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Anomalies and universality

Vieri Mastropietro

University of Milan

The massive Thirring model

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- The massless Thirring is exactly solvable, in the sense that one has an explicit set of correlations verifiying the Wightmann axioms (Johnson (1961), Klaiber (1968), Carey et al (1985)). Froehlich and Seiler (1976) constructed the Thirring-Schwinger model, equivalent to Thirring with non local interaction.

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- Gawedzki and Kupiainen (1985) and Feldman, Magnen, Rivasseau and Seneor (1986) constructed a generalization of it, the Gross-Neveu GN_N, $N > 1$ which is asymptotically free in the ultraviolet.

The massive Thirring model

- The construction of the Thirring model by RG was however lacking, and this was unfortunate as it has several basic features in common with more realistic models, like QED_{3+1} or YM_{3+1}; for instance the fact that it requires the implementation of Ward Identiites based on local symmetries to decrease the number of independent renormalizations.

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- At a formal level, what one has to do was understood by Johnson (1961) and Gomes-Lowenstein (1972); one needs to combine Schwinger-Dyson equation with Ward Identities to prove that the effective coupling is proportional to the square of the wave function renormalization $\lambda_h \sim \lambda_0 Z_h^2$.

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- However in implementing such strategy in a non perturbative Wilsonian RG one has to face basic difficulties; the momentum cut-offs breaks the formal invariance and produces corrections, which in principle could spoil WI from their utility

$$\mathbf{p}_\mu < j_{\mu,\mathbf{p}} \psi_{\mathbf{k},\omega} \bar\psi_{\mathbf{k}+\mathbf{p}} > = < \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}} > - < \psi_{\mathbf{k}+\mathbf{p}} \bar\psi_{\mathbf{k}+\mathbf{p}} > + \Delta(\mathbf{k},\mathbf{p})$$

where $\Delta = < \delta j_{\mathbf{p}} \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}+\mathbf{p}} >$ with
$\delta j_{\mathbf{p}} = \int d\mathbf{k}[(\chi^{-1}(\mathbf{k}+\mathbf{p})-1)(\not{k}+\not{p}) - (\chi^{-1}(\mathbf{k})-1)\not{k}]\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}}$

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- In BM (2005) we were finally able to overcome such problems and solved the infrared problem of the Thirring model with a cut-off (or Tomonaga model). We sketched the solution of the ultraviolet problem in a short appendix but to fully construct the model one has to verify the axioms and a lot of technical mwork was still necessary, in particular for the verification of the OS axioms. .

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- This was the point when Pierluigi came into the game. His first result was a very powerful and general bound for the *n*-point Schwinger function, which was strong enough to allow the axiom verifications (and later on in the proof of Coleman equivalence between Thirring and Sine Gordon or in interacting dimers).

The massive Thirring model

- The main conclusion of Falco PHD thesis was the following theorem, which finally provided the full construction of the massive Thirring model after so many years of attempts. One starts from $S_{n,N}$, the Schwinger function of a regularized Thirring model with ultraviolet momentum cut-off 2^N, wave function renormalization Z_N, coupling λ and mass μ_N.

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Theorem

(Benf, Falco, Mas CMP 2007) For λ small enough it is possible to choose

$$Z_N = 2^{-\eta_z N}(1 + O(\lambda)) \qquad \mu_N = 2^{-\eta_\mu N}\mu(1 + O(\lambda))$$

with $\eta_z = a_z\lambda^2 + O(\lambda^3)$ and $\eta_\mu = -a_\mu\lambda + O(\lambda)$, $a_z, a_\lambda > 0$ such that

$$\lim_{N\to\infty} S_{n,N}(\underline{x}) = S_n(\underline{x})$$

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- In Coleman paper it is proved the equivalence order by order in the expansion, but the proof of convergence was lacking. The limit $\Lambda\to\infty$ is still an open problem

Anomalies

- One of the most interesting consequences of BFM06 was the following formula for the WI of the massless Thirring model

$$\partial_\mu < \bar\psi_{\mathbf{z}}\gamma_\mu\psi_{\mathbf{z}};\psi_{\mathbf{x}}\bar\psi_{\mathbf{y}} >= A(\delta(\mathbf{x}-\mathbf{z})-\delta(\mathbf{y}-\mathbf{z})) < \psi_{\mathbf{x}}\bar\psi_{\mathbf{y}} >$$

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- Our analysis shown that such anomaly naturally comes out in a non-perturbative analysis of the functional integrals; indeed it comes from the the correction term Δ in the WI.

