UNIVERSITÀ DEGLI STUDI DI MILANO
DIPARTIMENTO DI FISICA

Bounds to precision for quantum interferometry with Gaussian states and operations

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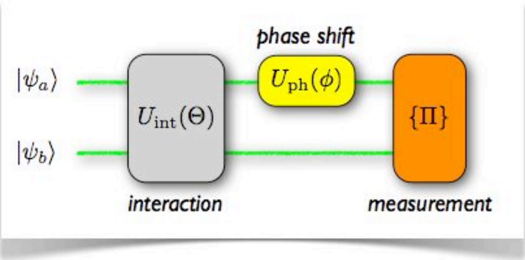


IQIS 2015 - 8th Italian Quantum Information Science Conference, Monopoli, 12 September 2015

Outline

一 Basic of phase estimation
- *Single parameter estimation*
- *Fisher and quantum Fisher information*

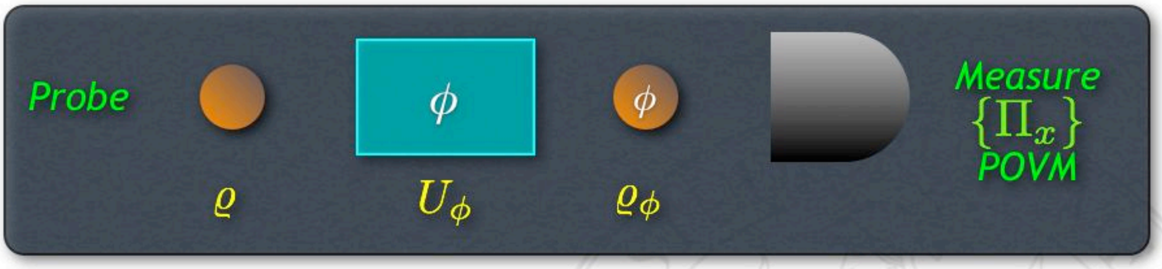
二 Interferometry with Gaussian states and operations (ideal)
- *Passive & active interferometers: ultimate bounds*

三 Realistic measurements
- *Passive & active detection stage, photon number detection*
- *Passive & active interferometers: sensitivity (quantum efficiency)*

四 Concluding remarks

Single parameter estimation

When a physical parameter ϕ (e.g., phase, entropy, entanglement,...) is not directly accessible, one has to resort to indirect measurements.



$$\chi = (x_1, x_2, \dots x_M)$$

sample

$$\hat{\phi} = \hat{\phi}(\chi)$$

estimator

minimize uncertainty

reach minimum uncertainty

<u>Goals</u>: optimal probe states & measurements

Classical estimation

Classically, optimal unbiased estimators saturate the Cramér-Rao inequality:

$$\mathrm{Var}(\phi) = E[\hat{\phi}^2] - E[\hat{\phi}]^2$$

$$\mathrm{Var}(\phi) \geq \frac{1}{M\,F(\phi)}$$

M is the statistical scaling due to the M outcomes.

$$F(\phi) = \int_\Omega dx \, \frac{[\partial_\phi \, p(x|\phi)]^2}{p(x|\phi)}$$

Fisher Information

$$\chi = (x_1, x_2, \dots x_M)$$

sample

$$\hat{\phi} = \hat{\phi}(\chi)$$

estimator

conditional probability

Quantum estimation

Starting from the Born rule: $p(x|\phi) = \mathrm{Tr}[\Pi_x \, \varrho_\phi]$

We have the FI: $F(\phi) = \int_\Omega dx \, \frac{\Re\mathrm{e}\left(\mathrm{Tr}[\varrho_\phi \, \Pi_x \, L_\phi]\right)^2}{\mathrm{Tr}[\varrho_\phi \, \Pi_x]}$

$2\,\partial_\phi \varrho_\phi = L_\phi \varrho_\phi + \varrho_\phi L_\phi$ *symmetric logarithmic derivative (SLD)*

Maximizing over all the possible measurements (POVMs), we obtain the quantum Cramér-Rao bound:

$$\mathrm{Var}(\phi) \geq \frac{1}{M\, H(\phi)}$$

$$H(\phi) = \mathrm{Tr}[\varrho_\phi \, L_\phi^2] \geq F(\phi)$$

Quantum Fisher Information

The eigenstates of SLD operator correspond to the optimal POVM!

