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SELZ

Operational resource theory of coherence

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[Joint work with Dong Yang, arXiv:1506.07975;
cf. Baumgratz/Cramer/Plenio, PRL 113:140401, 2014]

Outline

1. Superpositions as a resource; incoherent operations and monotones
2. Pure states: Majorization
3. Pure asymptotics: Entropy
4. Mixed states: Distillation & formation
5. Generic irreversibility
6. Outlook

1. Superposition

Consider a quantum system, i.e. a Hilbert space $\mathcal{H}$ (of finite dimension $d=|\mathcal{H}|$), with a distinguished orthogonal basis $\{|x\rangle\}$.

[Baumgratz/Cramer/Plenio, PRL 113:140401, 2014]

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States diagonal in the computational basis are "incoherent", all others have some "coherence". (Classical vs quantum!)

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"Incoherent operations" are those that do not create coherence from incoherence.

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1. Superposition

Decohering operator:

$$\Delta(\rho) = \sum |x\rangle\langle x|\, \rho\, |x\rangle\langle x|$$

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Incoherent operations have Kraus operators $K = \sum c_x |y(x)\rangle\langle x|$, where $y = y(x)$ is a det. function of x. Makes sure that $K|x\rangle \propto |y\rangle$.

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Example: Δ is an incoherent operation.

[Baumgratz/Cramer/Plenio, PRL 113:140401, 2014]

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Fundamental question: When can we transform a state ρ into a state σ by incoherent operations? $\rho \xrightarrow{IC} \sigma$

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Motivates "coherence monotones", i.e. functionals that cannot increase under incoherent operations, and are 0 on Δ. Shows that certain transformations are impossible...

[Baumgratz/Cramer/Plenio, PRL 113:140401, 2014]

1. Superposition

Examples of coherence monotones:

Relative entropy of coherence

$$C_r(\rho) = \min D(\rho \| \sigma) \text{ s.t. } \sigma \in \Delta$$

$$= S(\Delta(\rho)) - S(\rho)$$

[Åberg, arXiv:quant-ph/0612146, 2006]
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Coincide for pure states:

$$C_r(\varphi) = C_f(\varphi) = S(\Delta(\varphi)) =: C(\varphi)$$

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$$\left\{ (p_x) \succ (q_x) \text{ iff } \sum_1^n p_x \geq \sum_1^n q_x \text{ for } n=1,\ldots,d \right\}$$

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Cor. Every state ψ can be obtained from the "max. coherent state" $\Phi_d = \frac{1}{d} \sum_{xy} |x\rangle\langle y|$

[Du/Bai/Guo, PRA 91:052120, 2015]

3. Pure states asymptotics

Consider $\varphi^{\otimes n} \xrightarrow{IC} \approx \psi^{\otimes m}$, where approximation means that fidelity $\to 1$ as $n \to \infty$.

[See also Yuan et al., arXiv:1505.04032, 2015]

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Consider $\varphi^{\otimes n} \xrightarrow{IC} \approx \psi^{\otimes m}$, where approximation means that fidelity $\to 1$ as $n \to \infty$.

These states live of course on tensor product spaces. It seems natural to declare the tensor product basis of the "local" classical bases as the incoherent states.

[See also Yuan et al., arXiv:1505.04032, 2015]

3. Pure states asymptotics

Consider $\varphi^{\otimes n} \xrightarrow{IC} \approx \psi^{\otimes m}$, where approximation means that fidelity $\to 1$ as $n \to \infty$.

Theorem 2. $\varphi^{\otimes n} \xrightarrow{IC} \approx \Phi_2^{\otimes nR}$ if $R < C(\varphi)$;

$\Phi_2^{\otimes nR} \xrightarrow{IC} \approx \psi^{\otimes n}$ if $R > C(\psi)$.

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Cor. $\varphi^{\otimes n} \xrightarrow{IC} \approx \psi^{\otimes nR}$ is possible if $R < \frac{C(\varphi)}{C(\psi)}$,

and impossible if $R > \frac{C(\varphi)}{C(\psi)}$.

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Both these rates are optimal!

I.e. distillable coherence $= C_r$,

coherence cost $= C_f$.

5. Generic irreversibility

While for pure states,

$$C_r(\psi) = C_f(\psi) = C(\psi) = S(\Delta(\psi)),$$

for generic mixed states,

$$C_r(\rho) < C_f(\rho).$$

(Theorem 4 characterizes equality case - omit)

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Note however that there is no "bound coherence":

$$\rho \in \Delta \text{ iff } C_f(\rho) = 0 \text{ iff } C_r(\rho) = 0.$$

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$C_r(\rho) < C_f(\rho)$.

Contrast this is with the general framework of [Brandão/Gour, arXiv:1502.03149]:

There, transformations are from the wider class of incoherence-preserving operations, i.e. T such that $T(\Delta) \subseteq \Delta$. This makes both distillability and cost equal to $C_r(\rho)$, so the theory becomes reversible.

6. Outlook

- Resource theory of coherence quite complete, as all pure state questions and mixed state distillation & formation have explicit, simple answers.
- At the same time it's complex enough to exhibit irreversibility (unlike the theory à la Brandão/Gour).

6. Outlook

- Interestingly, the theory of coherence (under incoherent ops) seems to be a complete mirror of the theory of maximally correlated states (under LOCC), via the correspondence of $\rho = \sum \rho_{xy} |x\rangle\langle y| \iff \rho' = \sum \rho_{xy} |xx\rangle\langle yy|$ [Cf. Streltsov et al., arXiv:1502.05876].

So, coherence theory is a kind of "entanglement theory for babies" :-)

6. Outlook

- Extend resource theory to operations (=cptp maps), of which states are only a special case. First results include complete characterization when and how efficiently 2x2 unitaries can be transformed into each other by incoherent operations [Work in progress w/ Khaled Ben Dana].

6. Outlook

- Motivation of Åberg and Baumgratz et al. is formalization of classical-quantum divide – in finite dimension. In CV systems, coherent states $|\alpha\rangle$ are the "classical" (well, least quantum) states, all others are "quantum". Possible to construct an operational resource theory along the same lines as above?