Renormalziation and non Renormalization of the Anomalies

- Curiously the value of $A, \bar{A}$ were different with respect to the values found by Johnson $A^{-1} = 1 - \frac{\lambda}{4\pi}\lambda$, $\bar{A}^{-1} = 1 + \frac{\lambda}{4\pi}\lambda$, in which there were no higher order corrections.

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- Pierluigi computed in his thesis c_+ and he was very surprised to find such a non zero value; he started to think on the reasons on the difference between Johnson result and he convinced me that such difference was interesting and worthwhile to understand.
- We discovered a paper of Georgi Rawls (1971) in which the absence of higher order corrections in $A, \bar{A}$ was related to the validity of the anomaly non renormalization property analogous to the one predicted by Adler-Bardeen (1969) in QED. Therefore we were saying that such a property was instead somewhat violated.

- Similarly if you write the Thirring model in terms of bosons via an Hubbard-Stratonivich transformation, then the linearity of A follows from certain results on the fermionic determinants, or from Fujikkawa (1979) theory. Why we get instead higher order corrections to the anomaly?

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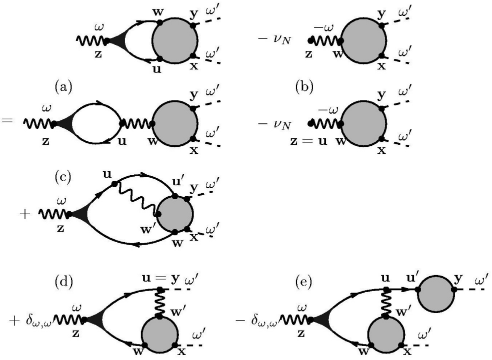
- The reason relyes in a subtle exchange of limits phenomenon. One can introduce an ultraviolet cut-off in the Thirring model both introducing a momentum cut-off and a transfer momentum cut-off in the interaction. If the limit of local interaction is taken before the removal of the momentum cut-off the anomaly has higher order corrections; if is taken after all higher order corrections are vanishing (Mastropietro JMP 007).



In a RG analysis $\Delta(\mathbf{k},\mathbf{p})$ the terms $\delta j\psi^+\psi^-$ are marginal; one subtracts a local term, and one can further decompose them in a sum of terms with have scaling negative dimension (see c,d,e) for the non locality of the interaction except *a*, which is compensated by the local term (b)

Non perturbative Adler-Bardeen theorem

- More in general one can consider a $d = 1 + 1$ photon-fermion model with interaction $eA_\mu \bar{\psi}\gamma_\mu\psi$, where A_μ is a massive (mass M) photon field with ultraviolet cut-off. If the fermionic cut-off is removed then the anomaly is not renormalized

$$\partial_\mu j^5_\mu = \frac{e}{4\pi}\varepsilon_{\mu,\nu}\partial^\nu A_\nu$$

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- If $M = 0$ this result implies the mass generation of the Schwinger model; in Falco 2010 $M > 0$ but the result is an important step toward the rigorous construction of the Schwinger model starting (a problem correctly considered fundamental by Pierluigi).