M. G. A. Paris, Int. J. Quantum Inform. **7**, 125 (2009)

Quantum estimation

If the probe state is a mixed state: $\varrho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$ *statistical model*

and $\varrho_\phi = U_\phi \, \varrho \, U_\phi^\dagger$ *with* $U_\phi = \exp\{-i\phi G\}$

$$H = 2 \sum_{n \neq m} \frac{(p_n - p_m)^2}{p_n + p_m} |\langle\psi_n|G|\psi_m\rangle|^2$$

the QFI is independent of ϕ.

If the probe is a pure state:

$$\varrho = |\psi_0\rangle\langle\psi_0| \quad \Rightarrow \quad H = 4\langle\psi_0|\Delta G^2|\psi_0\rangle$$

namely, the QFI corresponds to the fluctuations of the generator G on the probe state and is independent on ϕ.

Quantum estimation

In our case we have: $U_\phi = \exp\{-i\phi\, a^\dagger a\}$ *(phase shift)* $a^\dagger a|n\rangle = n|n\rangle$

Classical strategy *(the probe is a coherent state)*

$H \propto N$ *the QFI is proportional to the energy of probe state*

Quantum strategy

Bipartite entangled states & interferometric setup

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + |0,N\rangle\right) \quad \Rightarrow \quad H = N^2$$

P. Kok, *et al.*, PRA **63**, 063407 (2001)

NOON state *Heisenberg limit*

Phase estimation with Gaussian states

Gaussian states (states with Gaussian Wigner function: coherent, thermal, squeezed states,...) play a major role in quantum information with continuous variables:

- *Generated and manipulated by means of linear and bilinear interactions of modes implemented in quantum optics laboratories.*

$$\hat{\varrho} = \hat{\varrho}(\alpha, \xi, N) = \hat{D}(\alpha)\hat{S}(\xi)\hat{\nu}_{\rm th}(N)\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)$$

$$\hat{\nu}_{\rm th}(N) = \sum_{n=0}^{\infty} \frac{N^n}{(1+N)^{n+1}} |n\rangle\langle n|$$

thermal state

$$N = 0 \rightarrow \nu_{\rm th}(0) = |0\rangle\langle 0|$$

$$\hat{S}(\xi) = \exp\{\tfrac{1}{2}[\xi(\hat{a}^\dagger)^2 - \xi^* \hat{a}^2]\}$$

squeezing operator

$$\hat{D}(\alpha) = \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}]$$

displacement operator

Phase estimation with Gaussian states

Gaussian states (states with Gaussian Wigner function: coherent, thermal, squeezed states,...) play a major role in quantum information with continuous variables:

- *Generated and manipulated by means of linear and bilinear interactions of modes implemented in quantum optics laboratories.*

Pure single-mode GS:

$$D(\alpha)\,S(r)|0\rangle \qquad \beta = \frac{\sinh^2 r}{N}$$

squeezing photons: $\sinh^2 r$

squeezing fraction: β

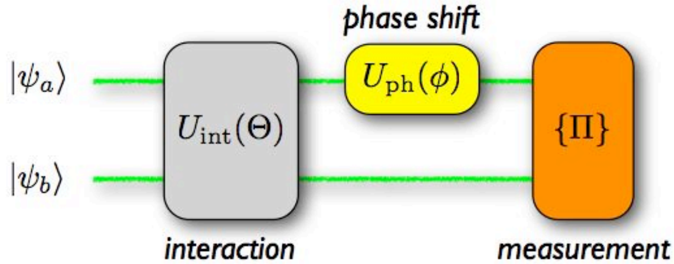
total number of photons: N

$$N = \sinh^2 r + |\alpha|^2$$

A. Monras, PRA **73**, 033821(2006)

$$S(r)|0\rangle \quad \Rightarrow \quad H = 8(N^2 + N)$$

Passive & active interferometers



$$U_{\rm ph}(\phi) = {\rm e}^{-i\phi\, a^\dagger a} \otimes \mathbb{I}$$

$$|\psi_a\rangle \otimes |\psi_b\rangle = |\alpha, \xi\rangle \otimes |\gamma, \zeta\rangle$$

$$|z, s\rangle = D(z)\,S(s)\,|0\rangle$$

$$\alpha, \xi, \gamma \in \mathbb{R} \qquad \zeta = r\,{\rm e}^{-i\theta} \in \mathbb{C}$$

