# Singular stochastic PDE and Dynamical field theory models

Pronob K. Mitter

Laboratoire Charles Coulomb

CNRS/Université de Montpellier

**Frascati**

June 9, 2015

# Plan of the presentation

# Singular stochastic PDE

SSPDE : Non-linear Langevin equations "whose solutions" are supposed to generate Markov processes with Euclidean field theory measures as invariant measures.
*A priori* ill defined problem.

- EFT measures require renormalization to be defined. They are realized on spaces with distributions $\mathcal{D}'(\mathbb{R}^d)$ or $\mathcal{S}'(\mathbb{R}^d)$.
- The SSPDE = Non-linear Langevin equations having coefficients inherited from EFT measures. These are typically distributions. Nonlinear $\leftrightarrow$ pointwise products $\leftrightarrow$ renormalization.
  If "solutions" exist, Markov process is distribution valued.

How to proceed ? Follow Dirac's advice. Introduce cut-offs, solve the equations and then take limits.

# What sort of solutions ?

Different points of view :

- Probabilistic weak solutions/ Martingale solutions : With cut-offs these are also strong solutions and Martingale solutions exist. This leads to Functional integral/RG point of view to construction of semigroup/invariant measures.

  This is the point of view I will adopt.

# What sort of solutions ?

- Pathwise approaches

  Dynamical solutions of super-renormalizable Euclidean Field theories. Counter-terms : those of super-renormalizable EFT's. (massive $\phi_d^4$ $d = 2, 3$.)

  a) M. Hairer (theory of regularity structures), M. Gubinelli *et. al.* (paracontrolled distributions) : These are theories of multiplication of distributions with counter-terms. Verified in low order perturbation theory. The remainder is controlled as a fixed point problem in a Banach space.

# What sort of solutions ?

- b) Antti Kupiainen : RG approach directly on the equation

  UV cut-off noise, finite volume. Rescale to unit cut-off with enlarged volume. This gives rise to a sequence of effective equations with rescaling at each step $\rightarrow$ Perturbative part + remainder. The limit of the sequence of remainders has been proved to exist by solving a Banach fixed point problem.

  These are all : **short time solutions with the upper bound on time dependent on noise**.

Long time solution, $D = 2$ by J.C. Mourrat and H.Weber.

BUT : Hairer gives optimal regularity for paths, and intial conditions.

# An example

Scalar field theory in dimension $D$, $\phi(x)$, $x \in \mathbb{R}^D$.

$C_0(x-y) =$ Fourier transform of $\hat{C}_0(k) = \frac{1}{k^2+m^2}$.
$\kappa$ : UV cut-off

$$\hat{C}_\kappa(k) = \frac{1}{(k^2+m^2)(1+\frac{k^2}{\kappa^2})^p} \text{ for suffieciently large p.}$$

The random gaussian field $\phi$ in $\mathbb{R}^d$ with covariance $C_0$ is a distributon for $D \geq 2$. For $p$ sufficiently large, $\phi$ distributed according to $C_x$ is locally sufficiently differentiable.

$$\mathrm{d}\mu_\kappa(\phi) = \frac{1}{Z_\kappa(\Lambda)} \int \mathrm{d}\mu_{C_\kappa}(\phi)\, e^{-V_\kappa(\phi,\Lambda)}.$$

$$V_\kappa(\phi,\Lambda) = \int_\Lambda \mathrm{d}^D x \, \{\lambda : \phi^4 :_{C_\kappa} (x) + \mathrm{counterterms}_\kappa\}$$

$\Lambda_L$ : cube side, periodic b.c., $\Lambda_L = \mathbb{R}^D/(L\mathbb{Z}^D)$. $C_\kappa$ : periodized covariance

# Dynamical models

## Nonlinear Langevin equations

Large class of equations available, such that if solutions exist, they have the same invariant measure.

Example : Let $0 < \rho \leq 1$.

$$
\begin{aligned}
\mathrm{d}\tilde{\phi}_t &= \mathrm{d}W_t - \frac{1}{2}\left(C_x^{-\rho}\tilde{\phi}_t + \lambda C^{1-\rho} : \tilde{\phi}_t^3 :_{C_\kappa}\right)\mathrm{d}t, \\
\tilde{\phi}_0 &= \phi,
\end{aligned}
$$

values in subspace of $\mathcal{D}'(\Lambda_L)$, (sufficiently differentiable functions). $f, g$ are test functions in $\mathcal{D}(\Lambda_L)$.

$$
E(W_t(f), W_t(g)) = (f, C^{1-\rho}g)\min(t, s).
$$

[Canonical choice $\rho = 1$.] Also counter-terms in the drift are omitted.

