The Coulomb gas in two dimensions

Contributions of Pierluigi Falco

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Energy of configuration ω:

$$H_\Lambda(\omega) = \lim_{m^2 \downarrow 0} \frac{1}{2} \sum_{i,j=1}^{n} \sigma_i \sigma_j w_\Lambda(x_i - x_j)$$

$$w_\Lambda(x - y) = (m^2 - \Delta)^{-1}_{x,y}$$

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Grand Canonical partition function:

$$Z_\Lambda(\beta, z) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\omega \in \Omega_n^0} e^{-\beta H_\Lambda(\omega)}$$

- Fractional test charges

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Fractional charge correlation function:

$$\rho_\eta(x - y) = \lim_{\Lambda \to \infty} \frac{Z_\Lambda^{p_1,p_2}(\beta, z)}{Z_\Lambda(\beta, z)}$$

Debye and Hückel 1923

Theorem (Wei-Shih Yang 1987)

For the grand canonical Coulomb system on $\mathbb{R}^2$ *with* β *small,* $\rho_\eta(x,y)$ *decays exponentially to zero as* $|x-y| \to \infty$.

Open problems: (1) implicit hypothesis on z; (2) free boundary conditions; (3) extend range of β.

Theorem (Fröhlich-Spencer 1981)

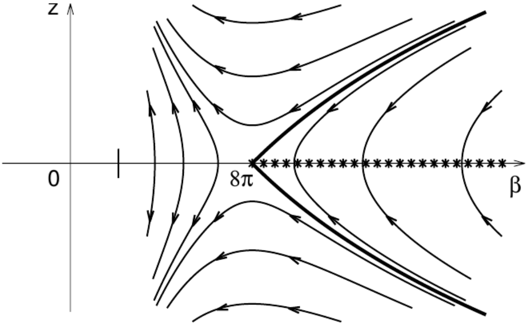
For the Coulomb system on $\mathbb{Z}^2$ *exponential screening does not hold for* β *large.*

Fröhlich, J. (1976). Classical and quantum statistical mechanics in one and two dimensions: two-component Yukawa- and Coulomb systems.
Comm. Math. Phys., 47(3):233–268

Yang, W.-S. (1987). Debye screening for two-dimensional Coulomb systems at high temperatures.
J. Statist. Phys., 49:1–32

Fröhlich, J. and Spencer, T. (1981). The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas.
Comm. Math. Phys., 81(4):527–602

KT Picture



β_{eff} =: where trajectories cross horizontal axis.

$\rho(a-b)$ decays as $|a-b|^{-2\kappa}$ for $\eta \in (0, \frac{1}{2}]$ and $\beta_{\text{eff}} \geq 8\pi$.

$\kappa = \frac{\beta_{\text{eff}}}{4\pi}\eta^2$ with log corrections for $\beta_{\text{eff}} = 8\pi$.

Berezinskiĭ, V. L. (1970). Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group. I. Classical systems.
Ž. Èksper. Teoret. Fiz., 59:907–920

Kosterlitz, M. and Thouless, D. J. (1973). Ordering, metastability and phase transitions in two-dimensions.
J. Phys. C, 6:1181–1203

Kosterlitz, J. M. (1974). The critical properties of the two-dimensional xy model.
Journal of Physics C: Solid State Physics, 7(6):1046

Theorem (Pierluigi Falco, 2013)

KT picture, including differential equations for trajectories, holds with explicit log corrections to κ for $\beta_{\mathit{eff}} = 8\pi$ and z small.

The FS result was improved to $\beta_{\mathrm{eff}} > 8\pi$, (see Marchetti-Klein 1991).

Falco, P. (2012). Kosterlitz-Thouless transition line for the two dimensional Coulomb gas. *Comm. Math. Phys.*, 312(2):559–609

Falco, P. (2013). Critical exponents of the two dimensional coulomb gas at the Berezinskii-Kosterlitz-Thouless transition.
http://arxiv.org/abs/1311.2237

Sine–Gordon transformation

Gaussian field:

$$\mathbb{E}_m[\varphi_x \varphi_y] = (-\Delta + m^2)^{-1}(x, y)$$

Sine–Gordon transformation:

$$Z_\Lambda(\beta, z) = \lim_{m \to 0} \mathbb{E}_m \; e^{z \sum_x 2 \cos \beta^{1/2} \varphi_x}$$

$$\rho_\eta(x - y) = \lim_{\Lambda \to \infty} \lim_{m \to 0} \left\langle e^{i\eta\beta^{1/2}\varphi_x} e^{-i\eta\beta^{1/2}\varphi_y} \right\rangle_{m,\Lambda} .$$

Generating functional and interaction

$$\Omega(J,\Lambda) = \lim_{m\to 0} \mathbb{E}_m \exp\left[z \sum_{x,\sigma=\pm} e^{i\sigma\beta^{1/2}\varphi_x} + \sum_{x\in\Lambda,\sigma=\pm} J_{x,\sigma} e^{i\eta\sigma\beta^{1/2}\varphi_x}\right]$$

$$= \lim_{m\to 0} \mathbb{E}_m \left[e^{\mathcal{V}(J,\varphi)}\right]$$

where

$$\mathcal{V}(J,\varphi) = \frac{s}{2}\sum_{x,\mu}(\partial^\mu \varphi_x)^2 + z\sum_{x,\sigma=\pm} e^{i\sigma\alpha\varphi_x} + \sum_{x,\sigma=\pm} J_{x,\sigma} e^{i\eta\alpha\sigma\varphi_x}$$

$$s \in (0,\frac{1}{2}), \qquad \alpha^2 = \beta(1-s)$$

Dropping multiplicative constants.

