Results and Conjectures about the XY Model

Dedicated to Pierluigi

Tom Spencer
Institute for Advanced Study
Princeton, NJ

June 9, 2015

Definition of X-Y model

$$s_j = (cos(\theta_j), sin(\theta_j)), \quad j \in \Lambda \subset \mathbb{Z}^d, \quad |j - j'| = 1$$

$$Z_\Lambda(\beta, \varepsilon) = \int e^{\beta\{\sum_{j\sim j'} cos(\theta_j - \theta_{j'}) + \varepsilon \sum_j cos(\theta_j)\}} \prod_\Lambda d\theta_j$$

$$< \cos(\theta_0 - \theta_x) >_\Lambda (\beta, \varepsilon) =$$

$$Z_\Lambda^{-1} \int cos(\theta_0 - \theta_x) e^{\beta\{\sum_{j\sim j'} cos(\theta_j - \theta_{j'}) + \varepsilon \sum_j cos(\theta_j)\}} \prod_\Lambda d\theta_j$$

Lim as $\Lambda \uparrow \mathbb{Z}^d$ well defined. $M(\beta) \equiv lim_{\varepsilon \downarrow 0} \langle cos(\theta_0) \rangle (\beta, \varepsilon)$.

Some 2D Results

- $M(\beta) = 0$ all β. There is a *unique* translation invariant state (Mer-Wag, Br-Fo-La, Mes-Mes-Pf)
- For $\beta \leq .88$ exponential decay of correlations. (Aiz-Si)
- For $\varepsilon > 0$, exponential decay of correlations at rate $\propto \varepsilon$. (Lee-Yang, Penrose, Lebowitz, Gu-Ro-Si, Fröhlich)
- $0 \leq \langle \cos(\theta_0 - \theta_x) \rangle(\beta) \leq C|x|^{-(2\pi\beta)^{-1}}$ (Mc-Sp)
- For $\beta \gg 1$, $\langle \cos(\theta_0 - \theta_x) \rangle(\beta) \geq C|x|^{-(2\pi\bar{\beta})^{-1}}$, $\bar{\beta} \approx \beta$ (Kosterlitz-Thouless, Fr-Sp)

X-Y Results in 3D

- If $\beta < .2 \Rightarrow$ exponential decay of correlations
- By Infrared bounds we have Long Range order (Fr-Si-Sp)

$$\langle \cos(\theta_0 - \theta_x) \rangle(\beta) \geq [1 - 2G(0,0)/\beta], \ \ G = (-\Delta)^{-1}$$

There are alternative proofs using duality (Guth, Fr-Sp), and by (Kennedy-King) using Gauge theory coupled to XY.

- There is an asymptotic expansion in powers of $1/\beta$ for the magnetization $M(\beta) = \lim_{\varepsilon \downarrow 0} \langle \cos(\theta_0) \rangle(\beta, \varepsilon)$ (Br-Fo-Le-Li-Sp)

Gaussian or Spin Wave Approximation

For large β, the Gaussian approximation:

$$\beta \sum_j \cos(\nabla\phi_j) + \varepsilon \sum_j \cos(\phi_j) \approx \beta \sum_j (\nabla\phi_j)^2/2 + \varepsilon \sum_j \phi_j^2/2$$

$$\langle cos(\phi_x - \phi_0)\rangle(\beta,\varepsilon) = \langle e^{(i(\phi_x-\phi_0))}\rangle(\beta,\varepsilon) = e^{-[G(0,0)-G(0,x)]/\beta}$$

$G(0,x) = (-\Delta + \varepsilon/\beta)^{-1}(0,x)$. When $\varepsilon \downarrow 0$

$$G(0,0) - G(0,x) = (2\pi\beta)^{-1} \ln|x| \quad 2D\,.$$

$$G(0,0) < \infty \ and \ G(0,x) \approx 1/|x| \quad 3D.$$

Conjectures for XY Model - Justify SW

For $\varepsilon > 0$,

$$0 \leq \langle \cos \phi_0 \, cos\phi_x \rangle - \langle \cos \phi_0 \rangle \langle cos\phi_x \rangle \leq C_\varepsilon e^{-m|x|}, \;\; m = \sqrt{\varepsilon/\beta} \; ?$$