# Dynamical models

If the solutions exist, then there is a generator $L_{\kappa,p}$ and it is easy to see

$$L_{\kappa,p} = \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, C^{1-\rho}(x-y) \left( \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} - \frac{\delta \mathcal{S}}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} \right).$$

is symmetric with respect to $L^2(\mathrm{d}\mu_{\kappa,\Lambda})$ and formally,

$$\int \mathrm{d}\mu_{\kappa,\Lambda}(\phi) \, e^{tL_{\kappa,p}} \, F(\phi) = \int \mathrm{d}\mu_{\kappa,\Lambda}(\phi) \, F(\phi).$$

$F$ : bounded $\mathcal{C}^2$ cylindrical function.

Each choice of $\rho \in (0,1]$ will, if equation can be solved, lead to Markov processes with same invariant measure $\mu_{\kappa,\Lambda}$.

# Linear processes

$$\mathrm{d}\phi_t = \mathrm{d}W_t - \frac{1}{2}\, C_\kappa^{-\rho}\phi_t \mathrm{d}t,$$
$$\phi_t = \phi$$

This is an Ornstein-Uhlenbeck process/Langevin equation.

This has a unique solution

$$\phi_t = e^{-t/2}\phi_0 + \int_0^t e^{-\frac{1}{2}\,(t-s)C_\kappa^{-\rho}}\,\mathrm{d}W_\kappa.$$

The O-U process has continuous sample paths.
Generator :

$$L_\kappa^{(0)} = \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}y \left[ C_\kappa^{1-\rho}(x-y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} - C_\kappa^{-\rho}(x-y)\phi(x)\frac{\delta}{\delta\phi(y)} \right].$$

# Linear processes

Transition probability : $p_t(\phi, \mathcal{B}) = \mu_{C_{t,\kappa}}\left(B - e^{-\frac{t}{2}C_\kappa^{-\rho}}\phi\right)$,
$\mathcal{B}$ : Borel set in $\mathcal{D}'(\Lambda_L)$.

$$C_{t,\kappa} = (1 - e^{-t\,C_\kappa^{-\rho}})C_\kappa.$$

$\mu_{C_\kappa}$ : invariant measure.

$P_\phi^{\mathrm{OU}}$ : O-U measure on path space : $\mathcal{C}^0([0,\infty), \mathcal{D}'(\Lambda_L))$.

In terms of the linear process, the full process must solve the integral equation :

$$\tilde{\phi}_t = \phi_t - \frac{\lambda}{2}\int_0^t \mathrm{d}s e^{-(t-s)C_\kappa^{-\rho}} C_\kappa^{1-\rho} :\tilde{\phi}_s^3:_{C_\kappa}.$$

# Probabilistic weak solution

Girsanov formula :

$$e^{t\,L_\kappa}F(\phi)=E^{(w)}_{\phi_0=\phi}\left(F(\phi_t)\;e^{\xi_0^{t,\kappa}}\right)\qquad(*)$$

O-U process : $\phi_t$ is a measurable function of $w_t$.

$$\xi^{0,(\kappa)}=-\frac{\lambda}{2}\int_0^t\left(:\phi_s^3:_{C_\kappa},\mathrm{d}w_s\right)-\frac{\lambda^3}{8}\int_0^t\mathrm{d}s\left(:\phi_s^3:_{C_\kappa},C^{1-\rho}:\phi_s^3:_{C_\kappa}\right).$$

Because of the cut-off $\kappa$ Ito's formula applied to $e^{\xi_0^{t,\kappa}}$ shows

$$E^{(w)}(e^{\xi_0^{t,\kappa}})=1.\qquad(**)$$

Martingale property :

$$E^{(w)}(e^{\xi_s^{t,\kappa}}|\mathcal{F}_s)=1,$$
$$E^{(w)}(e^{\xi_0^{t,\kappa}}|\mathcal{F}_s)=e^{\xi_0^{s,\kappa}}.$$