$X_- X_+ = 1$

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- After some thoughts and reading some Kadanoff paper I understood what he meant and I found it very interisting. Consider 2 Ising models coupled by a quartic interaction

$$H(\sigma, \sigma') = H_J(\sigma) + H_{J'}(\sigma') - \lambda V(\sigma, \sigma')$$

with $H = -J \sum_{j=0,1} \sum_{\mathbf{x} \in \Lambda} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j}$ $\sigma_{\mathbf{x}} = \pm$, Λ is a 2D square lattice, $\mathbf{x} \in \Lambda$, $\mathbf{e}_0 = (0,1)$, $\mathbf{e}_1 = (1,0)$. V is a short ranged, quartic in the spin and invariant in the spin exchange, like

$$V = \sum_{j=0,1} \sum_{\mathbf{x},\mathbf{y} \in \Lambda} v(\mathbf{x} - \mathbf{y}) \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma'_{\mathbf{y}} \sigma'_{\mathbf{y}+\mathbf{e}_j}$$

with $v(\mathbf{x})$ a short range potential.

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- I had the exponents $X_\pm$ in the form of convergent series, depending from all microscopic detail. Such expansions were so complicated that a direct proof of a model independent relation $X_- X + = 1$ seemed impossible.

$X_- X_+ = 1$

- Pierluigi proposal was clever and natural. We already used (for the proof of the positivity axiom) in the paper on Thirring that different regularizations (a momentum or a lattice cut-off) has as a limit the same correlations, provided that the bare parameters are suitably chosen.

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- One can follow a similar strategy to conclude that the exponents of coupled Ising are the same of effective models of interacting fermions with linear relativistic dispersion relation, if the bare parameters are properly tuned.
- Among such effective models there are some in which the anomaly is exactly known as higher order corrections are vanishimg (for instance if the interaction is short ranged), as we proved before. In such cases the exponents of the effective models have simple expressions in terms of the bare coupling of the effective model.

Universality relations

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Theorem

(Benfatto,Falco,Mastropietro CMP(2009)) If the coupling of the coupled Ising model is small enough

$$X_-(\lambda) = \frac{1}{X_+(\lambda)} \quad \nu = \frac{1}{2 - X_+(\lambda)} \quad \alpha = \frac{2 - 2X_+(\lambda)}{2 - X_+(\lambda)}$$

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The last relation is new; the others were proposed by Kadanoff (1977), Kadanoff and Wegner (1971) and imply the hyperscaling relation $2\nu = 2 - \alpha$.

Haldane relations for the conductivity

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- The Heisenberg XXZ spin chain

$$H_0 = -\sum_{x=1}^{L-1}[JS_x^1 S_{x+1}^1 + JS_x^2 S_{x+1}^2 + J_3 S_x^3 S_{x+1}^3 - hS_x^3]$$

where $S_x^\alpha = \sigma_x^\alpha/2$ for $i = 1, 2, \ldots, L$ and $\alpha = 1, 2, 3$, σ_x^α being the Pauli matrices ($J = 1$).

- The above model can be solved by Bethe ansatz, and it is interesting to add a next-to-nearest neighbor interaction breaking exact solvability, that is consider $H = H_0 + H_1$

$$H_1 = -\lambda\sum_{x=1}^{L-1}[S_x^1 S_{x+2}^1 + S_x^2 S_{x+2}^2 + S_x^3 S_{x+2}^3]$$

Linear response theory

- By the Peierls substitution $j_x = S_x^1 S_{x+1}^2 - S_x^2 S_{x+1}^1 + \lambda F_x$ where F_x is an expression *quartic* in the spin operators.

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$$\sigma_\lambda(\omega) = \lim_{\delta \to 0} \lim_{p \to 0} \lim_{\beta \to \infty} \frac{D_{\beta,\lambda}(\mathbf{p})}{ip_0} |_{ip_0 \to \omega + i\delta}$$

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Luttinger liquid conjecture

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- Haldane (1980) conjectured that the same relations is true in a wide class of systems, including non solvable models (Luttinger liquid conjecture; in particular for $\lambda \neq 0$.