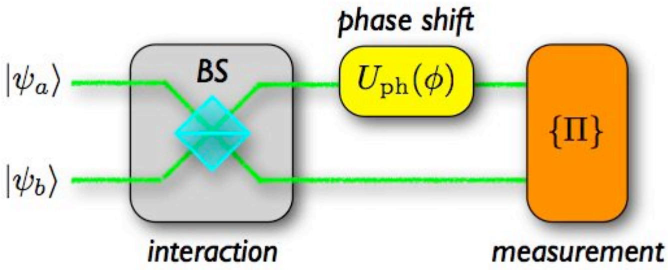
Passive devices, BS-like interaction, SU(2):

$$U_{\rm BS} \equiv U_{\rm int}(\Theta) = \exp\left[\frac{\pi}{4}(a^\dagger b - a b^\dagger)\right]$$

Active devices, two-mode squeezing, SU(1,1):

$$U_{\rm int}(\Theta) = \exp\left(\xi a^\dagger b^\dagger - \xi^* ab\right)$$

Passive interferometer: ultimate bounds



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Total number of photons

$$N_{\rm tot} = \alpha^2 + \gamma^2 + \sinh^2 r + \sinh^2 \xi$$

Signal fraction

$$\Delta = \frac{\alpha^2}{\alpha^2 + \gamma^2}$$

Total squeezing fraction

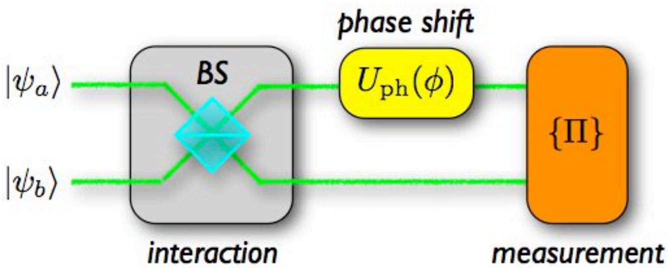
$$\beta_{\rm tot} = \frac{\sinh^2 r + \sinh^2 \xi}{N_{\rm tot}}$$

Partial squeezing fraction

$$\beta = \frac{\sinh^2 \xi}{N_{\rm tot}}$$

Optimization *over all* the involved parameters!!

Passive interferometer: ultimate bounds



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$$\alpha, \xi, \gamma \in \mathbb{R} \qquad \zeta = r\, {\rm e}^{-i\theta} \in \mathbb{C}$$

Total number of photons

$$N_{\rm tot} = 2(\alpha^2 + \sinh^2 r)$$

Signal fraction

$$\Delta = \frac{1}{2}$$

Total squeezing fraction

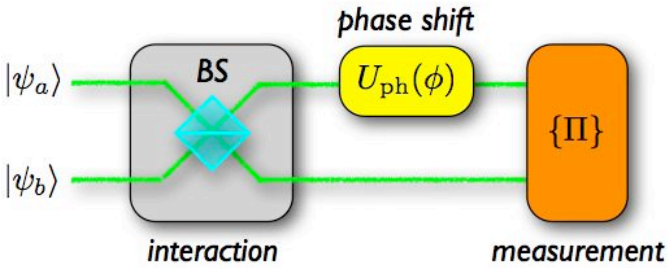
$$\beta_{\rm tot} = \frac{2\sinh^2 r}{N_{\rm tot}}$$

Partial squeezing fraction

$$\beta = \frac{\sinh^2 r}{N_{\rm tot}}$$

Optimization *over all* the involved parameters!! $\alpha = \gamma, \quad \xi = r$