# Probabilistic weak solution

Because of $(**)$ in the previous slide, the Gaussian representation of $(*)$ defines a semigroup with transition probabilities,

$$\hat{p}_t(\phi, \mathcal{B}) = E_\phi^{(w)} \left( \mathcal{X}\left(\phi_t \in \mathcal{B}\right) \, e^{\xi_0^{t,\kappa}} \right),$$

satisfies Chapman-Kolmogoroff equation (semigroup property).
$\implies$ new probability measure $\hat{P}_\phi$ on $C_0([0, T]), \mathcal{D}'(\Lambda) = \Omega_T$.
$\hat{P}_\phi$ solves the non-linear integral equation in weak sense.

$$Z_t = \hat{\phi}_t + \frac{\lambda}{2} \int_0^t \mathrm{d}s \, e^{-(t-s)\, C_\kappa^{-\rho}} C_\kappa^{1-\rho} \, : \hat{\phi}_s^3 :_{C_\kappa}, \quad \hat{\phi}_t \in \Omega_T$$

Under $\hat{P}_\phi$, $Z_t$ has probability distribution of an O-U process.

The drift is a gradient vector field.

Therefore we can get rid of the stochastic integral in the Girasanov exponent $\xi_0^t$, using Ito's formula.

The result is :

$$\xi_t = -\frac{1}{2}\, V_\kappa(\phi_t) + \frac{1}{2}\, V_\kappa(\phi_0) - \int_0^t \mathrm{d}s\, \tilde{V}_\kappa(\phi_s),$$

$$\tilde{V}_\kappa(\phi_s) = \frac{\lambda}{4} : \left(\phi_s^3, C_\kappa^{-\rho}\, \phi_s\right)_{L^2(\Lambda)} :_{C_\kappa} + \frac{\lambda^2}{8}\left(: \phi_s^3 :_{C_\kappa}, C_\kappa^{1-\rho} : \phi_s^3 :_{C_\kappa}\right)_{L^2}$$
$$-\frac{3\lambda}{4} : \left(\phi_s, C_\kappa^{1-\rho}\phi_s\right)_{L^2(\Lambda)},$$

$$\left(e^{t\, L_\kappa}\right)(\phi) = E_{\phi_0=\phi}\left(F(\phi_t)\, e^{-\frac{1}{2}\, V_\kappa(\phi_t) + \frac{1}{2}\, V_\kappa(\phi_0) - \int_0^t \mathrm{d}s\, \tilde{V}_\kappa(\phi_s)}\right).$$

$$\Big(G, e^{t\,L_\kappa} F\Big)_{L^2\,(\mathrm{d}\mu_\kappa,\lambda,\Lambda_L)}$$
$$= \int \mathrm{d}\mu_{\kappa,\frac{\lambda}{2},\Lambda_L}(\phi)\; G(\phi) E_{\phi_0=\phi}\;\Big(F(\phi_t) e^{-\frac{1}{2}\,V_\kappa(\phi_t) - \int_0^t \mathrm{d}s\,\tilde{V}_\kappa(\phi_s)}\Big)$$

**ULTRAVIOLET CUT-OFF REMOVAL** $(\kappa \to \infty)$ **.**

# $D = 2$ The weak solution of Jona-Lasinio, Mitter in the limit $\kappa \to \infty$(1985)

Here the parameter $\rho : 0 \leq \rho \leq 1$ plays an important role. In other words we are studying a modified non-linear Langevin (Glauber) dynamics. The initial choice of $\rho$ is very restricted for technical reasons : $0 < \rho < \frac{1}{10}$. Progressively, this restriction was removed by Rozovskii and Mikulevicius (1998) : $0 < \rho < 1$ and then $\rho = 1$. A strong solution was given by G. da Prato and Debussche (2003) for $\rho = 1$ in a very important work which introduced new technology (use of Besov spaces). Finally (2015) J-C Mourrat and H.Weber have extended da Prato's work to prove global in time ( and also in space) pathwise solutions.

# Estimates [J-L,M (CMP-1985), M(Spain-1986)], $\kappa \to \infty$

1) $D = 2 \quad C_\kappa \xrightarrow{\kappa\to\infty} C$ (for massive covariance).

$: \phi^n :_C (f) \in L^p(\mathrm{d}\mu_C), \ \forall p : 1 \leq p \leq \infty.$

2) The O-U transition probability $\mu_{C_t}(\phi, \cdot)$ is absolutely continuous with respect to $\mu_C(\cdot)$, $\mu_C$ a.e. in $\phi$. The Radon-Nikodym derivarive is in $L^2(\mathrm{d}\mu_C)$, $\mu_C$ a.e. in $\phi$.

