The rigorous construction of the 1D Extended Hubbard model by RG techniques

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A conference in memory of Pierluigi Falco
INFN, Frascati, 9-11 June 2015

Outline

Introduction

Abstract

References

The extended Hubbard model

The model

The free model

Anomalous exponents and logarithmic corrections

Universal relations

Borel summability

Main ingredients of the proof

The strategy

The multiscale expansion

Renormalization

Asymptotic Gauge Invariance

The lattice Ward identities

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References
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The free model
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The strategy
The multiscale expansion
Renormalization
Asymptotic Gauge Invariance
The lattice Ward identities

Abstract

In the last twenty-five years, many people in Rome have studied various types of Fermion models, by applying rigorous RG techniques. This line of research was open in 1990 by Giovanni Gallavotti and myself in a paper, published on JSP, on the weakly interacting Fermi gas in one and three dimensions.

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In this talk I will give a brief review of the results that have been obtained by Pierluigi Falco, Vieri Mastropietro and myself, in the case of the one dimensional extended Hubbard model, (a gas of fermions of spin 1/2 on the one dimensional lattice) at weak coupling and generic short range interaction, satisfying a *positivity condition*, to be defined later.

These results are the content of two long papers published on CMP on January 2014, after almost three years of work. This is the last work that Pierluigi could see published in his too brief life.

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I have to say that we could achieve this result mainly because Pierluigi first convinced Vieri and me to write a new paper on the subject, by completing the research we had done before, and, in the following, stimulated our efforts to make it as self-containing as possible, as requested by the CMP referees.

Results

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- Borel summability of perturbation theory.

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- References

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- Abstract
- References

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- The model
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The model

We study the Grand Canonical state

$$\langle A \rangle_{L,\beta} := \frac{\mathrm{Tr}[e^{-\beta H} A]}{\mathrm{Tr}[e^{-\beta H}]}, \quad \text{with}$$

$$H = -\frac{1}{2} \sum_{\substack{x \in \mathcal{C} \\ s=\pm}} (a^+_{x,s} a^-_{x+1,s} + a^+_{x,s} a^-_{x-1,s}) + \bar{\mu} \sum_{\substack{x \in \mathcal{C} \\ s=\pm 1}} a^+_{x,s} a^-_{x,s} +$$

$$+ \lambda \sum_{\substack{x,y \in \mathcal{C} \\ s,s'=\pm 1}} v_L(x-y) a^+_{x,s} a^-_{x,s} a^+_{y,s'} a^-_{y,s'}$$

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- $-\bar{\mu} \in (-1,+1)$ is the chemical potential

In particular, in the case

$$\bar{\mu} = \cos(\bar{p}_F) \neq 0, \quad \hat{v}(2\bar{p}_F) > 0, \quad 0 \leq \lambda \leq \varepsilon_0$$

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and $\lim_{\beta\to\infty} \lim_{L\to\infty}$ of the Schwinger functions

$$S_n^{\beta,L}(\mathbf{x}_1, \varepsilon_1, s_1; \ldots; \mathbf{x}_n, \varepsilon_n, s_n) = \langle \mathbf{T}\{a^{\varepsilon_1}_{\mathbf{x}_1,s_1} \cdots a^{\varepsilon_1}_{\mathbf{x}_n,s_n}\}\rangle^T_{\beta,L}$$

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- $a^{\pm}_{\mathbf{x},s} = e^{x_0 H} a^{\pm}_x e^{-Hx_0}, \quad \mathbf{x} = (x, x_0)\ , \quad 0 \leq x_0 < \beta$
- $\langle \cdot \rangle^T_{L,\beta}$ is the truncated expectation
- $\mathbf{T}$ is the operator of time ordering

We study also the response functions associated to the densities

$$\rho^C_{\mathbf{x}} = \sum_{s=\pm} a^+_{\mathbf{x},s} a^-_{\mathbf{x},s}, \quad \rho^{S_i}_{\mathbf{x}} = \sum_{s,s'=\pm} a^+_{\mathbf{x},s} \sigma^{(i)}_{s,s'} a^-_{\mathbf{x},s'}$$

$$\rho^{SC}_{\mathbf{x}} = \frac{1}{2} \sum_{\substack{s=\pm \\ \varepsilon=\pm}} s\, a^\varepsilon_{\mathbf{x},s} a^\varepsilon_{\mathbf{x},-s}, \quad \rho^{TC_i}_{\mathbf{x}} = \frac{1}{2} \sum_{\substack{s,s'=\pm \\ \varepsilon=\pm}} a^\varepsilon_{\mathbf{x},s} \tilde{\sigma}^{(i)}_{s,s'} a^\varepsilon_{\mathbf{x}+\mathbf{e},s'}$$