Passive interferometer: ultimate bounds



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Total number of photons

$$N_{\rm tot} = 2(\alpha^2 + \sinh^2 r)$$

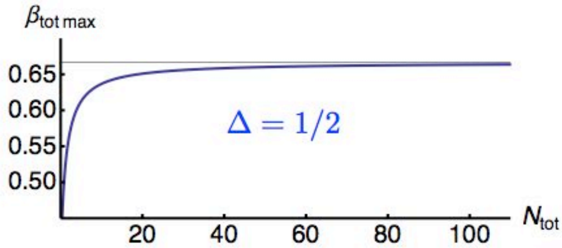
Total squeezing fraction

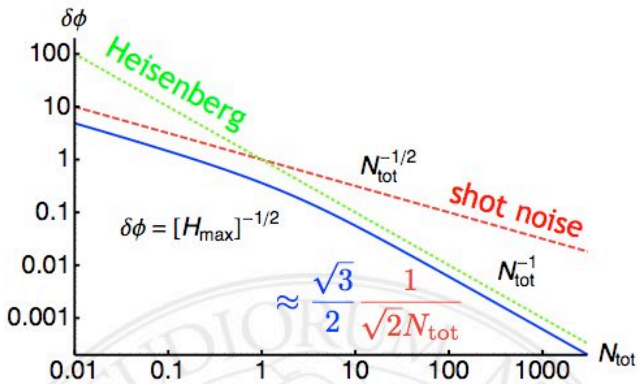
$$\beta_{\rm tot} = \frac{2\sinh^2 r}{N_{\rm tot}}$$

Optimized quantum Fisher information:

$$H_{\rm max}(N_{\rm tot}, \beta_{\rm tot}) = 2N_{\rm tot}\Big[2 + N_{\rm tot}\beta_{\rm tot}(2-\beta_{\rm tot}) + 2(1-\beta_{\rm tot})\sqrt{N_{\rm tot}\beta_{\rm tot}(2+N_{\rm tot}\beta_{\rm tot})}\Big]$$

Passive interferometer: ultimate bounds





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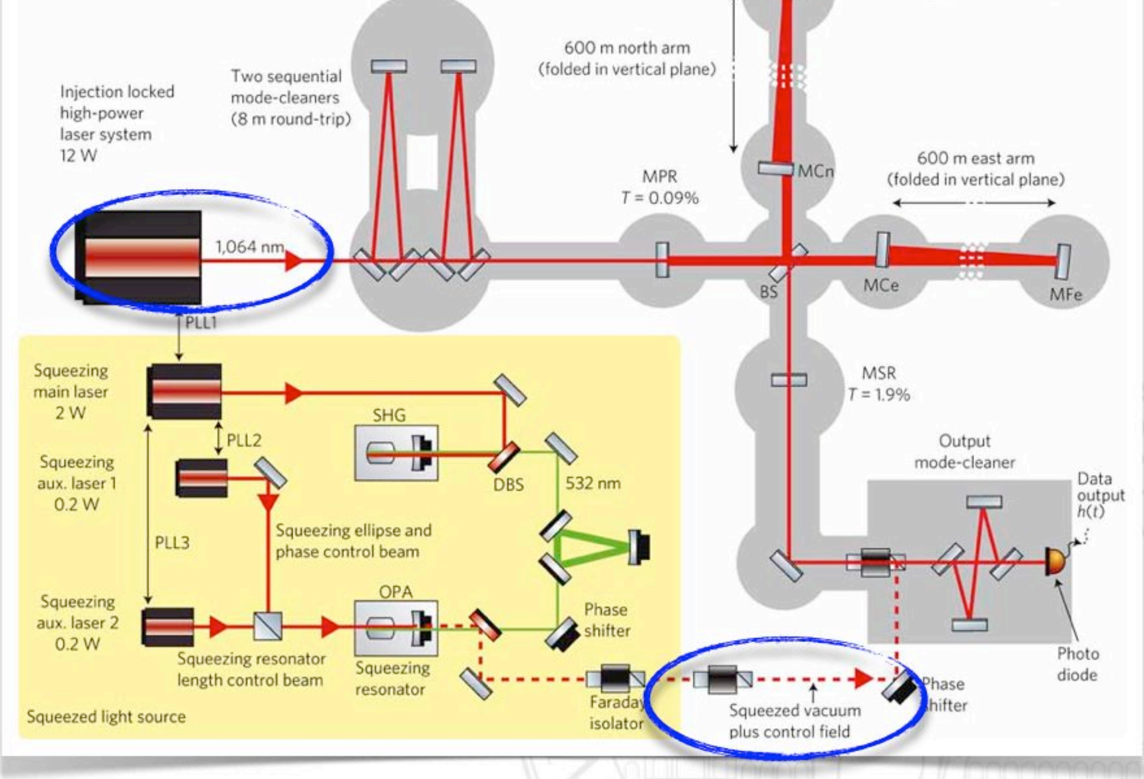
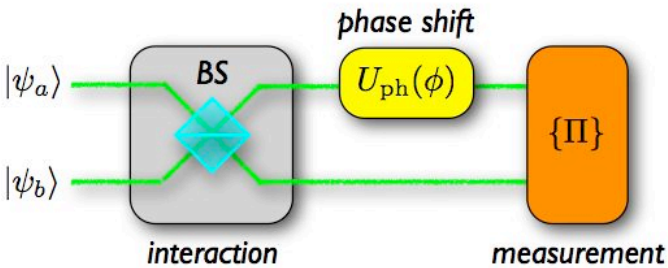
GEO 600 *(gw configuration)*