2D GFF

Assume period(Λ) $= L^R$.

$\exists$ multiscale covariance decomposition:

$$(-\Delta + m^2)^{-1} = \sum_{0 \le j < R} \Gamma_j + \Gamma'_R.$$

For $\zeta_j \sim N(\Gamma_j)$,

- Finite range property: $\zeta_j(x)$ independent of $\zeta_j(y)$ if $|x-y| \ge O(L^j)$
- Scaling estimates:

$$\nabla^\alpha \zeta_j \approx L^{-j|\alpha|_1}$$

$$\zeta_j \approx \sqrt{\log L}, \qquad \Gamma_j(0) \sim \frac{1}{2\pi} \log L.$$

RG

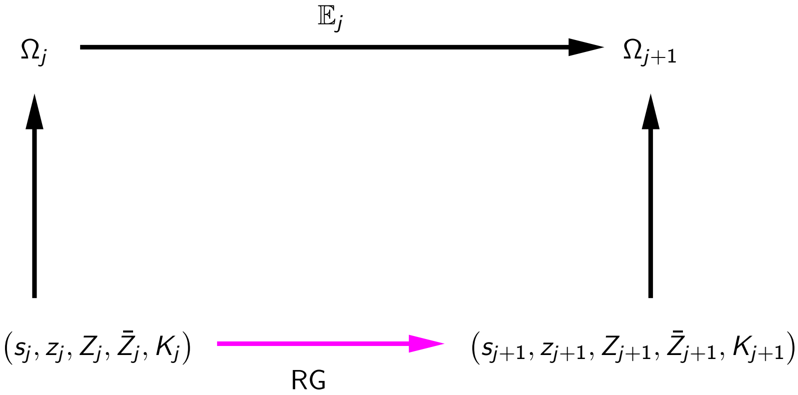
Evaluate Ω progressively:

$$\Omega_0(J,\varphi) = e^{\mathcal{V}(J,\varphi)}, \qquad \Omega_{j+1}(J,\varphi) = \mathbb{E}_j\left[\Omega_j(J,\varphi+\zeta_j)\right],$$

$$\Omega(J,\varphi) = \Omega_R(J,\varphi)$$

- $\mathbb{E}_j$ acts only on the fluctuation field ζ_j.

Begin definition of RG:



- Real valued bulk coupling constants s_j, z_j
- Real valued observable coupling constants $Z_j, \bar{Z}_j$
- K_j in Banach space

Definition of vertical arrows: step 1

Given $(s_j, z_j, Z_j, \bar{Z}_j) \in \mathbb{R}^4$, define functions of $\Phi = (J, \varphi)$.

$$V_{0,j}(\Phi, B) = \frac{s_j}{2} \sum_{x \in B, \mu} (\partial^\mu \varphi_x)^2 + z_j L^{-2j} \sum_{x \in B, \sigma = \pm} e^{i\sigma\alpha\varphi_x}$$

$$V_{1,j}(\Phi, B) = Z_j L^{-2j} \sum_{x \in B, \sigma = \pm} J_{x,\sigma} e^{i\eta\alpha\sigma\varphi_x}$$

$$+ \bar{Z}_j L^{-2j} \sum_{x \in B, \sigma = \pm} J_{x,\sigma} e^{i\bar{\eta}\alpha\sigma\varphi_x}, \qquad \bar{\eta} = 1 - \eta.$$

$$V_j(\Phi, B) = V_{0,j}(\Phi, B) + V_{1,j}(\Phi, B)$$

Definition of vertical arrows: step 2

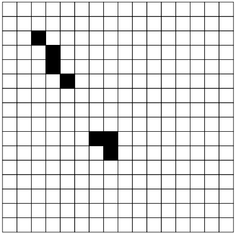
Let

$$U_j(\Phi, B) = V_j(\Phi, B) + W_j(\Phi, B)$$

where $W_j(\Phi, B)$ is another explicit function of $\Phi = (J, \varphi)$ defined by $(s_j, z_j, Z_j, \bar{Z}_j)$.

It is given by a LARGE formula obtained from second order perturbation theory.

Definition of vertical arrows: final step



Given $K_j : X \mapsto$ function of $(\varphi_x, J_x)_{x \in X^\square}$

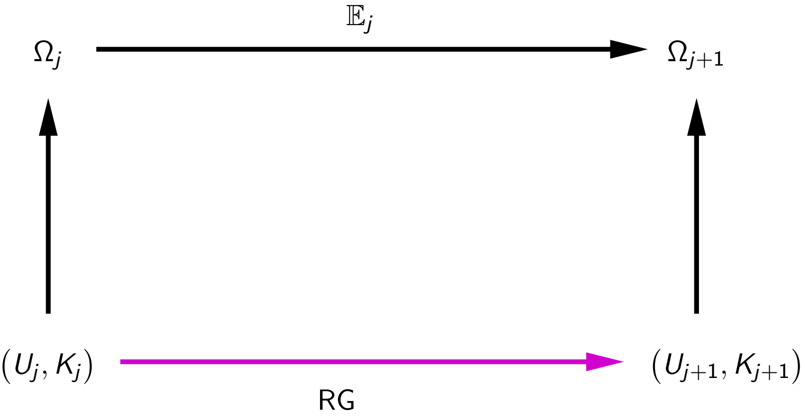
Ω_j is expressed in terms of (U_j, K_j)

using

$$\Omega_j(\Phi, \Lambda) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y).$$

K_j is there to include the remainder after second order perturbation theory.