Conductivity in the non integrable case

Theorem

(Benf Mas JSP 2012; Mas PRE 2013) There exists $\varepsilon < 1$ such that, if $|J_3|, |\lambda| \leq \varepsilon$ the zero temperature Drude weight is non vanishing and analytic in J_3, λ; moreover

$$D_\lambda = K \frac{v_{s,\lambda}}{\pi} \quad \kappa_\lambda = \frac{K}{\pi v_{s,\lambda}}$$

with $K = 1 - \frac{1}{\pi v_{s,\lambda}}[(J_3 + 2\lambda)(1 - \cos 2p_F) + \lambda(1 - \cos 4p_F) + F]$ *and* $v_s = \sin(p_F) + \tilde{F}$, $\sin p_F = h$ *and* $|F| \leq C\varepsilon^2, |\tilde{F}| \leq C\varepsilon.$

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- The zero temperature conductivity is still infinite (an interaction breaking integrability does not change qualitative behavior).
- The idea is to combine the lattice WI with the emerging WI; Again the non renormalization of anomalies and AB theorem for the effective model plays a crucial role.

Non equilibrium Luttinger model

- Universality properties for the conductivity emerge also in a non equilibrium context. I report a recent computation (preliminary results in collaboration with Langmann, Lebowitz, Moosavi) in the Luttinger model; we can exploit its exact solution to get information on non equilibrium dynamics.

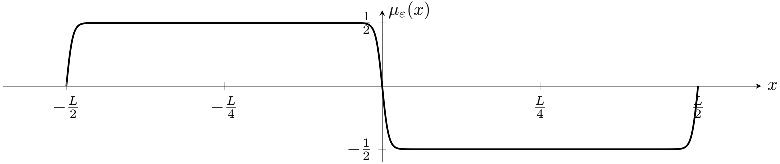
- Universality properties for the conductivity emerge also in a non equilibrium context. I report a recent computation (preliminary results in collaboration with Langmann, Lebowitz, Moosavi) in the Luttinger model; we can exploit its exact solution to get information on non equilibrium dynamics.
- The Luttinger model Hamiltonian is (antiperiodic b.c.)

$$H_\lambda = \sum_{\sigma=\pm} \int_{-L/2}^{L/2} dx : \tilde\psi^+_\sigma(x)(-i\sigma\partial_x - p_F)\tilde\psi^-_\sigma(x) : +\lambda \int_{-L/2}^{L/2} dxdyv(x-y)$$
$$\left(: \tilde\psi^+_+(x)\tilde\psi^-_+(x) :+: \tilde\psi^+_-(x)\tilde\psi^-_-(x) :\right)\left(: \tilde\psi^+_+(y)\tilde\psi^-_+(y) :+: \tilde\psi^+_-(y)\tilde\psi^-_-(y) :\right)$$

- We add a chemical potential term saying that there is an asymmetry of charge in the left or right hand side,
  $\rho(x) =: \tilde\psi^+_+(x)\tilde\psi^-_+(x) : + : \tilde\psi^+_-(x)\tilde\psi^-_-(x) :$

$$H_{\lambda,h} = H_\lambda - h_0 \int_{-L/2}^{L/2} dx\mu(x)\rho(x)$$

  We choose $\mu(x)$ to be positive on the left side ($x<0$) and negative on the right ($x>0$)



$\mu_\varepsilon(x)$
x
$\frac{1}{2}$
$-\frac{1}{2}$
$-\frac{L}{2}$
$-\frac{L}{4}$
$\frac{L}{4}$
$\frac{L}{2}$

- Remarkably the Luttinger model is solvable even with an external potential. We consider the ground state of H_{0,h_0}, called $|\Psi_{0,h_0}\rangle$; such a state has an excess of density in the right hand side.Then we switch off the external potential and let the system evolve with the interacting Hamiltonian.