Nature Physics **7**, 962 (2011)

Passive interferometer: ultimate bounds



$$U_{\rm ph}(\phi) = e^{-i\phi\, a^\dagger a} \otimes \mathbb{I}$$

$$|\psi_a\rangle \otimes |\psi_b\rangle = |\alpha, \cancel{\xi}\rangle \otimes |\cancel{\gamma}, \zeta\rangle$$

gw configuration

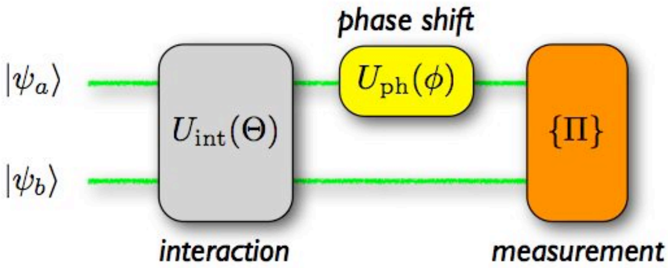
Quantum Fisher information (gw configuration):

$$H^{(\rm gw)}_{\max} \approx 2N^2_{\rm tot} \quad (N_{\rm tot} \gg 1, \beta_{\rm tot} \to 1)$$

Optimized quantum Fisher information:

$$H_{\max} \approx \frac{8}{3} N^2_{\rm tot} \approx 2.67\, N^2_{\rm tot} \quad (N_{\rm tot} \gg 1)$$

Passive & active interferometers



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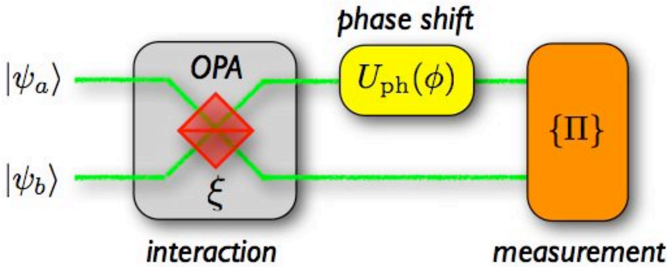
Active devices, two-mode squeezing, SU(1,1):

$$U_{\rm int}(\Theta) = \exp\left(\xi a^\dagger b^\dagger - \xi^* a b\right)$$

Active interferometer: ultimate bounds



$$U_{\rm ph}(\phi) = {\rm e}^{-i\phi\, a^\dagger a} \otimes \mathbb{I}$$

$$U_{\rm int}(\Theta) = \exp\left(\xi a^\dagger b^\dagger - \xi^* a b\right)$$

$$\xi = r\,{\rm e}^{-i\theta} \in \mathbb{C}$$

Two coherent states as input:

$$|\psi_a\rangle \otimes |\psi_b\rangle = |\alpha\rangle \otimes |\gamma\rangle \qquad \alpha, \gamma \in \mathbb{R}$$