If $h \in L^{2p}(\mathrm{d}\mu_C)$, the trivially,

$$E_\phi(|h(\phi_t)|^p) < \infty, \ \ 1 \leq p < \infty.$$

So $h(\phi_t) \in L^p(\mathrm{d}P_\phi, \Omega)$.

3) $\phi_t$ is continuous in $t$, $P_\phi$ a.s., $\mu_C$ a.e. in $\phi$.

Moreover the O-U semigroup is *hypercontractive*.

Using Riemann sum approximation, with $\kappa \to \infty$

$$\int_0^t \mathrm{d}s\, h(\phi_s) \in L^p(\mathrm{d}P_\phi, \Omega),\ \mu_C \text{ a.e. in } \phi.$$

Therefore provided $0 < \rho < 1$

$$E_\phi \left( \int_0^t \mathrm{d}s \left( :\phi^3:_C, C^{1-\rho} :\phi^3:_C \right)_{L^2} \right) < \infty,$$

$\mu_C$ a.e. in $\phi$.

## Ito isometry

$$E_\phi \left[ \left( \int_0^t (:\phi_s^3:, \mathrm{d}W_s) \right)^2 \right] = E_\phi \left( \int_0^t \mathrm{d}s (:\phi^3:_C, C^{1-\rho} :\phi^3:_C)_{L^2} \right) < \infty,$$

$\mu_C$ a.e. in $\phi$.

Because of the above, $\xi_0^{t,\infty}$ exists as a random variable.

4) We know

$$E_\phi(e^{\xi_0^{t,\kappa}}) = 1, \qquad \mu_C \ \text{a.e.}$$

By Fatou's lemma

$$E_\phi(e^{\xi_0^{t,\infty}}) \leq 1, \qquad \mu_C \ \text{a.e.}$$

For a martingale/weak solution we have to prove

$$E_\phi\left(e^{\xi_0^{t,\infty}}\right) \stackrel{?}{=} 1.$$

5)

$$e^{\xi_0^{t,\infty}} = e^{\xi_0^{t,\infty}} - e^{\xi_0^{t,\kappa}} + e^{\xi_0^{t,\kappa}}.$$

$$|E_\phi(e^{\xi_0^{t,\infty}}) - 1| \leq |E\left(e^{\xi_0^{t,\infty}} - e^{\xi_0^{t,\kappa}}\right)| \qquad (***)$$

$$|E\left(e^{\xi_0^{t,\infty}} - e^{\xi_0^{t,\kappa}}\right)| \leq E_\phi\left(|\xi_0^{t,\infty} - \xi_0^{t,\kappa}||\left(e^{\xi_0^{t,\infty}} + e^{\xi_0^{t,\kappa}}\right)|\right)$$
$$\leq \left(E_\phi(|\xi_0^{t,\infty} - \xi_0^{t,\kappa}|^2\right)^{\frac{1}{2}} \left[\left(E_\phi\left(e^{2\xi_0^{t,\infty}}\right)\right)^{\frac{1}{2}} + \left(E_\phi\left(e^{2\xi_0^{t,\kappa}}\right)\right)^{\frac{1}{2}}\right]$$

It is easy to show $|\xi_0^{t,\infty} - \xi_0^{t,\kappa}| \to 0$ in $L^2\left(\mathrm{d}P_\phi^W, \Omega\right)$, $\mu_C$ a.e. in $\phi$.

6) Lemma

Suppose $0 < \rho < \frac{1}{10}$. Then

$$E_\phi\left(e^{2\xi_0^{t,\infty}}\right) < \infty, \qquad \mu_C \text{ a.e. in } \phi.$$

$e^{2\xi_0^{t,\infty}}$ is uniformly bounded in $L^2(\mathrm{d}P_\phi^W, \Omega)$.

7) Therefore taking $\kappa \to \infty$ in $(***)$ we have

$$E_\phi\left(e^{\xi_0^{t,\infty}}\right) = 1.$$

# Proof of the lemma

The idea is to undo the stochastic integral in $\xi_0^{t,\infty}$ since the drift perturbation is a gradient. To undo the stochastic integral we use the Ito's formula. Then we see that each term exists as a random variable in $L^2(\mathrm{d}P_\phi^W, \Omega)$, $\mu_C$ a.e. in $\phi$ provided $0 < \rho < \frac{1}{2}$.

