beta)$ as $\beta \to \infty$ up to the first non-trivial contribution:

 $f(\beta) \le \beta^{-5/2} \left(C_0 + C_1' \beta^{-1} + C_2' \beta^{-2} + C_3' \beta^{-5/2} + o(\beta^{-5/2}) \right),$ where C_3' is still the spin-wave prediction.

(**X**) 2D?

(X) Spontaneous magnetization and breaking of the rotational symmetry.

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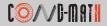
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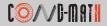
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Thank you for the attention!