Signal fraction

$$\Delta = \frac{\alpha^2}{\alpha^2 + \gamma^2}$$

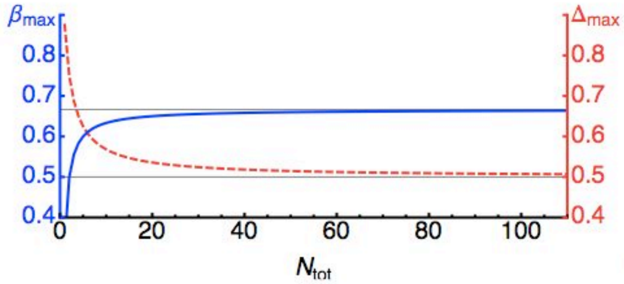
Partial squeezing fraction

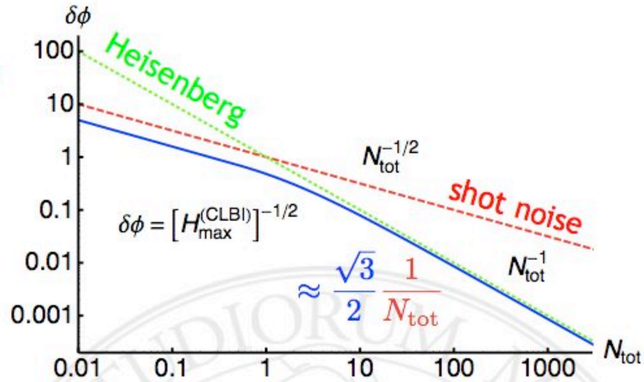
$$\beta = \frac{\sinh^2 r}{N_{\rm tot}}$$

Total number of photons:

$$N_{\rm tot} = (\alpha^2 + \gamma^2 + 1)\cosh(2r) + 2\alpha\gamma\cos(\theta)\sinh(2r) - 1$$

Active interferometer: ultimate bounds





Signal fraction

$$\Delta = \frac{\alpha^2}{\alpha^2 + \gamma^2}$$

Partial squeezing fraction

$$\beta = \frac{\sinh^2 r}{N_{tot}}$$

Total number of photons:

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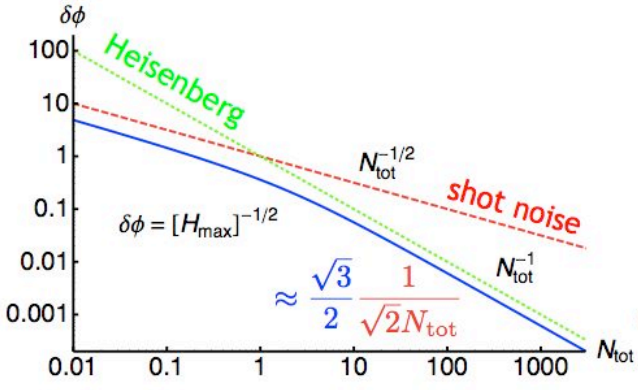
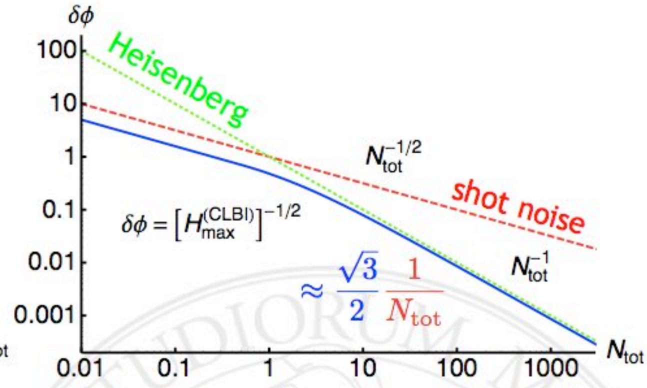
Passive vs active interferometer