where $i = 1, 2, 3$, $\mathbf{e} = (1, 0)$ and

$$\sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{\sigma}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \tilde{\sigma}^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tilde{\sigma}^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the paramagnetic current

$$J_{\mathbf{x}} = \frac{1}{2i} \sum_{s=\pm} [a^{+}_{\mathbf{x}+\mathbf{e},s} a^{-}_{\mathbf{x},s} - a^{+}_{\mathbf{x},s} a^{-}_{\mathbf{x}+\mathbf{e},s}]$$

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The density and current response functions are defined by

$$\Omega_{\alpha,\beta,L}(\mathbf{x}-\mathbf{y}) := \langle \mathbf{T} \rho^{\alpha}_{\mathbf{x}} \rho^{\alpha}_{\mathbf{y}} \rangle^{T}_{\beta,L} := \langle \mathbf{T} \rho^{\alpha}_{\mathbf{x}} \rho^{\alpha}_{\mathbf{y}} \rangle_{\beta,L} - \langle \rho^{\alpha}_{\mathbf{x}} \rangle_{\beta,L} \langle \rho^{\alpha}_{\mathbf{y}} \rangle_{\beta,L}$$

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If $\mathbf{x}-\mathbf{y} = (\xi, \tau)$, they are L-periodic in $\xi \in \mathbb{Z}$ and β-periodic in $\tau \in \mathbb{R}$; hence, if $F_{\beta,L}$ is any function of this type,

$$\hat{F}_{\beta,L}(\mathbf{p}) = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x \in \mathcal{C}} e^{i\mathbf{p}\mathbf{x}} \, F_{\beta,L}(\mathbf{x})$$

$\mathbf{p} = (p, p_0)$, $p = \frac{2\pi n}{L}$, $-[\frac{L}{2}] \leq n \leq [\frac{L-1}{2}]$, $p_0 \in \frac{2\pi}{\beta}\mathbb{Z}$.

We are interested in the zero temperature limit of the Schwinger functions, response functions and the vertex functions

$$G^{2,1}_{\rho,\beta,L}(\mathbf{x},\mathbf{y},\mathbf{z}) := \langle \mathbf{T}\rho^{(C)}_{\mathbf{x}} a^-_{\mathbf{y}} a^+_{\mathbf{z}} \rangle^T_{\beta,L}$$

$$G^{2,1}_{j,\beta,L}(\mathbf{x},\mathbf{y},\mathbf{z}) := \langle \mathbf{T} J_{\mathbf{x}} a^-_{\mathbf{y}} a^+_{\mathbf{z}} \rangle_{T,\beta,L}$$

calculated in the thermodynamic limit (same symbols, deprived of the β and L labels).

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Several important thermodynamic quantities can be deduced from the knowledge of the response functions. In particular the susceptibility, which is given by

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$$\kappa := \lim_{p\to 0}\lim_{p_0\to 0}\hat{\Omega}_C(\mathbf{p})$$

and the Drude weight, which is defined as

$$D = -\langle \tau_x\rangle - \lim_{p_0\to 0}\lim_{p\to 0} D(\mathbf{p}), \quad D(\mathbf{p}) \equiv \hat{\Omega}_{j,j}(\mathbf{p})$$

$$\tau_{\mathbf{x}} = -\frac{1}{2}\sum_{s=\pm}[a^+_{\mathbf{x},s}a^-_{\mathbf{x}+\mathbf{e},s} + a^+_{\mathbf{x}+\mathbf{e},s}a^-_{\mathbf{x},s}]$$

If one assumes analytic continuation in p_0 around $p_0 = 0$, one can compute the conductivity in the linear response approximation by the Kubo formula, that is

$$\sigma = \lim_{\omega \to 0} \lim_{\delta \to 0} \frac{\hat{D}(-i\omega + \delta, 0)}{-i\omega + \delta}$$

Therefore, a nonvanishing D indicates infinite conductivity.