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- We consider then the evolution of $|\Psi_{0,h_0}\rangle$ under the interacting Hamiltonian H_λ (no external field),

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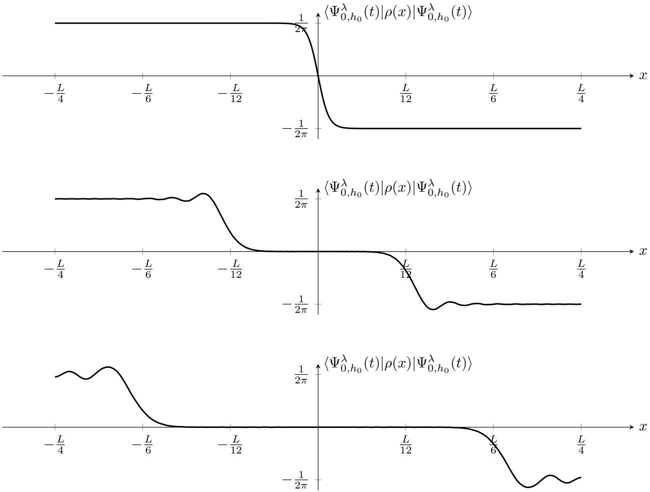
- The averaged density is, in the limit $L \to \infty$

$$\langle\Psi^\lambda_{0,h_0}(t)|\rho(x)|\Psi^\lambda_{0,h}(t)\rangle = \frac{h_0}{2\pi}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\hat{\mu}(p)\left(e^{ip(x-\varepsilon(p)t)} + e^{ip(x+\varepsilon(p)t)}\right)$$

where $\varepsilon(p) = \sqrt{\left(1+\frac{\lambda\hat{v}(p)}{\pi}\right)^2 - \left(\frac{\lambda\hat{v}(p)}{\pi}\right)^2}$ is an interaction dependent velocity. If the interaction is local $\varepsilon(p)$ is constant.

THE DENSITY AT DIFFERENT TIMES



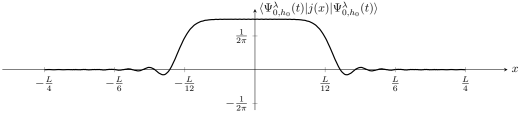
There is a region with zero density bounded by two fronts moving ballistically; the interaction changes the velocity of the two fronts and changes their shape. As $t \to \infty$ one reaches a zero density state.

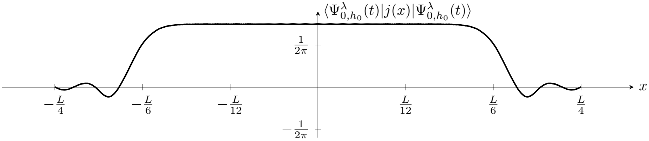
The current at different times

The averaged current $j(x) =: \tilde{\psi}_+^+(x)\tilde{\psi}_+^-(x) : - : \tilde{\psi}_-^+(x)\tilde{\psi}_-^-(x)$:is

$$\langle \Psi^{\lambda}_{0,h_0}(t)|j(x)|\Psi^{\lambda}_{0,h_0}(t)\rangle = \frac{h}{2\pi}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{\hat{\mu}(p)}{K(p)}\left(e^{ip(x-\varepsilon(p)t)} - e^{ip(x+\varepsilon(p)t)}\right)$$

where $K(p) = \sqrt{\frac{\pi}{\pi+2\lambda\hat{v}(p)}}$: there is a non zero region where the current is non vanishing. As $t \to \infty$ one reaches a state with a uniform non vanishing and finite current. The current is finite even without dissipation , as there is current without external field.





The averaged 2-point function is given by

$$\langle\Psi^{\lambda}_{0,h_0}(t)|\tilde{\psi}^{+}_{\sigma}(x)\tilde{\psi}^{-}_{\sigma}(y)|\Psi^{\lambda}_{0,h_0}(t)\rangle = e^{-i\sigma p_F(x-y)} e^{A_\sigma(x,y,t)} S_1(x,y,t),$$

where $S_1(x,y,t) = \langle\Psi_0|e^{i(H_\lambda t}\psi^{+}_{\sigma}(x)\psi^{-}_{\sigma}(y)e^{-i(H_\lambda)t}|\Psi_0\rangle_\infty$

$$S_1(x,y,t) = \frac{i}{2\pi\sigma(x-y)+i0^+}$$

$$\exp\left(\int_0^\infty dp\frac{\gamma(p)}{p}(\cos p(x-y)-1)(1-\cos 2\varepsilon(p)t)\right)$$

with $\gamma(p) = 4\sinh^2\varphi_p\cosh^2\varphi_p$, $\tanh 2\varphi_p = -\lambda\hat{v}(p)/2\pi$.

