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Spin-mixing Interferometry with Bose-Einstein Condensates

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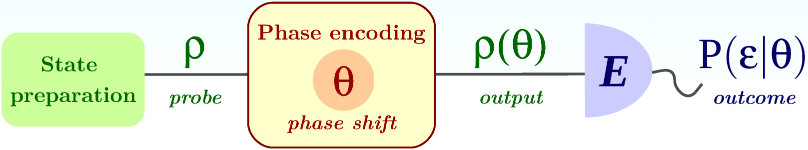
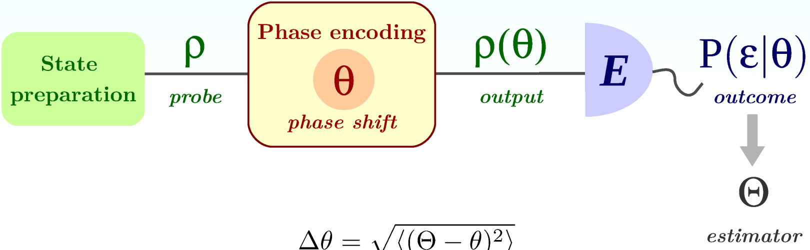
Interferometry and Phase Estimation

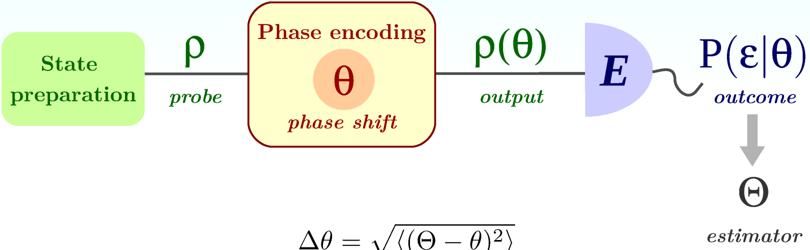


Interferometry and Phase Estimation



$$\Delta\theta = \sqrt{\langle(\Theta - \theta)^2\rangle}$$

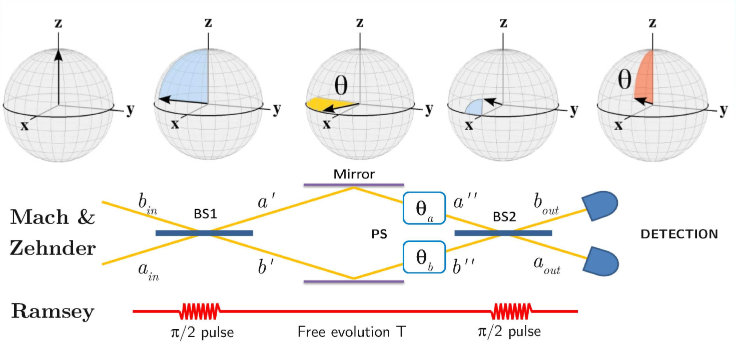
Interferometry and Phase Estimation



$$\Delta\theta = \sqrt{\langle(\Theta - \theta)^2\rangle}$$

- Fundamental aim of interferometry:
  - Cramér-Rao lower bound $\Delta\theta \geq \Delta\theta_{\text{min}} = \frac{1}{\sqrt{F}}$
  - Fisher Information $F = \sum_{\varepsilon} \frac{1}{P(\varepsilon|\theta)} \left(\frac{\partial P(\varepsilon|\theta)}{\partial\theta}\right)^2$
  - particle entanglement can come to our help

SU(2) interferometry



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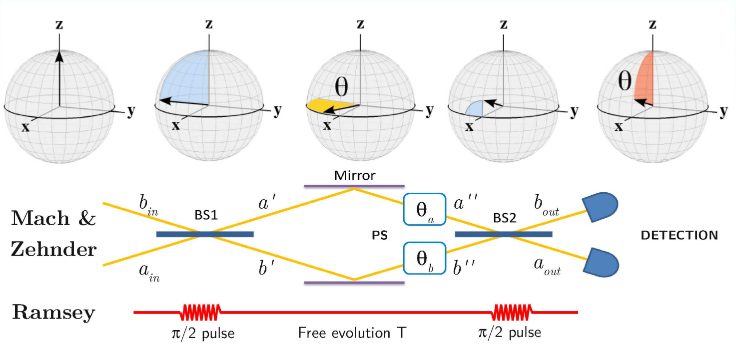
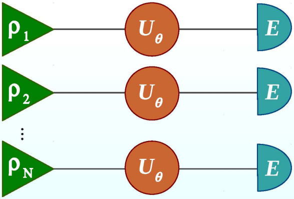