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The free model

In absence of interaction, the Hamiltonian looks like

$$H_0 = -\frac{1}{2} \sum_{\substack{x \in \mathcal{C} \\ s=\pm}} (a^+_{x,s} a^-_{x+1,s} + a^+_{x,s} a^-_{x-1,s}) + \mu \sum_{\substack{x \in \mathcal{C} \\ s=\pm 1}} a^+_{x,s} a^-_{x,s}$$

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Being H_0 quadratic, every correlation function can be easily calculated in terms of the free propagator

$$g^{\beta,L}(\mathbf{x}-\mathbf{y}) = \frac{\mathrm{Tr}\left[e^{-\beta H_0} \mathbf{T}(a^-_{\mathbf{x}} a^+_{\mathbf{y}})\right]}{\mathrm{Tr}[e^{-\beta H_0}]} =$$

$$= \lim_{N\to\infty} \frac{1}{\beta} \sum_{k\in\mathcal{D}_{L,\beta}, |k_0|\le N} \frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 + e(k)}, \quad e(k) = \mu - \cos k$$

$$\mathcal{D}_{L,\beta} := \mathcal{D}_L \times \mathcal{D}_\beta, \quad \mathcal{D}_L := \frac{2\pi}{L}\mathcal{C}, \quad \mathcal{D}_\beta := \frac{2\pi}{\beta}(\mathbb{Z} + \frac{1}{2})$$

$$\mathbf{x}-\mathbf{y} \neq (0, n\beta)$$

Ultraviolet singularity

If $\mu = \cos p_F$ and $g(\mathbf{x}) \equiv \lim_{\beta, L \to \infty} g^{\beta, L}(\mathbf{x})$

$$g(0,0) = g(0,0^-) = -p_F/\pi, \quad g(0,0^+) - g(0,0^-) = 1$$

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The Fermi momentum p_F appears also in the infrared singularity of $\hat{g}(\mathbf{k})$ at $\mathbf{k} = (0, \pm p_F)$, which produces a large distance behavior of the propagator of the form

$$g(\mathbf{x}) \sim \sum_{\omega=\pm} \frac{e^{-i\omega p_F x}}{v_F x_0 + i\omega x}, \quad v_F \equiv \sin p_F$$

where $\sim$ means up to faster decaying terms; v_F is usually called the Fermi velocity.

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Analogously, the response functions are sums of non oscillating and oscillating terms (with period π/p_F) decaying as $|\mathbf{x} - \mathbf{y}|^{-2}$.

Finally the susceptibility and the Drude weight are given by:

$$\kappa = \frac{1}{\pi v_F} \quad , \quad D = \frac{v_F}{\pi}$$

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Introduction
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The free model
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The strategy
The multiscale expansion
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Anomalous exponents and logarithmic corrections

Given $p_F \neq 0, \pi/2, \pi$ and an interaction $\lambda v(x)$ with $\hat{v}(2p_F) > 0$, there exists $\lambda_0 > 0$ and a unique chemical potential

$$-\bar{\mu} = -\mu - \nu(\lambda, \mu), \quad \mu = \cos p_F, \quad v_F = \sin p_F$$

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such that, if $0 \leq \lambda \leq \lambda_0$,

- $\nu(\lambda, \mu)$ is smooth and $O(\lambda)$ and this equation can be solved with respect to p_F: $p_F(\bar{\mu}, \lambda) = p_F + O(\lambda)$.

Anomalous exponents and logarithmic corrections

Given $p_F \neq 0, \pi/2, \pi$ and an interaction $\lambda v(x)$ with $\hat{v}(2p_F) > 0$, there exists $\lambda_0 > 0$ and a unique chemical potential

$$-\bar{\mu} = -\mu - \nu(\lambda, \mu), \quad \mu = \cos p_F, \quad v_F = \sin p_F$$

such that, if $0 \leq \lambda \leq \lambda_0$,

- $\nu(\lambda, \mu)$ is smooth and $O(\lambda)$ and this equation can be solved with respect to p_F: $p_F(\bar{\mu}, \lambda) = p_F + O(\lambda)$.
- if we take the limit $L = \infty$, followed from the limit $\beta = \infty$, then

$$S_2(\mathbf{x}) := \langle \mathbf{T}\{a^-_{\mathbf{x},s} a^+_{\mathbf{x},s}\}\rangle \sim \left[\bar{S}_0(\mathbf{x}) + R_2(\mathbf{x})\right] \frac{L(\mathbf{x})^{\zeta_z}}{|\tilde{\mathbf{x}}|^{1+\eta}}$$

$$\bar{S}_0(\mathbf{x}) := \frac{v_F}{\pi} \frac{x_0 \cos p_F - x}{|\tilde{\mathbf{x}}|}, \quad \tilde{\mathbf{x}} := (x, v_F x_0)$$