Then we use the method of Nelson and Glimm from the earliest days of constructive QFT. For the stability estimate to work we need to restrict : $0 < \rho < \frac{1}{10}$.

$$\int \mathrm{d}\mu(\phi)\, E_\phi\,(e^{2\xi_0^{t,\infty}}) = \int \mathrm{d}\mu_C(\phi)\, E_\phi\,\left(e^{-V(\phi_t)-2\int_0^t \mathrm{d}s\tilde{V}(\phi_s)}\right)$$

Where

$$\begin{aligned} 2\tilde{V}(\phi_s) &= \frac{\lambda}{2} : \left(\phi_s^3, C^{-\rho}\phi_s\right)_{L^2(\Lambda)} :_C + \frac{\lambda^2}{4}\left(: \phi_s^3 :_C, C^{1-\rho} : \phi_s^3 :_C\right)_{L^2} \\ &\quad - \frac{3\lambda}{2} : (\phi_s, C^{1-\rho}\phi_s) :_C \\ &\leq \left(\int \mathrm{d}\mu_C(\phi) E_\phi(e^{-2V(\phi_t)})\right)^{\frac{1}{2}} \left(\int \mathrm{d}\mu_C(\phi) E_\phi(e^{-4\int_0^t \mathrm{d}s\, \tilde{V}(\phi_s)})\right)^{\frac{1}{2}} \\ &\qquad (****) \end{aligned}$$

1) The first factor in $(* * **)$ is easily proven to be finite.

$$\int \mathrm{d}\mu_C(\phi) E_\phi(e^{-2V(\phi_t)}) = \int \mathrm{d}\mu_C(\phi)\, e^{t\,L_0}\, e^{-2V(\phi)}$$
$$= \int \mathrm{d}\mu_C(\phi) e^{-2V(\phi)} < \infty,$$

by Nelson's estimate ( have used that $\mu_C$ is invariant measure of O-U process).

2)

$$\int \mathrm{d}\mu_C(\phi) E_\phi^{(w)}(e^{-4\int_0^t \mathrm{d}s\, \tilde{V}(\phi_s)}) \leq \int \mathrm{d}\mu_C(\phi) e^{-4t\, \tilde{V}(\phi)}.$$

($\phi_s$ a.s. continuous, Riemann sum approximation, Hölder's inequality)

$\tilde{V}(\phi)$ : The non-local $\phi^6$ term is a positive random variable, can be dropped.

The negative sign mas term is dominated by the Gaussian measure, for small $\lambda$.

We are left with the estimate provided by the following proposition.

# Proposition

For $0 < \rho < \frac{1}{10}$

$$\int \mathrm{d}\mu_C(\phi) e^{-pt\lambda G(\phi)} < \infty$$

$$G(\phi) =: (\phi^3, C^{-\rho}\phi)_{L^2} :_C, \qquad \text{in } L^p(\mathrm{d}\mu_C) \text{ for } 0 < \rho < \frac{1}{2}.$$

- Step 1

  $(\phi^3, C^{-\rho}\phi)_{L^2(\Lambda)} \geq \int_\Lambda \mathrm{d}^2 x \phi^4(x), \quad 0 < \rho < 1.$

  Proved using spectral representation and Young's inequality.

- Step 2
  UV cut-off field (cut-off Fourier modes)

$$\phi_\kappa(x) = \int_{|k|\le\kappa} \frac{\mathrm{d}^2 k}{(2\pi)^2}\, e^{\mathrm{i}k\cdot x}\hat{\phi}(k).$$

  $C_\kappa(x,y)$ is $\mu_C$ covariance of $\phi_\kappa$.
  $G_\kappa(\phi) = G(\phi_\kappa)$.

  $G_\kappa \to G$ in $L^p(\mathrm{d}\mu_C)$ for $0 < \rho < \frac{1}{2}$.

  Undo Wick ordering in $G_\kappa$. The Wick constants $\to \infty$ when $\kappa \to \infty$.

Using Step 1 and and estimates of Wick constants get

$$G_\kappa \ge -\mathrm{const}\; \times \left(\kappa^{4\rho}\,(\ln\kappa)^2\right)$$

## Step 3

Define $\tilde{G}_\kappa = G - G_\kappa$.