- for separable input states

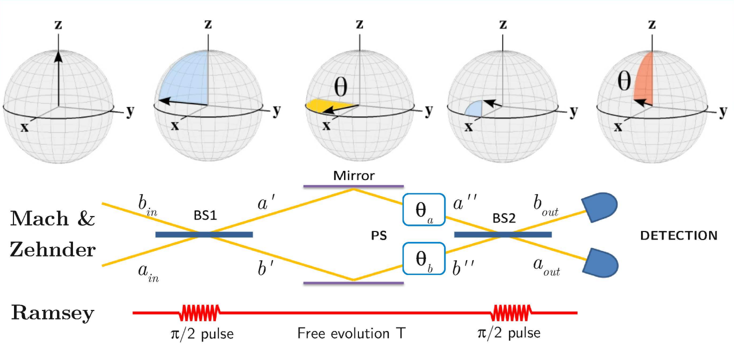
$$F \leq N$$

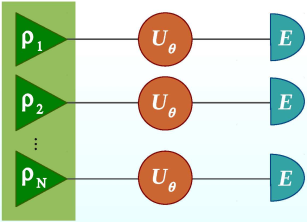
- minimum uncertainty for classical states

$$\Delta\theta \sim \frac{1}{\sqrt{N}} \quad \text{shot noise}$$

Giovannetti, Lloyd and Maccone, PRL (2006)

SU(2) interferometry





Pezzè and Smerzi, PRL (2009)

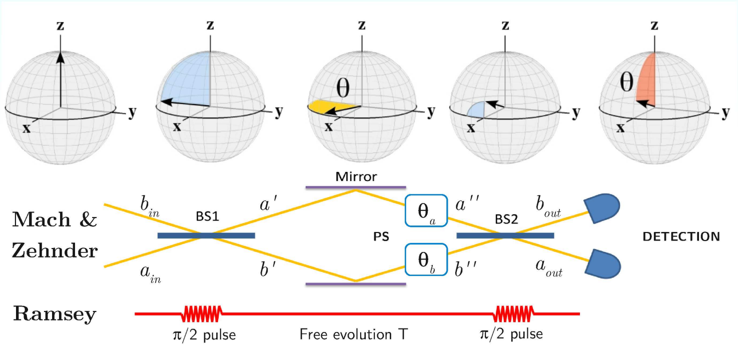
- for *useful* particle-entangled input states

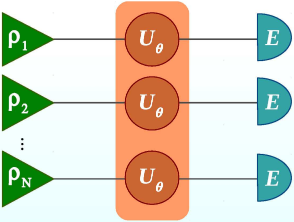
$$F > N$$

- minimum uncertainty for entangled states

$$\Delta\theta \sim \frac{1}{N}$$

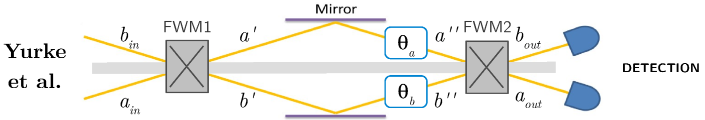
SU(2) interferometry



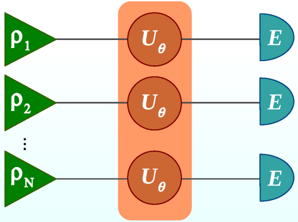


- quantum correlations created *inside* the interferometer

SU(1,1) interferometry



Yurke, McCall and Klauder, PRA (1986)



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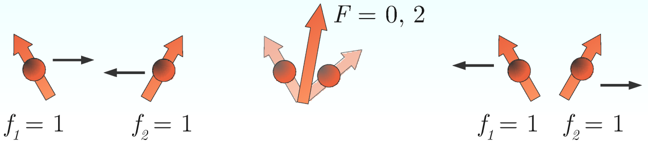
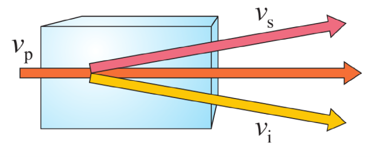
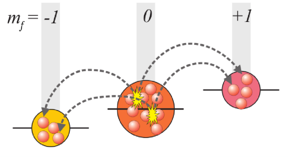
Physical system



- What a spinor BEC is?
  - spin degree of freedom *not frozen*
  - spin-mixing collisions occur in Zeeman subspace $|f, m_f\rangle$
  - conserved magnetization
  - analogy with degenerate four-wave mixing:





Stamper-Kurn and Ueda, RMP (2013)

Interferometric sequence

- To realize spin-mixing dynamics:
  - ▷ just wait under the nonlinear interaction Hamiltonian

$$\hat{H}_{\mathrm{SMD}} = \chi \left[\underbrace{\hat{a}_0^\dagger \, \hat{a}_0^\dagger \, \hat{a}_{+1} \, \hat{a}_{-1} + \hat{a}_0 \, \hat{a}_0 \, \hat{a}_{+1}^\dagger \, \hat{a}_{-1}^\dagger}_{\text{atomic four-wave mixing}} + \underbrace{\left(\hat{N}_0 - \tfrac{1}{2}\right)(\hat{N}_{+1} + \hat{N}_{-1})}_{\text{mean-field shift}} \right]$$