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For the charge and spin density response functions ($\alpha = C$ or $\alpha = S_i$), we get

$$\Omega_\alpha(\mathbf{x}) \sim \frac{\bar{\Omega}_0(\mathbf{x}) + R_\alpha(\mathbf{x})}{\pi^2|\tilde{\mathbf{x}}|^2} + \cos(2p_F x)\frac{L(\mathbf{x})^{\zeta_\alpha}}{\pi^2|\tilde{\mathbf{x}}|^{2X_\alpha}}\left[1 + \tilde{R}_\alpha(\mathbf{x})\right]$$

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Similar results for the other response functions.

Outline

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- Abstract
- References

The extended Hubbard model

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Universal relations

There are two functions of λ,

$$K(\lambda) = 1 - c\lambda + O(\lambda^2), \quad \bar{K}(\lambda) = 1 - c\lambda + O(\lambda^2)$$

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such that the critical exponents of the model satisfy the extended scaling relations

$$4\eta = K + K^{-1} - 2 \,, \qquad 2X_C = 2X_{S_i} = K + 1 \,,$$
$$2X_{TC_i} = 2X_{SC} = K^{-1} + 1 \,, \qquad 2\tilde{X}_{SC} = K + K^{-1} \,.$$

and the equations

$$\hat{\Omega}_C(\mathbf{p}) = \frac{\bar{K}}{\pi v} \frac{v^2 p^2}{p_0^2 + v^2 p^2} + A(\mathbf{p})$$

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Luttinger liquid relation $\qquad v^2 = D/\kappa$

Outline

Introduction
- Abstract
- References

The extended Hubbard model
- The model
- The free model
- Anomalous exponents and logarithmic corrections
- Universal relations
- Borel summability

Main ingredients of the proof
- The strategy
- The multiscale expansion
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- Asymptotic Gauge Invariance
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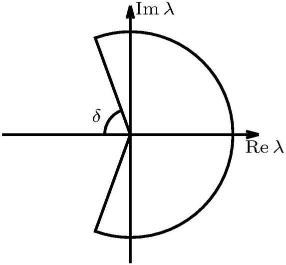
Borel summability

Given $\delta \in (0, \pi/2)$, there exists $\varepsilon = \varepsilon(\delta) > 0$, such that the free energy, the two-points Schwinger functions and the density correlations are analytic in the set

$$D_{\varepsilon,\delta} = \{\lambda \in \mathbb{C} : 0 < |\lambda| < \varepsilon, \\ |\text{Arg } \lambda| < \pi - \delta\}$$



continuous in the closure $\bar{D}_{\varepsilon,\delta}$ and satisfy the hypotheses of Watson Theorem on Borel summability at $\lambda = 0$.

$$f(\lambda) = \sum_{k=0}^{n} a_k \lambda^k + R_k(\lambda)$$

$$|a_k| \le C\sigma^k k!, \quad |R_k(\lambda)| \le C(\sigma|\lambda|)^{k+1}(k+1)!$$

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Let us consider, for example, the free energy $E(\lambda) = \lim_{h\to-\infty}\sum_h^0 E_j(\lambda)$. We have to check that, for any h, $E_h(\lambda)$ is analytic in a set

$$D^{(h)}_{\varepsilon,\delta} = D_{\varepsilon,\delta} \cup \left\{ |\lambda| \le \frac{c_0}{1+|h|} \right\}$$

and that $|E_h(\lambda)| \le c_1 e^{-\kappa|h|}$. This condition is a very simple consequence of our analysis.

Outline

Introduction
Abstract
References

The extended Hubbard model
The model
The free model
Anomalous exponents and logarithmic corrections
Universal relations
Borel summability

Main ingredients of the proof
The strategy
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Gauge Invariance in presence of an infrared cutoff for a solvable *reference model* with the same asymptotic behavior.

The correlations and the critical indices can be exactly computed in the reference model, when the infrared cutoff is removed, in terms of the coupling. This allows to prove in this model some simple scaling relations.

It is possible to choose the parameters in the reference model, so that its asymptotic behavior is exactly the same as that of the Hubbard model, up to logarithmic corrections. The critical indices are the same, then satisfy the same scaling relations.

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The charge and current correlations of the Hubbard model and of the reference model are asymptotically the same (no logarithmic corrections), up to some renormalization constants, which can not be explicitely calculated. The Luttinger Liquid relation follows from some relations between these constants, which derive both from the Gauge Invariance of the reference model and the exact Ward identities verified by the Hubbard model.