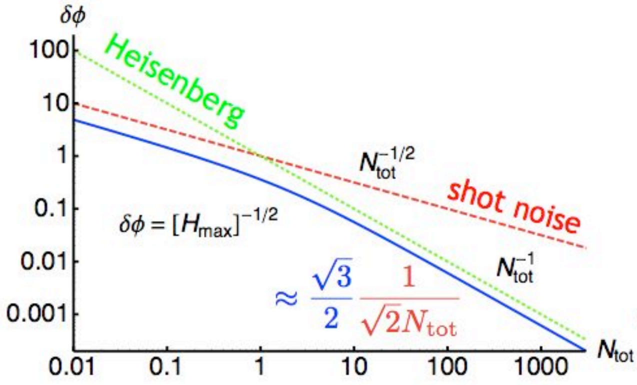


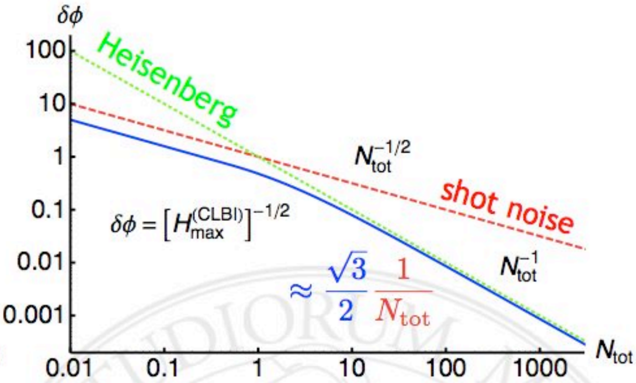
Passive interferometers
better performances

Active interferometers

C. Sparaciari, S. Olivares and M. G. A. Paris, J. Opt. Soc. Am. B **32**, 1354 (2015)

Passive vs active interferometer





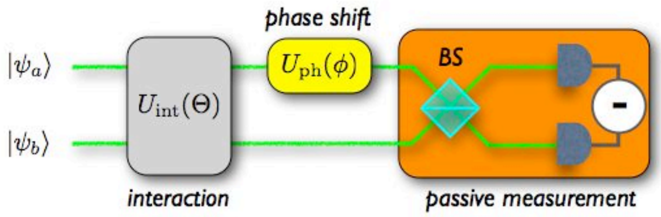
Passive interferometers
better performances

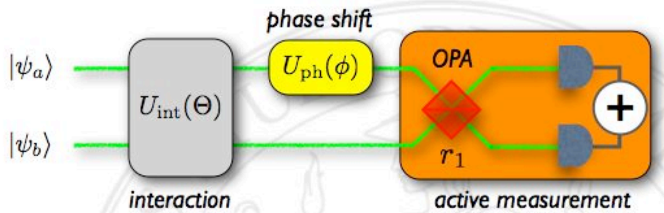
Active interferometers

However, *amplification* can be useful at the *detection stage* to *fight losses (non-unit quantum efficiency)...*

C. Sparaciari, S. Olivares and M. G. A. Paris, J. Opt. Soc. Am. B **32**, 1354 (2015)

Photon number detection

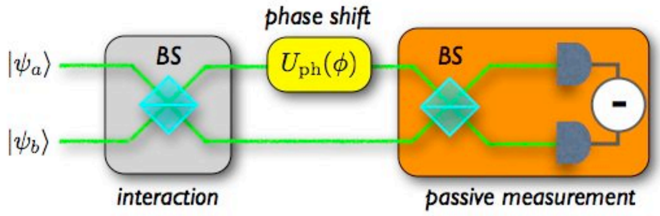




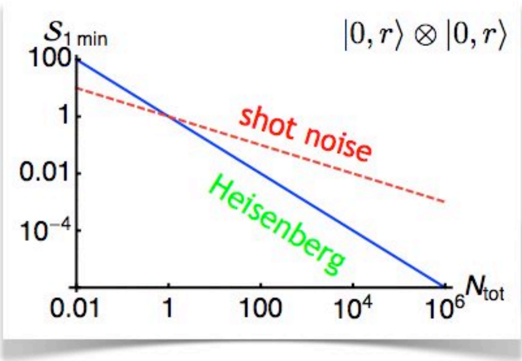
Observable: $D_\pm = N_a \pm N_b$

Sensitivity: $\mathcal{S} = \frac{\sqrt{\mathrm{Var}[D_\pm]}}{|\partial_\phi D_\pm|} \approx \frac{1}{\sqrt{F(\phi)}}$ ← *Fisher information*