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$$\chi = \sum_{F/2=0}^{f} A_F \, a_F$$

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$$\hat{H}_{\mathrm{PS}} = q \left(\hat{N}_{+1} + \hat{N}_{-1} \right)$$

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- To encode phase shift:
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$$q \propto B^2$$

Interferometric sequence

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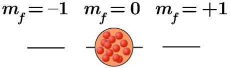
initial state preparation



$$|0, \alpha, 0\rangle \quad \rightarrow \quad \bar{n} = |\alpha|^2$$

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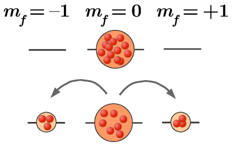
initial state preparation

spin-mixing dynamics



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$$e^{-i\,t\,H_{\mathrm{SMD}}} \quad \rightarrow \quad \mathcal{N} = \eta \; \bar{n}$$

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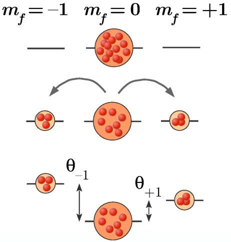
phase encoding



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$e^{-i\,t\,H_{\mathrm{SMD}}} \quad \rightarrow \quad \mathcal{N} = \eta \, \bar{n}$

$e^{-i\,t_{\mathrm{PS}}\,H_{\mathrm{PS}}} \quad \theta = \begin{cases} \theta_{+1} + \theta_{-1} \\ 2\,q\,t_{\mathrm{PS}} \end{cases}$

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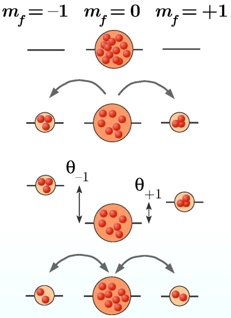

initial state preparation: $|0, \alpha, 0\rangle \quad \rightarrow \; \bar{n} = |\alpha|^2$

spin-mixing dynamics: $e^{-i\,t\,H_{\mathrm{SMD}}} \quad \rightarrow \; \mathcal{N} = \eta \, \bar{n}$

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measurement on output

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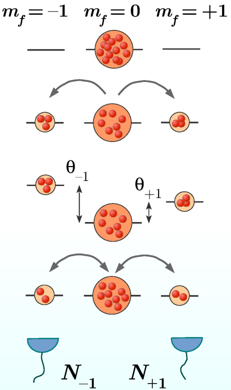

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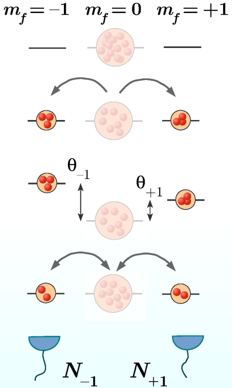
measurement on output		$\rightarrow \; N_{\pm 1} = N_{\mathrm{output}}$

Parametric approximation

$$\boxed{\hat{H}_{\mathrm{SMD}} = \chi\left[2\bar{n}\left(\hat{K}_x\,\cos 2\phi + \hat{K}_y\,\sin 2\phi\right) \;+\; \left(2\bar{n}-1\right)\hat{K}_z\right]}$$

Mean-field approximation: $\hat{a}_0 \to \sqrt{\bar{n}}\,e^{i\,\phi}$



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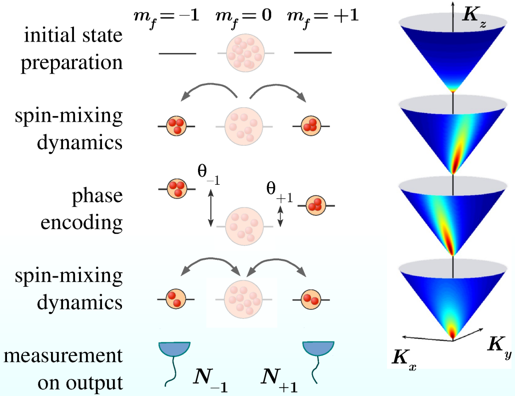

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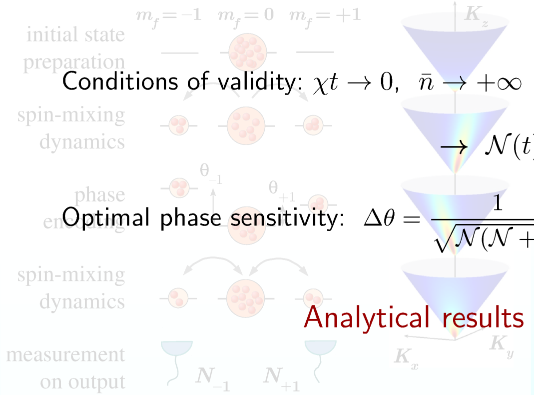