Outline

Introduction
- Abstract
- References

The extended Hubbard model
- The model
- The free model
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- Universal relations
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Main ingredients of the proof
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The generating functional

To control the perturbation expansion, it is convenient to use the functional representation, which allows very simply to make the needed resummations, before performing the bounds.

$$\mathcal{W}(J,\eta) = \log \int P(d\psi) \exp[-\mathcal{V}(\psi) + \mathcal{B}(\psi, J, \eta)]$$

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$$\mathcal{V}(\psi) = \lambda \sum_{s,s'=\pm} \int d\mathbf{x} d\mathbf{y}\, \psi^+_{\mathbf{x},s}\psi^-_{\mathbf{x},s} v(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{y},s'}\psi^-_{\mathbf{y},s'} + \nu \sum_{s=\pm} \int d\mathbf{x}\, \psi^+_{\mathbf{x},s}\psi^-_{\mathbf{x},s}$$

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$$\mathcal{B}(\psi, J, \eta) = \sum_\alpha \int d\mathbf{x} J^\alpha_{\mathbf{x}} \rho^\alpha_{\mathbf{x}} + \sum_s \int d\mathbf{x} [\eta^+_{\mathbf{x},s} \psi^-_{\mathbf{x},s} + \psi^+_{\mathbf{x},s} \eta^-_{\mathbf{x},s}]$$

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$$v(\mathbf{x}-\mathbf{y}) = \delta(x_0 - y_0) v_L(x-y), \quad \int d\mathbf{x} := \sum_{x\in\mathcal{C}} \int_{-\beta/2}^{\beta/2}$$

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$$\int P(d\psi) \psi^-_{\mathbf{x},s} \psi^+_{\mathbf{y},s} = g(\mathbf{x}-\mathbf{y})$$

The decomposition of the free measure

There is no IR problem at β finite, since $|k_0| \geq \pi/\beta$, but there is a mild UV problem, since $[-ik_0 + e(k)]^{-1}$ is not $\mathbf{L^1}$ in k_0.

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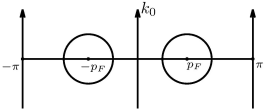
To get *good bounds* on the perturbation theory at finite β, one has to introduce an UV cutoff on k_0 and use a multiscale expansion.

If $\chi(\mathbf{k}')$ and $\chi_0(k_0)$ are two suitable compact support cutoff functions, $L >> \beta$, $\gamma > 1$ and $\pi/\beta = \gamma^{h_\beta}$



$$1 = \lim_{M\to\infty} \sum_{h=1}^{M} \hat{f}_1(\mathbf{k})\tilde{f}_h(k_0) + \sum_{h=h_\beta}^{0} \sum_{\omega=\pm 1} \hat{f}_h(k - \omega p_F, k_0)$$

$$\hat{f}_1(\mathbf{k}) := [1 - \chi(k - p_F, k_0) - \chi(k + p_F, k_0)]$$

$$\tilde{f}_h(k_0) = \chi(\gamma^{-h}k_0) - \chi(\gamma^{-h+1}k_0), \quad \hat{f}_h(\mathbf{k}') = \chi(\gamma^{-h}\mathbf{k}') - \chi(\gamma^{-h+1}\mathbf{k}')$$

The effective potentials

$$g(\mathbf{x}) = g^{(1)}(\mathbf{x}) + \sum_{h=h_\beta}^{0} \sum_{s,\omega=\pm 1} e^{-i\omega p_F x} g^{(h)}_{\omega,s}(\mathbf{x})$$

$$\psi^{\pm}_{\mathbf{x}} = \psi^{(1)\pm}_{\mathbf{x}} + \sum_{h=h_\beta}^{0} \sum_{s,\omega=\pm 1} e^{-i\omega p_F x} \psi^{(h)\pm}_{\mathbf{x},\omega,s}$$

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The limit $M \to \infty$ is essentially trivial; one has only to resum the tadpole graphs and to adjust their values by some finite counterterms to control the multiscale expansion and get the right perturbation theory.