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$$A_\sigma(x,y,t) = -\sigma h_0 \int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{\hat{\mu}(p)}{p}\Big(\frac{K(p)+1}{2K(p)}(e^{ip(x-\sigma\varepsilon(p)t)} - e^{ip(y-\sigma\varepsilon(p)t)})$$

$$+\frac{K(p)-1}{2K(p)}(e^{ip(x+\sigma\varepsilon(p)t)} - e^{ip(y+\sigma\varepsilon(p)t)})$$

Translation invariance is recovered in the $t\to\infty$ limit.

The local case

1. For definiteness we consider the case of delta interactions. The limiting value of the current is, $K \equiv K(0)$

$$\lim_{t \to \infty} \langle \Psi^{\lambda}_{0,h_0}(t) | j(x) | \Psi^{\lambda}_{0,h_0}(t) \rangle = \frac{h_0}{2\pi} \frac{1}{K} = I$$

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2. The 2-point function becomes

$$\lim_{t\to\infty}\lim_{a\to 0} Z_a \langle \Psi^{\lambda}_{0,h_0}(t)|\psi^{+}_{\sigma}(x)\psi^{-}_{\sigma}(y)|\Psi^{\lambda}_{0,h_0}(t)\rangle = \frac{i e^{-i\sigma\mu_\sigma(x-y)}|x-y|^{-\gamma}}{2\pi\sigma(x-y)+i0^+}$$

$$\mu_\sigma = p_F + \sigma\frac{h_0}{2K}$$

with $\gamma = \gamma(0)$ a critical exponent which is different from the exponent found in the average over the ground state of H_λ, that is $|\Psi_{\lambda,0}>$. Note that indeed $\lim_{t\to\infty} e^{iH_\lambda t}|\Psi_{0,0}> \neq |\Psi_{\lambda,0}>$.

Universality of conductance

1. The evolution of the domain wall state $\lim_{t\to\infty} e^{iH_\lambda t}|\Psi_{0,h_0}>$ converges to a state with different chemical potential $\mu_\pm$ for right and left going fermions (which is not the groundstate of $H_\lambda + \sum_\sigma \mu_\sigma \rho_\sigma$) ; that is asymtotically the excess of charge produces an effective potential. $\mu_+ - \mu_- = V$.

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2. Note that $\mu_\pm$ and the limiting current depend from λ but the Landauer conductance is λ independent and equal to the conductivity quantum

$$G = \frac{I}{\mu_+ - \mu_-} = \frac{h_0}{2\pi K}\frac{K}{h_0} = \frac{1}{2\pi}.$$

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Now η is the Luttinger liquid exponent; therefore $\lim_{t\to\infty}|\Psi_{\lambda,h_0}>$ tends to the ground state of a Luttinger Hamiltonian with different chemical potentials $H_\lambda + \sum_\sigma \tilde{\mu}_\sigma \rho_\sigma$. Again the conductance is universal ($\tilde{\mu}_\pm$ and I are different with respect to the previous case).

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4. Alekseev, Cheianov, and Froehlich (1996) computed The conductance of the GS of $H_\lambda + \sum_\sigma \mu_\sigma\rho_\sigma$ finding $G = \frac{1}{2\pi}$, using the anomaly non renormalization AB theorem.

Non equilibrium properties and universality

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- Such universality emerges at non equilibrium by a dynamical computation. In particular, the second case is related to the non renormalization of the anomaly but it is unclear if this is true also in the first.
- A goal for the future is to develop an RG for non equilibrium properties for non solvable models, as it was done for the equilibrium properties. How much the above results are related to the exact solvability ?
- It is really sad that we can not discuss of this with Pierluigi, profiting of his deep intuition!