Conditions of validity: $\chi t \to 0,\ \bar{n} \to +\infty$ with $\chi t \sqrt{\bar{n}} \ll 1$

$$\to\ \mathcal{N}(t) \ll \bar{n}\ \leftarrow$$

Optimal phase sensitivity: $\Delta\theta = \dfrac{1}{\sqrt{\mathcal{N}(\mathcal{N}+2)}}$

Analytical results ✓

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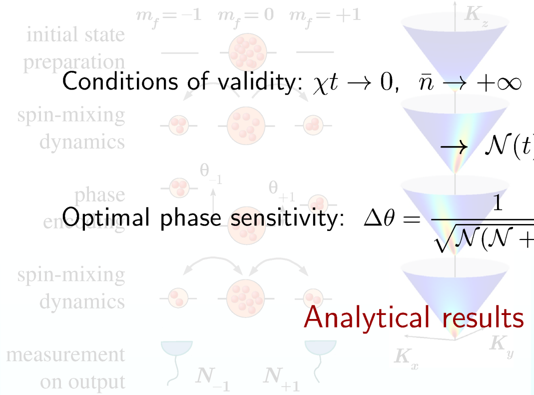

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Full quantum approach

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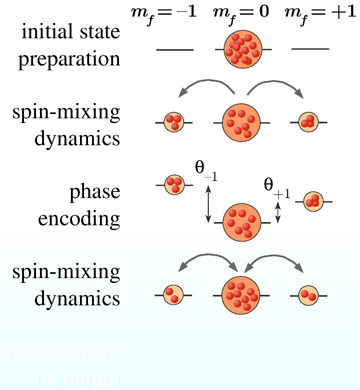

sub shot noise

?

Gabbrielli, Pezzè and Smerzi, arXiv:1503.08582

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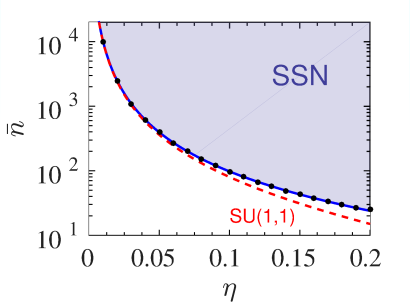
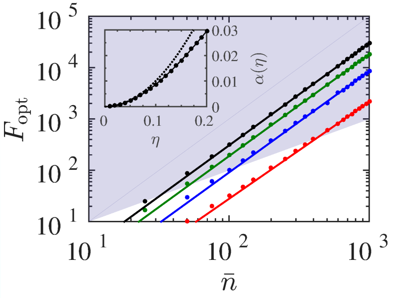


sub shot-noise sensitivity

in $(\bar{n}, \eta)$-parameter space ✓



scaling with respect to $\bar{n}$

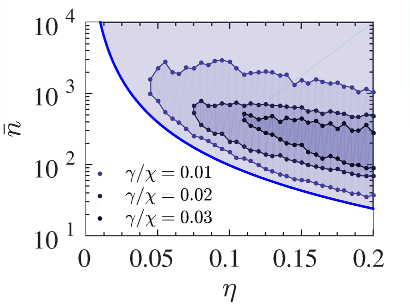
$$F_{\rm opt} \approx \alpha(\eta)\, \bar{n}^2 \quad \text{for} \quad \bar{n} \gg 1$$

$$\alpha(\eta) = \eta^2\,(1 - 1.3\,\eta)$$

$$\Downarrow$$

$$\Delta\theta_{\text{min}} = \frac{\beta(\eta)}{\bar{n}} \quad \text{with} \quad \beta = \frac{1}{\sqrt{\alpha}}$$

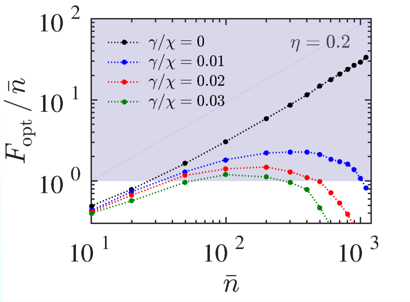
Adding decoherence...



- Robust against two-body losses

$$\bar{n}(t) = \frac{\bar{n}}{1 + 2\gamma\,\bar{n}\,t}$$

  - if *low enough* scattering rate
  - ok for ^{87}Rb in $f = 2$ ✓



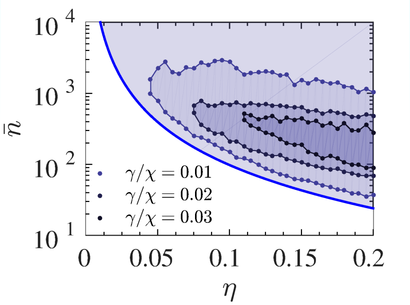
- Scaling for Fisher Information:
  - lossless scaling for