Hence, we consider the model with an UV cutoff on the scale $h = 0$ and an IR cutoff on scale $h_{L,\beta}$, such that $\gamma^{h_{L,\beta}} = \min\{\pi/L, \pi/\beta\}$. Hence we define, for any h such that $h_{L,\beta} \leq h \leq 0$:

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$$e^{\mathcal{W}_{L,\beta}} = e^{-L\beta E_h + \mathcal{S}_h(J,\eta)} \int P(d\psi^{\leq h})\, e^{-\mathcal{V}^{(h)}(\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\psi^{(\leq h)}, J, \eta)}$$

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In the limit $L, \beta \to \infty$, the free energy is given by $\lim_{h\to-\infty} E_h$, while the Schwinger functions and the correlation functions of the ρ^α densities are obtained by suitable functional derivatives at $J = \eta = 0$ of $\mathcal{S}_{-\infty}(J, \eta)$.

The tree expansion for $\mathcal{V}^{(h)}$

In order to describe the tree expansion, it is sufficient to consider only the case $J = \eta = 0$, which is all we need to calculate the free energy.

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$$\mathcal{V}^{(h-1)}(\psi^{(\leq h-1)}) + L\beta e_h = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{s!} \mathcal{E}_h^T \left[\mathcal{V}^{(h)}(\psi^{(\leq h)}); s \right]$$

$$\psi^{(\leq h)} = \psi^{(\leq h-1)} + \psi^{(h)}, \quad E_h = \sum_{j=h}^{0} e_j$$

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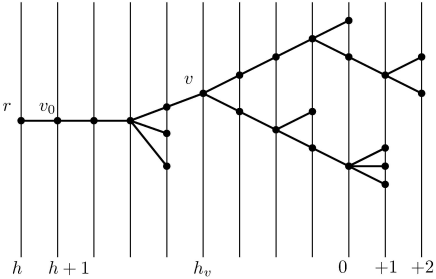
By iteration, we see that the r.h.s. can be written as

$$\sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_n} \bar{\mathcal{V}}^{(h)}(\tau, \psi^{(\leq h)})$$

$\mathcal{T}_n$ is a family of labeled trees with a root and n ordered end-points, each one associated with one of the two terms in the interaction.



There are trivial and non trivial vertices. With each vertex v, we associate a scale label h_v, a set of external legs P_v and a Kernel $K_v(\mathbf{x}_v)$, where $\mathbf{x}_v$ is the set of vertices where the external legs are based on .

The dimensional bound without renormalization

$$\bar{\mathcal{V}}^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{P_{v_0}, \underline{\varepsilon}, \underline{\mathbf{s}}, \underline{\omega}} \int d\mathbf{x}_{v_0} G(\mathbf{x}_{v_0}) \tilde{\psi}(P_{v_0}, \underline{\varepsilon}, \underline{\mathbf{s}}, \underline{\omega}, \mathbf{x}_{v_0})$$

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$$(L\beta)^{-1} \int d\mathbf{x}_{v_0} |G(\mathbf{x}_{v_0})| \leq (C\varepsilon_0)^n \sum_{\{h_v, P_v, v > v_0\}} \sum_{T} \cdot$$

$$\cdot\, \gamma^{-(D_{v_0} + n_{2,v_0})h} \Big[\prod_{v \text{ non trivial}} \frac{1}{s_v!} \gamma^{-(h_v - h_{v'})(D_v + n_{2,v})} \Big]$$

where $\varepsilon_0 = \max\{|\lambda|, |\nu|\}$, $n_{2,v}$ is the number of endpoints of type ν following v, and

$$D_v = -2 + \frac{1}{2}|P_v| \qquad \textit{scaling dimension}$$

Outline

Introduction
Abstract
References

The extended Hubbard model
The model
The free model
Anomalous exponents and logarithmic corrections
Universal relations
Borel summability

Main ingredients of the proof
The strategy
The multiscale expansion
Renormalization
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Renormalization

$$e^{\mathcal{W}(J,\eta)} = e^{-L\beta E_j+\mathcal{S}_j(J,\eta)} \int P_{Z_j}(d\psi^{\leq j})\, e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{(\leq j)})+\mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{(\leq j)},J,\eta)}$$

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$$P_Z(d\psi) \sim \frac{d\psi}{\mathcal{N}} \exp\left\{-\frac{Z}{L\beta}\sum_{\omega,s}\sum_{\mathbf{k}\in\mathcal{D}} C_{h,h_\beta}(\mathbf{k})(-ik_0 + \omega v_F k)\hat{\psi}^+_{\mathbf{k},\omega,s}\hat{\psi}^-_{\mathbf{k},\omega,s}\right\}$$

$$C_{h,h_\beta}(\mathbf{k}) = \left[\sum_{j=h_\beta}^{h} \hat{f}_j(\mathbf{k})\right]^{-1}$$