In 3D for $\varepsilon \downarrow 0$

$$0 \leq \langle \cos \phi_0 \, cos\phi_x \rangle - \langle \cos \phi_0 \rangle \langle cos\phi_x \rangle \leq Const 1/|x|^2 \; ?$$

Holds in the Gaussian approximation

Main Problem with the Gaussian Approximation:

Ignores vortices and No Phase Transition.

The Villain X-Y model

$$e^{\beta cos(\theta_j - \theta_{j'})} \approx \sum_{m_{j,j'} \in \mathbb{Z}} e^{-\beta/2(\theta_j - \theta_{j'} - 2\pi m_{j,j'})^2}$$

$$= const \sum_{n_{j,j'} \in \mathbb{Z}} e^{-n^2_{j,j'}/2\beta} e^{i\, n_{j,j'}(\theta_j - \theta_{j'})}$$

After explicit integration over θ obtain dual representation:

$$Z_\Lambda(\beta) = \sum_{div\; n = 0} e^{-\sum n^2_{j,j'}/2\beta}$$

In 2D, since $div\, n = 0$, $n = \nabla \times \phi$, $\phi \in \mathbb{Z}$ hence

$$Z_\Lambda(\beta) = \sum_\phi e^{-\sum_j (\nabla\phi(j))^2/2\beta} .$$

By Poisson summation (with a =1)

$$Z_\Lambda(\beta) = \int e^{-\sum_j (\nabla\phi(j))^2/2\beta} \times \prod_j [1 + 2a \sum_{q \in Z} \cos(2\pi q \phi(j))]\, d\phi(j)$$

If $a = 0$, the spin-spin correlation is $|x|^{-(2\pi\beta)^{-1}}$.

Is a irrelevant ? Yes, if $2\pi\beta_{eff}(\beta, a) > 4$

Vortices bind to form Dipoles for large β

$$e^{\pm 2\pi i\, q_j} \;=\; \text{vortex at j of charge} \pm q$$

Non-neutral charge configurations have 0 probability

For a dipole at 0 and x

$$\langle e^{2\pi i q(\phi_0 - \phi_x)} \rangle(\beta) = e^{-2\pi\beta q^2 \ln(|x|)} \le |x|^{-2\pi\beta}$$

Note that $q = \pm 1$ are the dominant configurations.

The dipoles give rise to $\beta_{eff}(\beta, a) < \beta$.

However, $2\pi\beta_{eff} \geq 4$ for the dipoles to be irrelevant.

Remarks: The dual spin-spin is nonlocal

$$e^{1/\beta \sum_0^x (\partial_2 \phi(j) - 1/2)}$$

Sine Gordon representation of Coulomb Gas:

$$e^{-\sum_j (\nabla \phi(j))^2 / 2\beta + a \sum_j cos(\phi(j))}$$

$$\beta_{c,eff} = 8\pi$$

Remarks on Dipole gas

Consider the action:

$$\sum_j (\nabla\phi(j))^2/2\beta - \sum a_{jk} cos(\phi_j - \phi_k)$$

Note that if $|a_{jk}| \ll |j-k|^{-4-\delta}$ then the action is convex:

Hessian $\approx -\Delta/\bar{\beta}$ since $\sum_j a_{jk}|j-k|^2 < \infty$

Hence one can get apply Brascamp-Lieb or its stronger version Helffer-Sjöstrand to see that it has free field behavior.

3D is simpler: For large β, a single renormalization group step converts the Villain action to a **convex** action. Vortices are replaced by vortex loops. (Fr-Sp).