Then :

$$\int \mathrm{d}\mu_C |\tilde{G}_\kappa|^{2j} \le (j!)^4 \, b^j \left( (\ln \kappa)^m \, \kappa^{-2+4\rho} \right)^j \qquad \forall j, \text{ some } m > 0.$$

Hypercontractivity to reduce $L^p(\mathrm{d}\mu_C)$ estimates to $L^2(\mathrm{d}\mu_C)$ estimates, then Feynman graph computation.

## Step 4

$$\mu_C\left(\{G \leq -\mathrm{const}\,(\kappa^{4\rho}(\ln\kappa)^2) - 1\}\right) \leq \mu_C\left(\{|\tilde{G}_\kappa|^{2j} \geq 1\}\right)$$
$$\leq \int \mathrm{d}\mu_C\,|\tilde{G}_\kappa|^{2j} \leq (j!)^4 b^j (\ln\kappa)^m (\kappa^{-2+4\rho})^j$$

Then Stirling's approximation and optimal $\kappa$-dependent choice of $j$ gives

$$\mu_C\left(\{G \leq -\mathrm{const}\,(\kappa^{4\rho}(\ln\kappa)^2) - 1\}\right)$$
$$\leq e^{-\mathrm{const}\left(\kappa^{\frac{2-4\rho}{4}}(\ln\kappa)^{-\frac{m}{4}}\right)}$$

## Step 5

$$\kappa_n = 2^n \to \infty,\ C_{\kappa_n} = \mathrm{const}\, \kappa_n^{4\rho} (\ln \kappa_n)^2.$$

$$\begin{aligned} \int \mathrm{d}\mu_C e^{-G} &= \int \mathrm{d}\mu_C e^{-G} \left( \mathcal{X}\{G > -C_{\kappa_0} - 1\} + \mathcal{X}\{G \le -C_{\kappa_0} - 1\} \right) \\ &\le \quad e^{C_{\kappa_0}+1} + \mathcal{I}_0 \end{aligned}$$

## Step 5 continued

$$
\begin{aligned}
\mathcal{I}_0 &= \int \mathrm{d}\mu_C e^{-G} \mathcal{X}\{G \leq -C_{\kappa_0} - 1\} \\
&= \int \mathrm{d}\mu_C e^{-G} \sum_{n=0}^{\infty} \mathcal{X}\{-C_{\kappa_{n+1}} - 1 < G \leq -C_{\kappa_n} - 1\} \\
&\leq \sum_{n=0}^{\infty} e^{C_{\kappa_{n+1}}+1} \mu_C\{G \leq -C_{\kappa_n} - 1\} \\
&\leq \sum_{n=0}^{\infty} e^{C_{\kappa_{n+1}}+1} e^{-\mathrm{const}\left(\kappa^{\frac{2-4\rho}{4}} (\ln \kappa)^{-\frac{m}{4}}\right)},
\end{aligned}
$$

and using the definition of $C_{\kappa_n}$, this series converges for $\rho < \frac{1}{10}$.

□

# Stationary O-U process

$\tilde{P}$ : measure on path space $\Omega = \mathcal{C}^0([0,\infty), \mathcal{D}')$.

$$\tilde{P}(B) = \int \mathrm{d}\mu_{C_\kappa}(\phi)\ \tilde{P}_\phi(B), \qquad \tilde{P}_\phi : \text{O-U measure.}$$

- Stationarity : $E^{\tilde{P}}(\phi_t(f)\,\phi_s(g)) = E^{\tilde{P}}(\phi_{t-s}(f)\phi_0(g)), \quad t > s.$
- Symmetry : $E^{\tilde{P}}(\phi_t(f)\,\phi_s(g)) = E^{\tilde{P}}(\phi_t(g)\,\phi_s(f)).$
- Covariance : $E^{\tilde{P}}(\phi_t(f)\,\phi_s(g)) = C_\kappa(f, e^{-\frac{t-s}{2}C_\kappa^{-\rho}} g).$

Because of stationarity : path space $\Omega \to \tilde{\Omega} = \mathcal{C}^0((-\infty,\infty), \mathcal{D}')$
All of this is true when $\kappa \to \infty$.

# Side remark

A Euclidean formalism for the canonical choice $\rho = 1$.