Summary



- U_j determined by coupling constants $(s_j, z_j, Z_j, \bar{Z}_j)$
- $\Omega_j(\Phi, \Lambda) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y)$,

Theorem ($\exists$ RG)

For all j such that (s_j, z_j) is small, K_j is also small, $O(s_j, z_j)^3$ uniformly in j, and (s_j, z_j) follows the KT picture:

$$s_{j+1} \approx s_j - a z_j^2, \qquad z_{j+1} \approx L^2 e^{-\frac{\alpha^2}{2}\Gamma_j(0)}[z_j - b s_j z_j]$$

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Theorem (Best choice of s)

For $\alpha^2 = 8\pi$, for $z_0 = z$ small, there is a unique $s_0 = s_0(z)$ such that (s_j, z_j, K_j) is in the domain of RG for $j \leq R$ and (s_R, z_R, K_R) tends to zero.

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$z_R \to 0$ means there are no dipoles at macroscopic scales.

$s_R \to 0$ means that $\alpha\varphi$ is "the best" gaussian approximation to the Coulomb gas at the KT transition.

Since $\Gamma_j(0) \sim \frac{1}{2\pi} \log L$ as $j \to \infty$,

$$L^2 e^{-\frac{\alpha^2}{2}\Gamma_j(0)} \sim L^{2-\frac{\alpha^2}{4\pi}}$$

So if

$$\alpha^2 = 8\pi, \qquad \text{KT point}$$

then z_j is marginal.

To have $\alpha^2 = 8\pi$, by the definition $\alpha^2 = (1 - s_0(z))\beta$,

$$\beta = \frac{8\pi}{1 - s_0(z)}$$

Calculation of $\rho(a, b)$

After R steps Λ becomes a single block so that

$$\Omega_R(\Phi, \Lambda) = e^{U_R(\Phi,\Lambda)} + K_R(\Phi, \Lambda).$$

Put this into

$$\rho_\eta(x, y) = \frac{1}{\Omega_R(\Phi, \Lambda)} \frac{\partial^2 \Omega_R(\Phi, \Lambda)}{\partial J_x \partial J_y}\Big|_{J=0}.$$

In the infinite volume limit $R \to \infty$, K_R becomes zero and makes no contribution!

In fact $\rho(a, b)$ is completely determined by the double derivative of W_R and the $(s, z, Z, \bar{Z})$ flow.

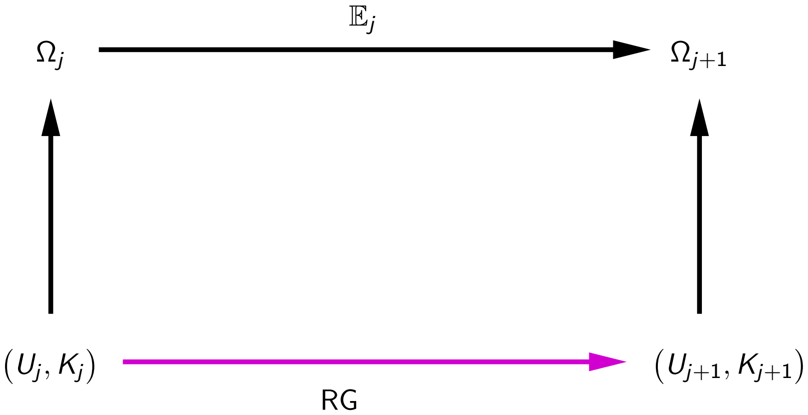
- First paper [CMP 2012]: External field $J = 0$ ("bulk").

$$(s_j, z_j, K_j^{J=0}) \mapsto (s_{j+1}, z_{j+1}, K_{j+1}^{J=0}).$$

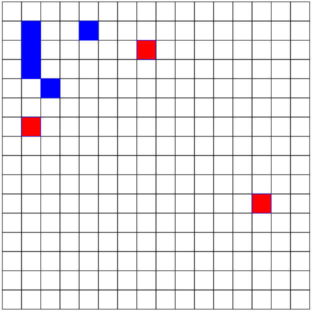
- Second paper [arXiv 2013]: Extension to $J \neq 0$.

  This is really an extension, in the sense that $Z_j, \bar{Z}_j$ and J do not feed back into the bulk coordinates.