$$\bar{n} \ll (\chi/2\gamma)^2$$

  - no sub shot noise for

$$\gamma/\chi \gtrsim 0.04$$

...and other experimental imperfections



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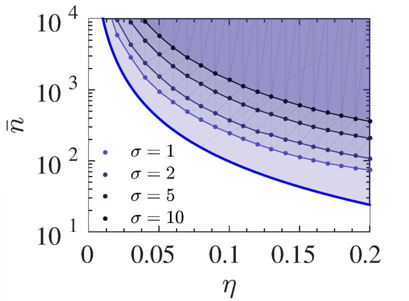
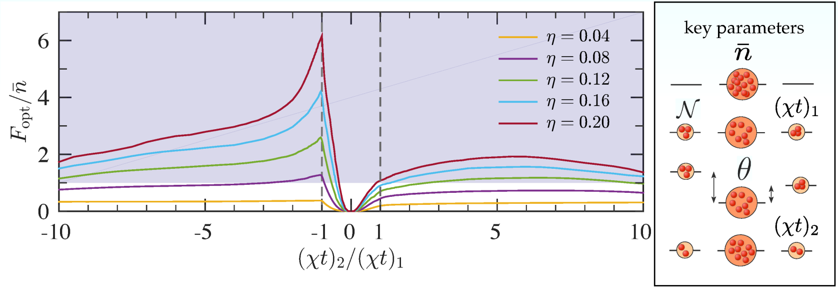


- Robust against detection noise
  - if *low enough* fluctuation σ
  - Fisher Information scaling

$$F_{\rm opt} \approx \frac{\alpha(\eta)}{\sigma}\,\bar{n}^2$$

Variable dynamics



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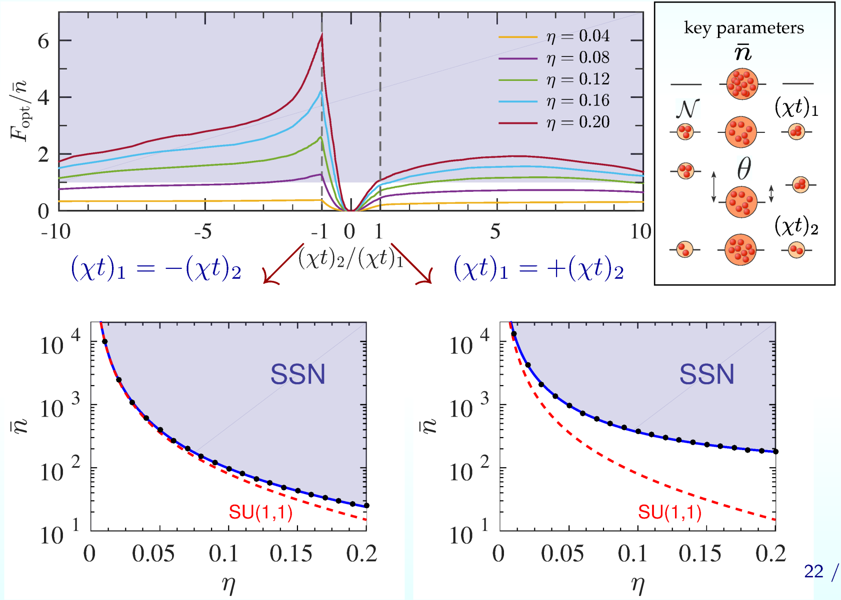

Conclusions

- Spin-mixing interferometry with spinor BEC:
  - ▷ sub shot-noise sensitivity with respect to total number of atoms
  - ▷ beyond mean field
- Applications:
  - ▷ ultraprecise magnetometry
  - ▷ experimental feasibility (work in progress in Heidelberg)

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Thanks for your attention!

References

V. Giovannetti, S. Lloyd and L. Maccone, *Phys. Rev. Lett.* **96**, 010401 (2006)
L. Pezzè and A. Smerzi, *Phys. Rev. Lett.* **102**, 100401 (2009)
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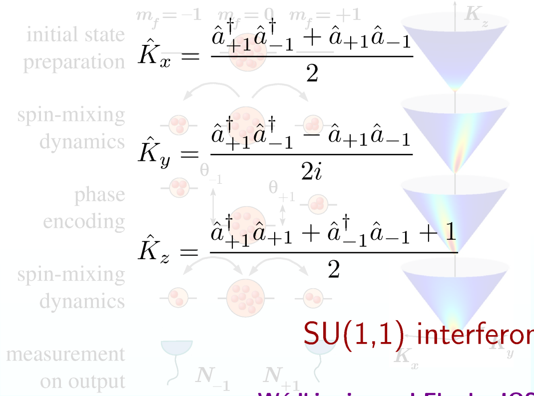
D. M. Stamper-Kurn and M. Ueda, *Rev. Mod. Phys.* **85**, 1191 (2013)
M. Gabbrielli, L. Pezzè and A. Smerzi, arXiv:1503.08582 (accepted for publication)

APPENDICI

Parametric approximation

$$\boxed{\hat{H}_{\text{SMD}} = \chi\left[2\bar{n}\left(\hat{K}_x\,\cos 2\phi + \hat{K}_y\,\sin 2\phi\right) \;+\; \left(2\bar{n}-1\right)\hat{K}_z\right]}$$