We take any *cluster* appearing in the expansion of

$$\bar{\mathcal{V}}^{(j)}(\tau, \psi^{(\leq j)}) = \sum_{P_{v_0}, \underline{\varepsilon}, \underline{s}, \underline{\omega}} \int d\mathbf{x}_{v_0} G(\mathbf{x}_{v_0}) \tilde{\psi}(P_{v_0}, \underline{\varepsilon}, \underline{s}, \underline{\omega}, \mathbf{x}_{v_0})$$

and, if $|\mathbf{x}_{v_0}| = 2, 4$, we *localize* it by choosing a point $\mathbf{x}$ in the set $\mathbf{x}_{v_0}$ and by doing a Taylor expansion with rest of order $z(|P_{v_0}|)$ at $\mathbf{x}$ of the fields based on the other points of $\mathbf{x}_{v_0}$, with

$$z(4) = 1, \quad z(2) = 2$$

If we include in the interaction the functional $\mathcal{B}(\psi, J, \eta)$, we have to consider clusters with J and η fields, whose scale dimension is

$$D_v = -2 + \frac{1}{2}|P_v| + m_{J,v} + \frac{3}{2} m_{\eta,v}$$

so that we have to renormalize only a new cluster, that with $m_{J,v} = 1$ and $m_{\eta,v} = 0$, through a Taylor expansion of order 0.

Let us come back to the tree expansion for the free energy. If we sum over all trees, we get

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$$\mathcal{L}\mathcal{V}^{(j)}(\sqrt{Z_j}\psi) = \gamma^j n_j F_\nu(\sqrt{Z_j}\psi) + a_j F_\alpha(\sqrt{Z_j}\psi) + z_j F_z(\sqrt{Z_j}\psi)$$
$$+ l_{1,j} F_1(\psi) + l_{2,j} F_2(\sqrt{Z_j}\psi) + l_{4,j} F_4(\sqrt{Z_j}\psi)$$

$$F_\nu = \sum_{\omega,s} \int d\mathbf{x}\, \psi^+_{\mathbf{x},\omega,s}\psi^-_{\mathbf{x},\omega,s}, \qquad F_1 = \frac{1}{2}\sum_{\omega,s,s'} \int d\mathbf{x}\, \psi^+_{\mathbf{x},\omega,s}\psi^-_{\mathbf{x},-\omega,s}\psi^+_{\mathbf{x},-\omega,s'}\psi^-_{\mathbf{x},\omega,s'}$$

$$F_\alpha = \sum_{\omega,s} \int d\mathbf{x}\, \psi^+_{\mathbf{x},\omega,s}\mathcal{D}\psi^-_{\mathbf{x},\omega,s}, \qquad F_2 = \frac{1}{2}\sum_{\omega,s,s'} \int d\mathbf{x}\, \psi^+_{\mathbf{x},\omega,s}\psi^-_{\mathbf{x},\omega,s}\psi^+_{\mathbf{x},-\omega,s'}\psi^-_{\mathbf{x},-\omega,s'}$$

$$F_z = \sum_{\omega,s} \int d\mathbf{x}\, \psi^+_{\mathbf{x},\omega,s}\partial_0\psi^-_{\mathbf{x},\omega,s}, \qquad F_4 = \frac{1}{2}\sum_{\omega,s} \int d\mathbf{x}\, \psi^+_{\mathbf{x},\omega,s}\psi^-_{\mathbf{x},\omega,s}\psi^+_{\mathbf{x},\omega,-s}\psi^-_{\mathbf{x},\omega,-s}$$

- $z_j\,[F_\alpha(\sqrt{Z_j}\psi) + F_z(\sqrt{Z_j}\psi)]$ is moved to the free measure:

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$$\nu_j = \frac{\sqrt{Z_j}}{\sqrt{Z_{j-1}}}\, n_j, \quad \delta_j = \frac{\sqrt{Z_j}}{\sqrt{Z_{j-1}}}\,(a_j - z_j), \quad g_{i,j} = \left(\frac{\sqrt{Z_j}}{\sqrt{Z_{j-1}}}\right)^2 l_{i,j}$$

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- $P_{Z_{j-1}}(d\psi^{\le j}) = P_{Z_{j-1}}(d\psi^{(\le j-1)})\tilde{P}_{Z_{j-1}}(d\psi^{(j)})$
- $D_v + n_{2,v} \to D_v + z(|P_v|) > 0, \quad \forall v > v_0$

Beta function

The tree expansion is an analytic function of the r.c.c. $\{v_h := (g_{1,h}, g_{2,h}, g_{4,h}, \delta_h, \nu_h); h_{L,\beta} < h \le 0\}$ in a small neighborhood of 0. Hence, the problem is to show that the r.c.c. stay in this small neighborhood of 0 for all h, if $\lambda \in D_{\varepsilon,\delta}$.