Recall the magenta arrow



Provisional definition of $(U_j, U_{j+1}, K_j) \mapsto K_{j+1}$



Expand $\Omega_j =$

$$\sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y)$$

using, in each small block,

$$\varphi = \varphi' + \zeta_j$$

$$e^{U_j(\varphi' + \zeta_j)} = e^{U_{j+1}(\varphi')} + \text{difference}.$$

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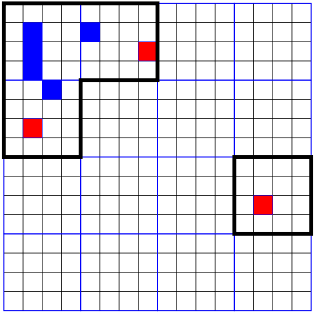

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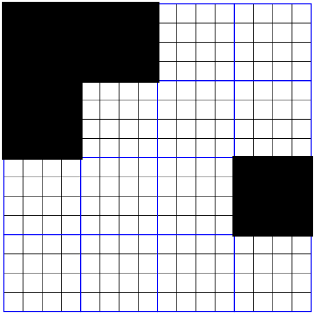

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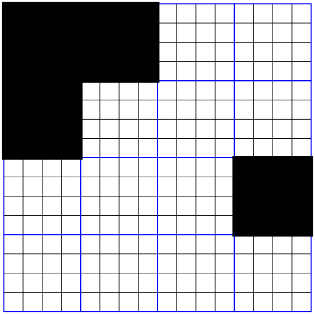

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Finite range: expectation factors over connected components.

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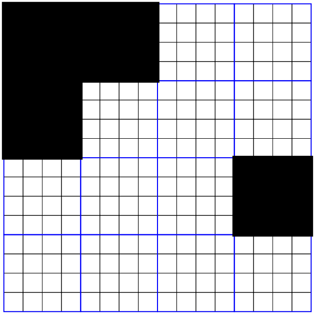

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Finite range: expectation factors over connected components.

For a connected union X of big blocks,

$$K_{j+1}(X) = \mathbb{E}_j\Big(\text{sum over ways to fill } X\Big).$$

Linear part on small sets

$X \mapsto K_{j+1}(X)$ is a power series in K_j. The linear term in this series is

$$X \mapsto \sum_{Y : \overline{Y} = X, |Y|_j \leq 2^d} \mathbb{E} K_j(Y)$$

when coupling constants are zero.

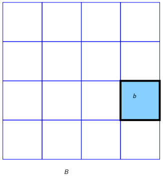
By very general arguments, the theorems above reduce to showing that this part of K_{j+1} is contractive as a function of K_j.

Brydges, D. C. (2009). Lectures on the renormalisation group.
In *Statistical Mechanics*, volume 16 of *IAS/Park City Math. Ser.*, pages 7–93. Amer. Math. Soc., Providence, RI

Brydges, D. and Yau, H.-T. (1990). Grad ϕ perturbations of massless Gaussian fields.
Comm. Math. Phys., 129(2):351–392

Example

Consider a scale $j+1$ block B.



The linearisation of $K_j \mapsto K_{j+1}(B)$ is

$$\sum_{b \in \mathcal{B}_j(B)} K_j(b)$$

Making no assumptions on K_j, it would expand by L^2 because there are L^2 little blocks b inside B.

Progressive integration

- Represented Coulomb gas as lattice Sine-Gordon model
- Goal is to understand the generating functional

$$\Omega(J) = \lim_{m \to 0} \mathbb{E}_m \exp\left[z \sum_{x,\sigma=\pm} e^{i\sigma\beta^{1/2}\varphi_x} + \sum_{x\in\Lambda,\sigma=\pm} J_{x,\sigma} e^{i\eta\sigma\beta^{1/2}\varphi_x} \right]$$

- Evaluate progressively:

$$\Omega_{j+1}(J,\varphi) = \mathbb{E}_j\left[\Omega_j(J,\varphi+\zeta^{(j)})\right], \qquad \Omega_0(J,\varphi) = e^{\mathcal{V}(J,\varphi)}$$

using finite range decomposition

$$(-\Delta + m^2)^{-1} = \Gamma_1 + \cdots + \Gamma_R.$$

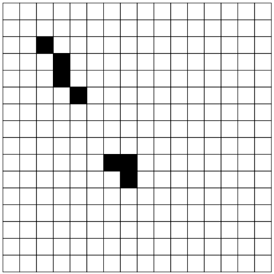
Then $\Omega(J) = \Omega_R(J,0)$.

Local coordinates

Represented $\Omega_j \mapsto \Omega_{j+1}$ via $(U_j, K_j) \mapsto (U_{j+1}, K_{j+1})$ and

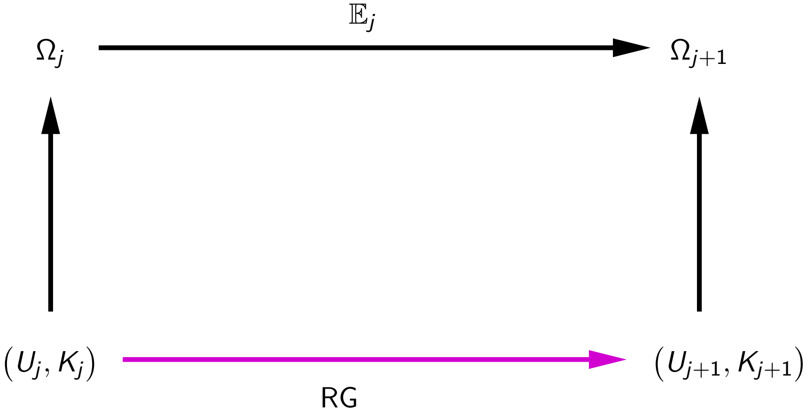
$$\Omega_j(\Phi) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y), \qquad \Phi = (J, \varphi).$$

- $U_j \equiv (s_j, z_j, Z_j, \bar{Z}_j)$ coupling constants $\rightsquigarrow$ KT picture;
- K_j remainder coordinate.
- $\mathcal{P}_j$: unions of blocks of side L^j;
- $\mathcal{C}_j$: connected unions of blocks;



So far: Evolution $(U_j, K_j) \mapsto K_{j+1}$ still contained relevant/marginal directions. Not contractive.