Falco's Theorem:

For a small, dipoles remain irrelevant for $2\pi\beta_{c,eff}(\beta, a) = 4$ with explicit log corrections.

Falco's result is also *expected* to hold for $a = 1$ and if so it implies that at the KT edge (β_c)

$$\langle \cos(\theta_0 - \theta_x) \rangle_{\beta_c} \approx \ln^{1/8}(|x|)\, |x|^{-1/4}$$

Earlier Renormalization group analysis Dimock and Hurd but required $2\pi\beta > 4$

Universality of Mean Field Theory for $D \geq 3$?

Consider **O(n) invariant** interacting spins:

eg. X-Y model O(2), Heisenberg O(3)

Let $s_j \in \mathbb{R}^n$, $j \in \Lambda_L \cap Z^d$, periodic box of side L, $h \in R^n$

Conjecture:

$$\langle e^{\sum_{j\in\Lambda} h\cdot s_j/|\Lambda|} \rangle(\beta) \ as \ L \Rightarrow \infty$$

$$= \int_{S^{n-1}} e^{h\cdot S_0 M(\beta)} d\mu(S_0) \times [1 + O(\frac{1}{\beta L^{d-2}})]$$

$M(\beta) =$ Magnetization, $d\mu(S_0)$ is uniform measure on S^{n-1} .

The leading term is like a **Law of Large Numbers** and the correction is **CLT.**

Theorem (J. Fröhlich and T.S.) Holds for **O(2) symmetric systems**

Main ideas: Pure states are parametrized by S^1, Fröhlich-Pfister, + IR bounds.

Explanation of Conjectured Universality of Wigner-Dyson statistics in 3D:

Apply SUSY statistical mechanics, (Kravstov and Mirlin (1994))

$$s_j \in U(1,1|2)/U(1|1) \times U(1|1), \quad M(\beta) \approx \rho(E),\ DOS$$

Conclusion: Wigner Dyson should describe RBM and Random Schrödinger in 3 Dim in the extended region

T. Shcherbina: Example with SU(2) symmetry:

Let H be an $N \times N$ Gaussian matrix such that

$$\langle H_{ij} \bar{H}_{i'j'} \rangle = \frac{e^{-|i-j|/W}}{W} \delta_{i,i'} \delta_{j,j'} \quad 1 \leq i,j \leq N$$

Define

$$F_N(E,u) = \frac{\langle \det(H - E + u/N) \det(H - E - u/N) \rangle}{\langle \det(H - E)^2 \rangle}$$

If the width $W^2 \gg N \gg 1$ then

$$F_N(E,u) \Rightarrow \int_{S^2} e^{iu\rho(E) S_0^{(3)}} d\mu = \frac{\sin(2u\rho(E))}{2u\rho(E)}$$

Main Idea

In 1D, F(E,u) is mapped onto classical Heisenberg spin chain of length N, $\beta = \rho(E)^2 W^2, \quad s_j \in S^2$

In one dimension the chain is ordered if $N \ll \beta \approx W^2$

For $d \geq 3$ it is ordered for $\beta \geq 1$.

Conjecture: The same Mean field result to hold for fixed W and large N in 3D.

More Problems and Conjectures

• Prove that there is a K-T transition for the 2D Quantum XY model: obtain power law lower bound.

• Show that if $2\pi\beta_{eff} < 4$ there is exponential decay of correlations

Or if $\langle S_0 S_x \rangle \leq |x|^{-(1/4+\varepsilon)}$, then it decays exponentially fast.

• Prove that $\langle \cos(\phi_0 - \phi_1) \rangle(\beta)$ is continuously differentiable for all β - Finite specific heat.

• Show staggered sine-Gordon

$$\sum_j (-1)^j \cos(\phi_j) \approx \sum_j \cos(2\phi_j)$$

so that it flows to free field in 2D if $\beta_{eff} > 2\pi$ instead of 8π. Related to six vertex. In 3D show that it has exponential decay.