Mean-field approximation: $\hat{a}_0 \to \sqrt{\bar{n}}\,e^{i\,\phi}$



$$\hat{K}_x = \frac{\hat{a}^\dagger_{+1}\hat{a}^\dagger_{-1} + \hat{a}_{+1}\hat{a}_{-1}}{2} \qquad [\hat{K}_x, \hat{K}_y] = -i\hat{K}_z$$

$$\hat{K}_y = \frac{\hat{a}^\dagger_{+1}\hat{a}^\dagger_{-1} - \hat{a}_{+1}\hat{a}_{-1}}{2i} \qquad [\hat{K}_y, \hat{K}_z] = i\hat{K}_x$$

$$\hat{K}_z = \frac{\hat{a}^\dagger_{+1}\hat{a}_{+1} + \hat{a}^\dagger_{-1}\hat{a}_{-1} + 1}{2} \qquad [\hat{K}_x, \hat{K}_z] = -i\hat{K}_y$$

SU(1,1) interferometry

Wódkiewicz and Eberly, JOSAB (1985)

Parametric approximation

$$\hat{H}_{\mathrm{SMD}} = \chi \left[2\bar{n} \left(\hat{K}_x \cos 2\phi + \hat{K}_y \sin 2\phi \right) + (2\bar{n} - 1) \hat{K}_z \right]$$

Mean-field approximation: $\hat{a}_0 \to \sqrt{\bar{n}}\, e^{i\phi}$

Conditions of validity: $\chi t \to 0, \;\; \bar{n} \to +\infty$ with $\chi t \sqrt{\bar{n}} \ll 1$

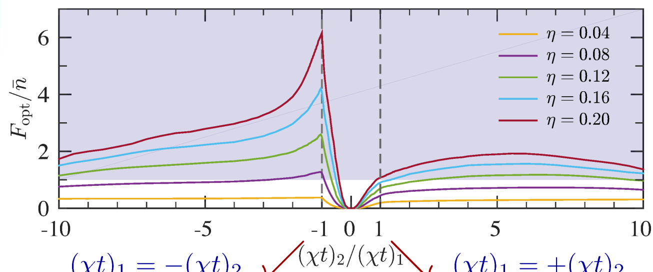
Number of transferred particles: $\mathcal{N}(t) = \dfrac{8\bar{n}^2}{4\bar{n}-1} \sinh^2 \left(\dfrac{\sqrt{4\bar{n}-1}}{2} \chi t \right) \ll \bar{n}$

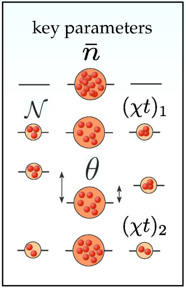
Fisher Information: $F(\theta) = \dfrac{\mathcal{N}(\mathcal{N}+2)}{\mathcal{N}(\mathcal{N}+2) \sin^2 \frac{\theta}{2} + 1} \cos^2 \frac{\theta}{2}$

Optimal phase sensitivity: $\Delta\theta = \dfrac{1}{\sqrt{\mathcal{N}(\mathcal{N}+2)}}$

Analytical results ✓

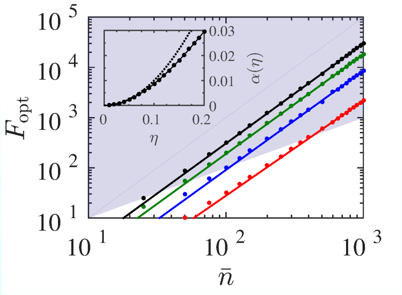
Variable dynamics

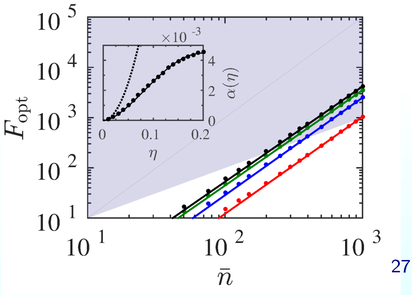




$(\chi t)_1 = -(\chi t)_2$

$(\chi t)_1 = +(\chi t)_2$





Likelihood

- m uncorrelated subsystems: $\hat{\rho}_m \equiv \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)} \otimes \ldots \otimes \hat{\rho}^{(m)}$
- m independent measurements:

  $\hat{E}(\boldsymbol{\varepsilon}) \equiv \hat{E}^{(1)}(\varepsilon_1) \otimes \hat{E}^{(2)}(\varepsilon_2) \otimes \ldots \otimes \hat{E}^{(m)}(\varepsilon_m)$