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$$v_{\alpha,j-1} = A_\alpha v_{\alpha,j} + \beta_\alpha^{(j)}(v_j; ..., v_0; \lambda, \nu)$$

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$$g_{2,-\infty} = g_{2,0} - \frac{1}{2} g_{1,0} + O(|\lambda|^{3/2}), \quad g_{4,-\infty} = g_{4,0} + O(|\lambda|^2)$$

Outline

Introduction
- Abstract
- References

The extended Hubbard model
- The model
- The free model
- Anomalous exponents and logarithmic corrections
- Universal relations
- Borel summability

Main ingredients of the proof
- The strategy
- The multiscale expansion
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Asymptotic Gauge Invariance

The three properties that allow us to control the flow are:

1 - Asymptotic freedom of the g_1 interaction

$$g_{1,j-1} \simeq g_{1,j} - a g_{1,j}^2, \quad a > 0$$

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2 - Partial vanishing of the beta function

$$v_{\alpha,j-1} \sim v_{\alpha,j} + O(g_{1,j}^2), \quad \alpha = 2, 4, \delta$$

thanks to the local gauge invariance of the *effective model*

$$\psi^{\pm}_{\mathbf{x},\omega,s} \to e^{\pm i\alpha_{\mathbf{x},\omega,s}} \psi^{\pm}_{\mathbf{x},\omega,s}$$

a non Hamiltonian model with linear dispersion relation and non local interaction, whose beta function is asymptotically vanishing.

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We solve this problem by writing explicitly the *correction term* ad showing that its effect does not vanish as the cutoffs are removed.

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We solve this problem by writing explicitly the *correction term* ad showing that its effect does not vanish as the cutoffs are removed.
On the contrary, it is essential to explain the anomalies.

Outline

Introduction

- Abstract
- References

The extended Hubbard model

- The model
- The free model
- Anomalous exponents and logarithmic corrections
- Universal relations
- Borel summability

Main ingredients of the proof

- The strategy
- The multiscale expansion
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Lattice Ward Identities

The Schwinger, response and vertex functions are not all independent one from the other, but there are exact Ward identities (WI), following from the conservation law

$$\frac{\partial \rho^C_{\mathbf{x}}}{\partial x_0} = e^{Hx_0}[H, \rho_x]e^{-Hx_0} = -i\partial^{(1)}_x J_{\mathbf{x}} \equiv -i[J_{x,x_0} - J_{x-1,x_0}]$$

where $\partial^{(1)}_x$ denotes the lattice derivative.

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where $\partial^{(1)}_x$ denotes the lattice derivative.
Our results on the susceptibility and the Drude weight depend on the WI's

$$-ip_0\hat{G}^{2,1}_{\rho,\beta,L}(\mathbf{k},\mathbf{k}+\mathbf{p}) - i(1-e^{-ip})\hat{G}^{2,1}_{j,\beta,L}(\mathbf{k},\mathbf{k}+\mathbf{p}) = \hat{S}^{\beta,L}_2(\mathbf{k}) - S^{\beta,L}_2(\mathbf{k}+\mathbf{p})$$

$$-ip_0\hat{\Omega}_{C,\beta,L}(\mathbf{p}) - i(1-e^{-ip})\hat{\Omega}_{j,\rho,\beta,L}(\mathbf{p}) = 0$$

$$-ip_0\hat{\Omega}_{\rho,j,\beta,L}(\mathbf{p}) - i(1-e^{-ip})\hat{\Omega}_{j,j,\beta,L}(\mathbf{p}) = 0$$

where $\Omega_{\rho,j,\beta,L}(\mathbf{x},\mathbf{y}) = \langle \rho^C_{\mathbf{x}} J_{\mathbf{y}}\rangle^T_{\beta,L}$ and $\Omega_{j,\rho,\beta,L}(\mathbf{x},\mathbf{y}) = \langle J_{\mathbf{x}} \rho^C_{\mathbf{y}}\rangle^T_{\beta,L}$.