Now: How to make $(U_j, K_j) \mapsto K_{j+1}$ contractive.



- Vertical arrows:

$$\Omega_j(\Phi, \Lambda) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y), \qquad \Phi = (J, \varphi)$$

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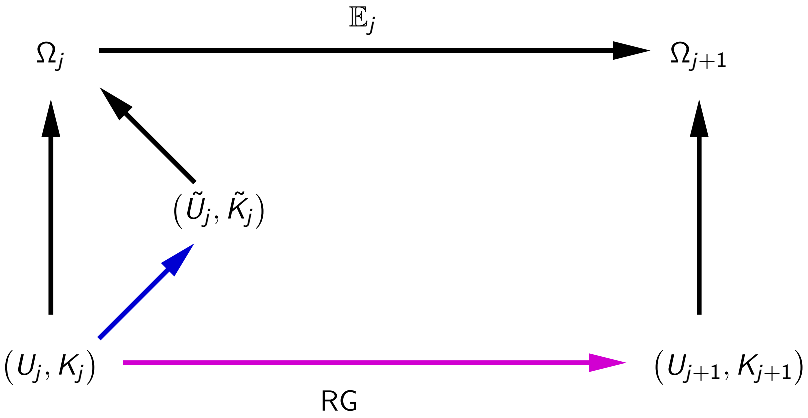
$$U_j(\Phi, X) = \frac{s_j}{2} \sum_{x \in X, \mu} (\partial^\mu \varphi_x)^2 + z_j L^{-2j} \sum_{x \in X, \sigma = \pm} e^{i\sigma\alpha\varphi_x} + \cdots$$



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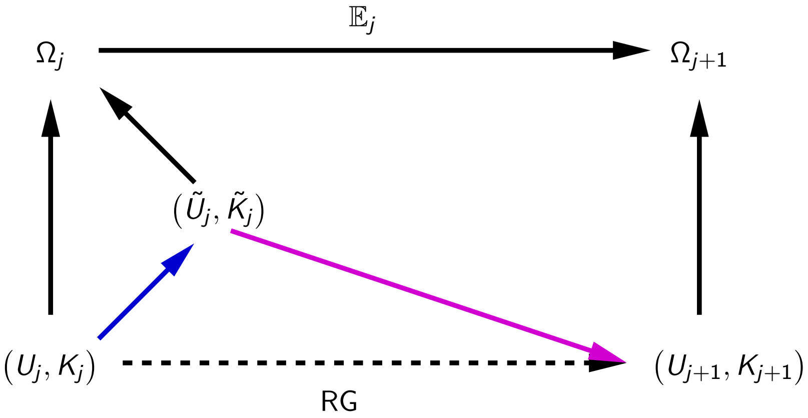

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Charge decomposition isolates expanding parts

K has the property that it is invariant under

$$\varphi_x \mapsto \varphi_x + \frac{2\pi}{\alpha}.$$

Any function $F(\varphi)$ with this property can be written as

$$F(\varphi) = \sum_{q \in \mathbb{Z}} \widehat{F}(q, \varphi),$$

such that, for all constants ϑ,

$$\widehat{F}(q, \varphi) = e^{iq\alpha\vartheta} \widehat{F}(q, \varphi - \vartheta).$$

Charge decomposition isolates expanding parts

K has the property that it is invariant under

$$\varphi_x \mapsto \varphi_x + \frac{2\pi}{\alpha}.$$

Any function $F(\varphi)$ with this property can be written as

$$F(\varphi) = \sum_{q \in \mathbb{Z}} \widehat{F}(q, \varphi),$$

such that, for all constants ϑ,

$$\widehat{F}(q, \varphi) = e^{iq\alpha\vartheta} \widehat{F}(q, \varphi - \vartheta).$$

Fix a base point x_0 and set $\vartheta = \varphi_{x_0}$. Then

$$\widehat{F}(q, \varphi) = e^{iq\alpha\varphi_{x_0}} \widehat{F}(q, \varphi - \varphi_{x_0}).$$

Interpretation: q is charge and $\widehat{F}(q, \varphi - \varphi_{x_0})$ represents dipoles.

Charge power counting

In Sine–Gordon picture: charge q at site x is represented by $e^{iq\alpha\varphi_x}$.

$$\mathbb{E}_{\Gamma_j}[e^{iq\alpha\varphi_x}] = e^{-\frac{1}{2}q^2\alpha^2\Gamma_j(x,x)} \sim L^{-q^2\frac{\alpha^2}{4\pi}}$$

- $L^{-q^2\frac{\alpha^2}{4\pi}}$ beats the volume factor L^2 if $|q| \geq 2$ or $\alpha^2 > 8\pi$.
- Along KT line $q = \pm 1$ is marginal. Recall the KT line:

$$\frac{\alpha^2}{4\pi} = 2.$$

Estimates by complex translation

Recall $\mathbb{E}_\Gamma$ applies to fluctuation field ζ. For F analytic, by

$$\zeta \mapsto \zeta + i\Gamma f \qquad \text{in } \varphi = \varphi' + \zeta$$

$$\mathbb{E}_\Gamma[F(\varphi)] = e^{\frac{1}{2}(f,\Gamma f)}\mathbb{E}_\Gamma[e^{-i(\zeta,f)}F(\varphi + i\Gamma f)]$$

If F behaves like $e^{iq\alpha\varphi_x}$ choose f so that $iq\alpha\zeta_x - i(\zeta, f) = 0$.