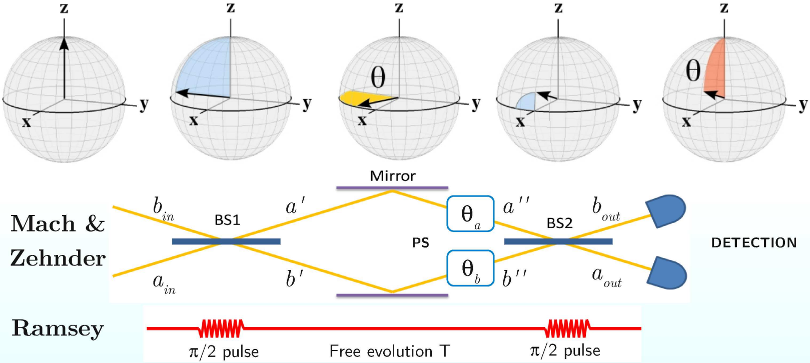
$\Rightarrow P(\boldsymbol{\varepsilon}|\theta) = \mathrm{Tr}\left[\hat{\rho}_m(\theta)\hat{E}(\boldsymbol{\varepsilon})\right] = \prod_{i=1}^{m} P_i(\varepsilon_i|\theta)$ Likelihood

$\Rightarrow \Theta_{\mathrm{ML}}(\boldsymbol{\varepsilon}) = \arg\max_{\varphi} P(\boldsymbol{\varepsilon}|\varphi)$ Maximum Likelihood Estimator

SU(2) interferometers

$$[\hat{J}_1, \hat{J}_2] = i\,\hat{J}_3 \qquad [\hat{J}_2, \hat{J}_3] = i\,\hat{J}_1 \qquad [\hat{J}_3, \hat{J}_1] = i\,\hat{J}_2$$

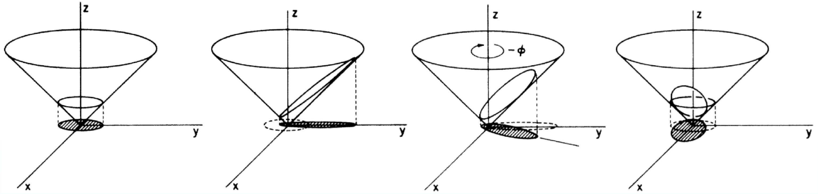
$$\mathfrak{su}(2) \cong \mathfrak{so}(3)$$



SU(1,1) interferometers

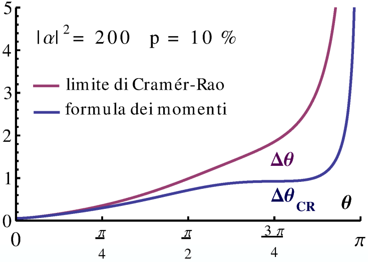
$$[\hat{K}_1, \hat{K}_2] = -i\,\hat{K}_3 \qquad [\hat{K}_2, \hat{K}_3] = i\,\hat{K}_1 \qquad [\hat{K}_3, \hat{K}_1] = i\,\hat{K}_2$$

$$\mathfrak{su}(1,1) \cong \mathfrak{so}(2,1)$$



Error propagation

Alternative protocol



$$\Delta\theta = \frac{(\Delta \hat{J}_z)_{out}}{\left|\frac{\partial \langle \hat{J}_z \rangle_{out}}{\partial \theta}\right|} \geq \Delta\theta_{\mathrm{CR}} \ .$$

Sensitivity and comparison with "moment formula"

Ipotesi e semplificazioni

- gas diluito $a \ll d$ $\longrightarrow$ urti binari
- interazioni dipolari trascurabili $\longrightarrow$ assenza rilassamento Zeeman
- bassa energia $a \ll \lambda_{\mathrm{T}}$ $\longrightarrow$ scattering in onda s
- trappola ottica $\longrightarrow$ indipendente dallo spin
- healing length $\xi_{spin} \gtrsim V^{1/3}$ $\longrightarrow$ singolo modo spaziale

$\Longrightarrow$ Hamiltoniana efficace di interazione spin-mixing:

$$\hat{H} = \chi \left[\tfrac{1}{2}(2\hat{N}_c - \hat{\mathbb{I}})(\hat{N}_a + \hat{N}_b) + (\hat{a}\hat{b}\hat{c}^\dagger\hat{c}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\hat{c}\hat{c}) \right] + q\,(\hat{N}_a + \hat{N}_b)$$

con $\chi \propto a_{F=2} - a_{F=0}$ e $q \propto B^2$

Altre ipotesi

Ulteriori approssimazioni adottate nella trattazione dello scattering fra bosoni identici e relative conseguenze sulle proprietà fisiche del fenomeno:

- campo $B \ll 1\,\mathrm{T} \longrightarrow$ spin iperfine f buon numero quantico
- invarianza rotazionale $\longrightarrow$ conservazione momento angolare
- ingresso $|\mathrm{in}\rangle = |0, 0, \alpha\rangle \longrightarrow$ magnetizzazione longitudinale nulla