$x = (x_1, ..., x_D) \in \mathbb{R}^D \quad x_0 \in \mathbb{R}$,

$\tilde{x} = (x_0, x) \in \mathbb{R} \times \Lambda_L \subset \mathbb{R}^{D+1}$,

$\phi(\tilde{x}) = \phi(x_0, x)$, $\quad x_0 =$ "time coordinate" $=$ "Langevin time etc.

Define :

$$\tilde{C}_\kappa(\tilde{x}, \tilde{y}) = \int \frac{\mathrm{d}k_0}{2\pi} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{e^{\mathrm{i}k_0(x_0-y_0)+\mathrm{i}(k,x-y)_{\mathbb{R}^D}}}{k_0^2 + \hat{C}^{-2}(k)}.$$

$\mu_{\tilde{C}_\kappa}$ : Gaussian measure, covariance $\tilde{C}_\kappa$ on $\mathcal{D}'(\mathbb{R} \times \Lambda_L)$.

# Side remark

$\mathcal{D}_+(\mathbb{R} \times \Lambda_L)$ : positive time subspace.

$$f_t(\tilde{x}) = f(x)\delta(x_0 - t),\ t > 0,\ \in\ \mathcal{D}_+(\mathbb{R} \times \Lambda_L).$$

Take $t > s$. Then an easy computation shows

$$\int \mathrm{d}\mu_{\tilde{C}_\kappa}\, \phi(f_t)\phi(f_s) \,=\, \left(f, C_\kappa e^{-|t-s|C_\kappa^{-1}} g\right)_{L^2}$$
$$= E^{\tilde{P}_{\mathrm{OU}}}(\phi_t(f)\phi_s(g)).$$

Thus the analogy is

$\tilde{C}_\kappa$ : Euclidean covariance.

$C_\kappa$ : Fock space covariance.

"Time" reflection $\theta$ :

$$f_t(\tilde{x}) = f(x)\,\delta(x_0 - t) \quad \in \mathcal{D}_+(\mathbb{R} \times \Lambda_L),$$
$$(\theta f_t)(\tilde{x}) = f(x)\delta(x_0 + t).$$

Reflection positivity :

$$\langle \phi(f_t), \phi(f_t) \rangle \;=\; \int \mathrm{d}\mu_{\tilde{C}}\;(\theta\phi)(f_t)\phi(f_t) \;=\; \int \mathrm{d}\mu_{\tilde{C}}\;\phi(\theta f_t)\phi(f_t)$$
$$= E^{\tilde{P}}_{\mathrm{OU}}\left(\phi_t(f)\,\phi_{-t}(f)\right) \;\geq\; 0.$$

You can apply the O-S construction of Hilbert space, semigroup etc. This could be the beginning of an Euclidean formalism for the non linear process.

# References


- Catellier, Rémi and Chouk, Khalil, Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equation. arxiv : 1310.6869, Oct. 2013
- Da Prato, Giuseppe and Debussche, Arnaud, Strong solutions to the stochastic quantization equations, Ann. Prob. 31(4) : 1900-1916 , 2003.
- Gubinelli, M. and Imkeller, P. and Perkowski, N., Paracontrolled distributions and singular PDEs, arxiv : 1210.2684, *math-ph* , Oct. 2012
- Hairer, M., A theory of regularity structures, Invent. Math.,198(2) : 269-504, 2014

# References


- Hairer M., Solving the KPZ equation , Annals of Mathematics, 178(2) : 559-664, June 2013.
- Hairer, Martin and Weber, Hendrik, Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions, arXiv:1404.5863.
- Hairer M. and Shen H., The dynamical sine-gordon model, arxiv : 1409.5724 *math-ph*, September 2014.
- Jona-Lasinio, G. and Mitter, P. K., On the stochastic quantization of field theory, *Comm. Math.Phys.*, 101(3) : 409-436, 1985.

# References


- Jona-Lasinio, G. and Mitter, P. K., Large deviation estimates in the stochastic quantization of $\phi_2^4$, *Comm. Math.Phys.*, 130(1) : 111-121, 1990.
- Kupiainen Antti, Renormalization Group and Stochastic PDE's, arxiv : 1410.3094, October 2014.
- Mikulevicius, R. and Rozovskii, B. L., Martingale problems for stochastic PDE's. Stochastic partial differential equations: six perspectives, Volume 64 of *Math Surveys Mongr.*, pages 243-325. Amer. Math. Society, Providence, RI, 1999.