McBryan, O. A. and Spencer, T. (1977). On the decay of correlations in $\mathrm{SO}(n)$-symmetric ferromagnets. *Comm. Math. Phys.*, 53(3):299–302

Fröhlich, J. and Spencer, T. (1981). On the statistical mechanics of classical Coulomb and dipole gases. *J. Stat. Phys*, 24:617–701

Dimock, J. and Hurd, T. R. (2000). Sine-Gordon revisited. *Ann. Henri Poincaré*, 1(3):499–541

Complex translation $\varphi \to \varphi + i\Gamma f$ applied to

$$\widehat{F}(q,\varphi) = e^{iq\alpha\varphi_{x_0}} \widehat{F}(q, \varphi - \varphi_{x_0})$$

gives

$$\mathbb{E}_\Gamma[\widehat{F}(q,\varphi)] = e^{\frac{1}{2}(f,\Gamma f) - \alpha q (\Gamma f)_{x_0}} \mathbb{E}_\Gamma[e^{iq\alpha\varphi_{x_0} - i(\zeta,f)} \widehat{F}(q, \underbrace{\varphi - \varphi_{x_0}}_{\delta\varphi} + \underbrace{i\Gamma f - i(\Gamma f)_{x_0}}_{i\delta\psi})] \, .$$

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The optimal choice is $f_x = \alpha q \delta_{x_0, x}$, but it would require analyticity of F in a strip of width $O(q)$ — which is unbounded.

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$$\widehat{F}(q,\varphi) = e^{iq\alpha\varphi_{x_0}} \widehat{F}(q, \varphi - \varphi_{x_0})$$

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The optimal choice is $f_x = \alpha q \delta_{x_0,x}$, but it would require analyticity of F in a strip of width $O(q)$ — which is unbounded.

Instead choose $f_x = \alpha \, \mathrm{sign}(q) \delta_{x_0,x}$. Then

$$e^{\frac{1}{2}(f,\Gamma f) - \alpha q(\Gamma f)_{x_0}} = e^{-(|q| - \frac{1}{2})\alpha^2 \Gamma_j(0)} \sim L^{-2(2|q|-1)\frac{\alpha^2}{8\pi}} .$$

- Still decays faster than volume factor L^2 if $|q| \geq 2$ or $\alpha^2 > 8\pi$.
- Using q instead of sign(q) would have recovered $L^{-q^2 \frac{\alpha^2}{8\pi}}$.

Estimates by complex translation (ii)

Still need to estimate:

$$e^{iq\alpha\varphi'_{x_0}} \mathbb{E}_\Gamma[e^{iq\alpha\zeta_{x_0} - i(\zeta,f)} \widehat{F}(q, \delta\zeta + \delta\varphi' + i\delta\psi)] \,.$$

This is a function of the field at the next scale φ'.

Estimates by complex translation (ii)

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This is a function of the field at the next scale φ'.

Use carefully chosen norm on such functions.

- Product property
  $\rightarrow$ charged and dipolar parts can be estimated individually.
- Norm guarantees analyticity in a strip.
- Tests against fields of typical size of Gaussian fluctuation field.
  $\rightarrow$ implements scaling heuristics for $\nabla\varphi$

Norm (i): small fields

The fluctuation covariance obeys

$$\nabla^\alpha \Gamma_j(x) = O(L^{-j|\alpha|_j}).$$

Thus: $\|\zeta\|_{C_j^2(B)}$ is bounded for a typical fluctuation field ζ.

First attempt: for $F = F(\varphi, X)$ define

$$\|F\|_{h,T_j} = \sup_{\varphi} \sum_{n \geq 0} \frac{h^n}{n!} \sup_{\|\zeta_k\|_{C_j^2(X) \leq 1}} |D_\varphi^n F(\varphi, X) \cdot (\zeta_1, \ldots, \zeta_n)|.$$

- Guarantees analyticity in a strip of width h.
- Product property (Taylor series of product is product of Taylor series).

Issue: sup over φ too strong.

Norm (ii): large fields

Fluctuation fields typically obeys $\nabla\zeta = O(L^{-j})$.

Implement this using a weight:

$$G_j(\varphi, X) = e^{c_1 \kappa_L \|\nabla_j \varphi\|^2_{L^2_j(X)} + \cdots}$$

Lemma

If $\kappa_L = c(\log L)^{-1}$ *with* $c > 0$ *and small enough: for* X *small,*

$$\mathbb{E}_j \left[G_j(\varphi' + \zeta, X) \right] \leq C G_{j+1}(\varphi', \bar{X}).$$

Norm weighted in the field:

$$\|F(X)\|_{h,T_j} = \sup_{\varphi} \frac{1}{G_j(\varphi, X)} \left[\sum_{n \geq 0} \frac{h^n}{n!} (\cdots) \right]$$

- Effectively reduces estimates to $\nabla\varphi = O(\frac{1}{\sqrt{\kappa_L}})$.