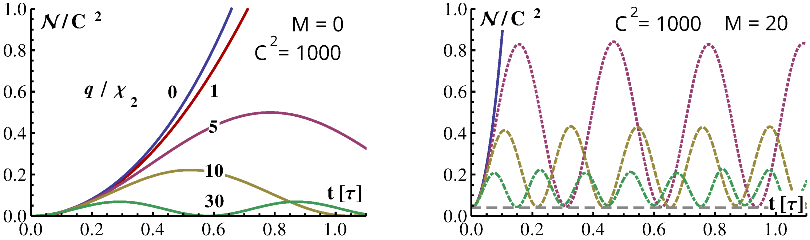
$\Longrightarrow$ Hamiltoniana efficace di interazione spin-mixing:

$$\hat{H} = \chi\left[(\hat{N}_a - \hat{N}_b)^2 + (2\hat{N}_c - \hat{\mathbb{I}})(\hat{N}_a + \hat{N}_b) + 2(\hat{c}^\dagger\hat{c}^\dagger\hat{a}\hat{b} + \hat{a}^\dagger\hat{b}^\dagger\hat{c}\hat{c})\right] + q\,(\hat{N}_a + \hat{N}_b)$$

con $\quad \chi \propto a_{F=2} - a_{F=0} \quad$ e $\quad q \propto B^2$

Approssimazione parametrica

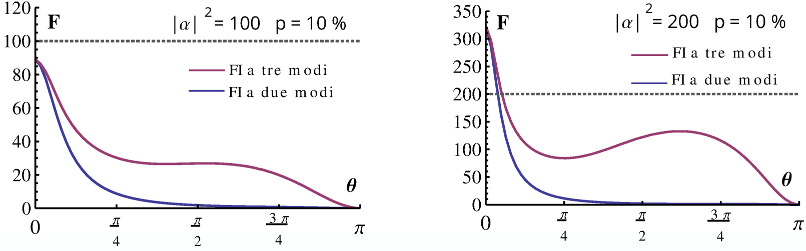
$$\hat{c} \to C\hat{\mathbb{I}} \text{ con } |C|^2 = |\alpha|^2 \quad \text{se} \quad |\alpha|^2 \gg 1$$



$\hat{H}$ schematizza sia FWM sia PS a seconda del rapporto fra parametri q/χ

$$\tau = \frac{\hbar}{\chi\sqrt{|\alpha|^2}} \quad \text{time scale}$$

Fisher a due o tre modi

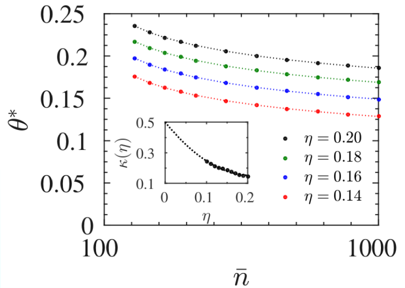


Confronto fra F.I. a due modi e a tre modi

Ampiezza regione sub shot noise

Per $\theta_{opt} - \theta^* < \theta < \theta_{opt} + \theta^*$

$$\Delta\theta\Big|_\theta < \frac{1}{\sqrt{|\alpha|^2}} \Leftrightarrow F(\theta) > |\alpha|^2$$



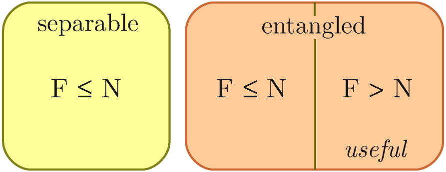
Linear interferometers

$$\hat{\mathcal{U}}(\theta) = e^{-i\,\theta\,\hat{H}} \qquad \text{con} \quad \hat{H} = \sum_{j=1}^{N} \hat{H}_j$$

- Correlazioni:

  $\hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2 \otimes \ldots \otimes \hat{\rho}_N$
  (separabilità)

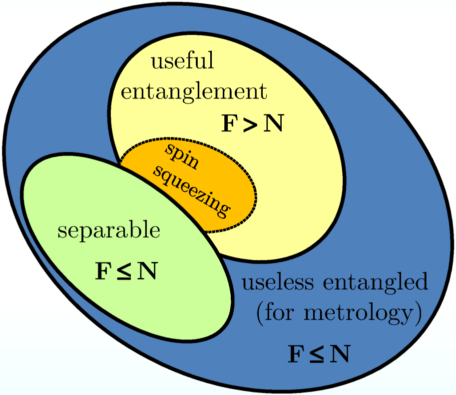
  $\hat{\rho} \neq \hat{\rho}_1 \otimes \hat{\rho}_2 \otimes \ldots \otimes \hat{\rho}_N$
  (entanglement)



Pezzé, Smerzi (2009)

- Limiti fondamentali:
  - stato separabile → shot noise $\Delta\theta_{\mathrm{SN}} = \frac{1}{\sqrt{N}}$
  - stato non separabile (utile) → sub shot noise $\frac{1}{N} \leq \Delta\theta < \frac{1}{\sqrt{N}}$

State space


useful
entanglement
F > N
spin
squeezing
separable
F ≤ N
useless entangled
(for metrology)
F ≤ N