Norm (iii): large sets

Given a parameter $A > 1$

$$\|F\|_{h,T_j} = \sup_X A^{|X|_j} \|F(X)\|_{h,T_j}$$

- Only small X are important (locality).

Estimate of charged part

Have so far expressed (recall: $\varphi = \varphi' + \zeta$)

$$\mathbb{E}_\Gamma[\widehat{F}(q,\varphi)] = L^{-2(2|q|-1)\frac{\alpha^2}{8\pi}} e^{iq\alpha\varphi'_{x_0}} \mathbb{E}_\Gamma[e^{iq\alpha\zeta_{x_0} - i(\zeta,f)} \widehat{F}(q, \delta\zeta + \delta\varphi' + i\delta\psi)]\,.$$

The charge is potentially dangerous:

$$\|e^{iq\alpha\varphi'_{x_0}}\|_{h,T_{j+1}(\varphi',X)} \leq e^{h|q|\alpha}.$$

For $|q| \geq 2$ this is okay since h is independent of L and the good prefactor can be made arbitrarily small by choosing L large.

Summary:

- $|q| \geq 2$: Irrelevant.
- $|q| = 1$: Marginal.
- $|q| = 0$: Dipole gas [do not do complex translation].

Dipolar part

After restricting to a charge sector, $\widehat{F}(q,\varphi)$ is effectively a function of $\nabla\varphi$ (gradient field). Only need to consider $q = 0, \pm 1$.

- $|q| = 0$: constants are relevant, $(\nabla\varphi)^2$ is marginal;
- $|q| = 1$: constants marginal.

For example consider $|q| = 1$. Then by Taylor expansion in $\delta\varphi'$:

$$\left|\widehat{F}(q, \delta\zeta + \delta\varphi' + i\delta\psi) - \widehat{F}(q, \delta\zeta + 0 + i\delta\psi)\right| \\ \leq O(L^{-1}) \left(1 + L\|\delta\varphi'\|_{C_j^2(X)}\right) G_j(\varphi, X) \, \|F\|_{h,T_j}$$

Taylor expansion:

$$\left|\widehat{F}(q, \delta\zeta + \delta\varphi' + i\delta\psi) - \widehat{F}(q, \delta\zeta + 0 + i\delta\psi)\right|$$
$$\leq O(L^{-1}) \left(1 + L\|\delta\varphi'\|_{C_j^2(X)}\right) G_j(\varphi', X) \, \|F\|_{h, T_j}$$

Factor $L^{-1} \ll 1$ from change of test fields:

$$\varphi' \text{ is smoother than } \varphi = \zeta + \varphi' \, ,$$

$$\nabla\varphi' \sim L^{-(j+1)} \quad \text{vs.} \quad \nabla\varphi \sim L^{-j} \, .$$

Taylor expansion:

$$\left|\widehat{F}(q, \delta\zeta + \delta\varphi' + i\delta\psi) - \widehat{F}(q, \delta\zeta + 0 + i\delta\psi)\right|$$
$$\leq O(L^{-1}) \left(1 + L\|\delta\varphi'\|_{C_j^2(X)}\right) G_j(\varphi', X) \, \|F\|_{h,T_j}$$

Analyticity strip is uniform:

$$L^{-1}h + \|\delta\psi\|_{C_j^2(X)} \leq h \,.$$

Taylor expansion:

$$\left|\widehat{F}(q, \delta\varphi' + i\delta\psi) - \widehat{F}(q, 0 + i\delta\psi)\right|$$
$$\leq O(L^{-1})\left(1 + L\|\delta\varphi'\|_{C_j^2(X)}\right) G_j(\varphi', X)\ \|F\|_{h,T_j}$$

Due to weight G_j, can effectively assume

$$1 + L\|\delta\varphi'\|_{C_j^2(X)} = O\left(\frac{1}{\sqrt{\kappa_L}}\right) = O(\sqrt{\log L}) .$$

In fact, for small X,

$$\mathbb{E}_j\left[\left(1 + L\|\delta\varphi'\|_{C_j^2(X)}\right) G_j(\varphi, X)\right] \leq O\left(\frac{1}{\sqrt{\kappa_L}}\right) G_{j+1}(\varphi', \bar{X}) .$$

Upshot

Recall:

$$\Omega_j(\Phi) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y)$$

Upshot

Recall:

$$\Omega_j(\Phi) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y)$$

Theorem

There exist $\tilde{U}_j$ and $\tilde{K}_j$ such that

$$\Omega_j(\Phi) = \sum_{X \in \mathcal{P}_j} e^{\tilde{U}_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} \tilde{K}_j(\Phi, Y)$$

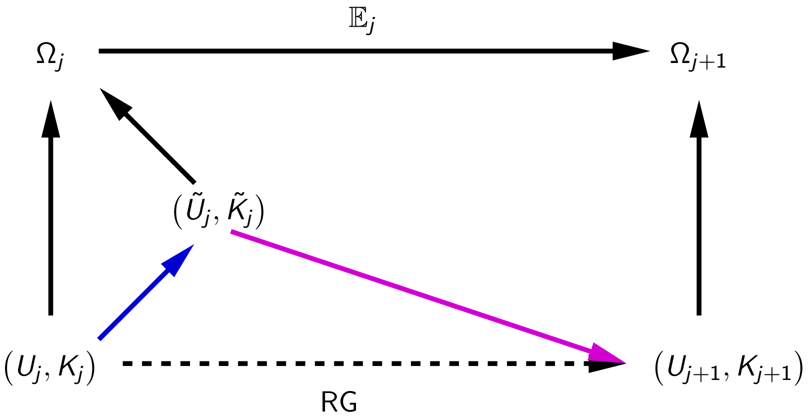
with $\tilde{K}_j$ given by the irrelevant parts of K_j .

- $(U_j, K_j) \mapsto \tilde{U}_j$ are nonperturbative third-order adjustments to coupling constants (corrections in $\approx$ in KT equations).

Brydges, D. C. (2009). Lectures on the renormalisation group.
In *Statistical Mechanics*, volume 16 of *IAS/Park City Math. Ser.*, pages 7–93. Amer. Math. Soc., Providence, RI

Brydges, D. C. and Slade, G. (2015). A Renormalisation Group Method. V. A Single Renormalisation Group Step.
J. Stat. Phys., 159(3):589–667

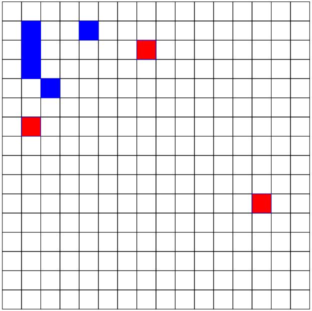
The renormalisation group



Theorem

For the combination of the blue and magenta arrows $(U_j, K_j) \mapsto K_{j+1}$ *is contractive.*

Definition of $(U_j, K_j) \mapsto K_{j+1}$



Expand $\Omega_j =$

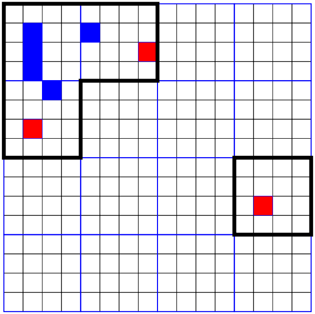
$$\sum_{X \in \mathcal{P}_j} e^{\tilde{U}_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} \tilde{K}_j(\Phi, Y)$$

using, in each small block,

$$\varphi = \varphi' + \zeta_j$$

$$e^{\tilde{U}_j(\varphi' + \zeta_j)} = e^{U_{j+1}(\varphi')} + \text{difference}.$$

Definition of $(U_j, K_j) \mapsto K_{j+1}$



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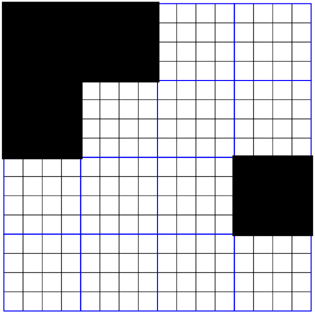
Sum over configurations with fixed closure X.

Definition of $(U_j, K_j) \mapsto K_{j+1}$



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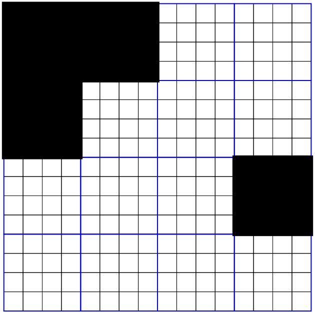
Sum over configurations with fixed closure X. For a small X,

$$K_{j+1}(X) \approx \mathbb{E}_j\Big(\text{sum over ways to fill } X\Big) \approx \sum_{\overline{Y} = X, |Y|_j \le 2^d} \mathbb{E}\tilde{K}_j(Y)$$

$$\approx O(L^2) O(L^{-3} (\log L)^{3/2}) \|K_j\| \, .$$

Definition of $(U_j, K_j) \mapsto K_{j+1}$



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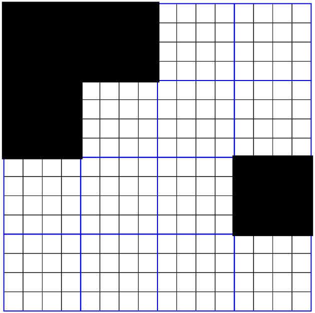

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$$\approx O(L^2) O(L^{-3} (\log L)^{3/2}) \|K_j\| .$$

Pierluigi's list of open problems

In a talk given in 2011 (video on IAS website), Pierluigi mentioned the following open problems.

- Correlation functions including logarithmic corrections [solved]
- Analyticity in z inside the dipole phase and Borel summability on the KT line.
- Extension to other models discussed by Fröhlich–Spencer? XY, Villain, discrete Gaussian, Z_n-clock, and solid-on-solid.
- Equivalence of Coulomb gas and other 2D probabilitic models at criticality: Ashkin–Teller, six-vertex, Q-state and antiferromagnetic Potts model, $O(n)$-models including self-avoiding walk.

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Fröhlich, J. and Spencer, T. (1981). The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas.
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Nienhuis, B. (1987)). Coulomb gas formulation of two-dimensional phase transitions.
In Domb, C. and Lebowitz, J., editors, *Phase Transitions and Critical Phenomena*, volume 11, New York. Academic Press

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