Passive int. & photon number detection

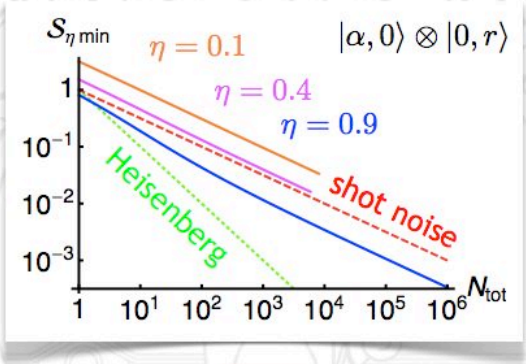


Unit quantum efficiency
(after optimization)



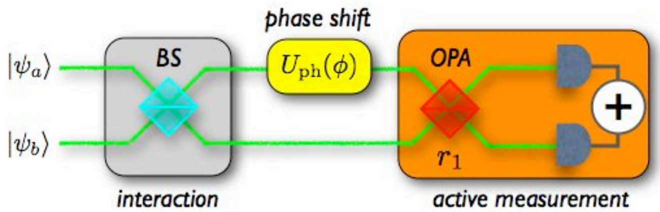
$$\mathcal{S} \approx \frac{1}{N_{\rm tot}}$$

Non-unit quantum efficiency
(after optimization)



$$\mathcal{S} \approx \sqrt{\frac{1-\eta}{\eta}}\,\frac{1}{\sqrt{N_{\rm tot}}}$$

Passive int. & photon number detection



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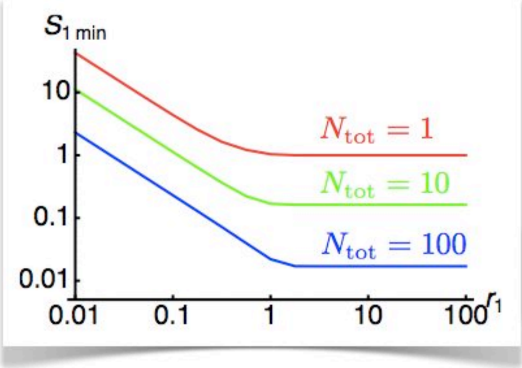
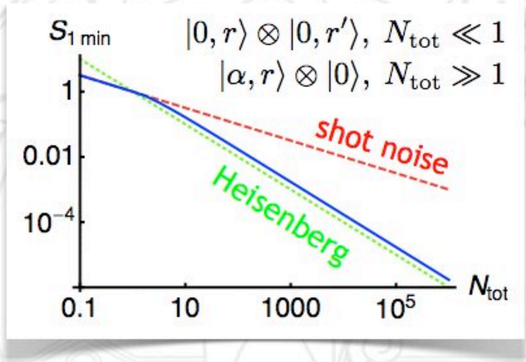
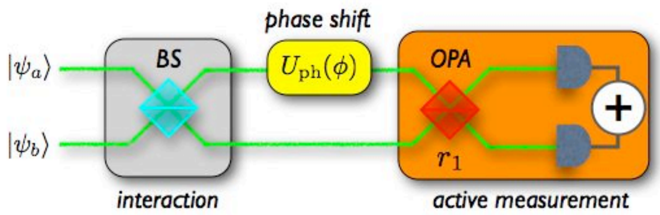




$$\mathcal{S} \approx \left(1 + \frac{1}{\sqrt{2}}\right)\frac{1}{N_{\rm tot}}$$

Passive int. & photon number detection



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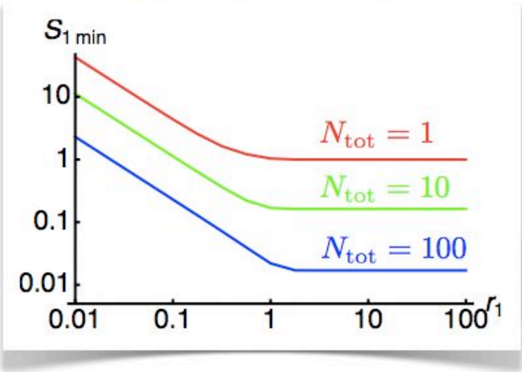

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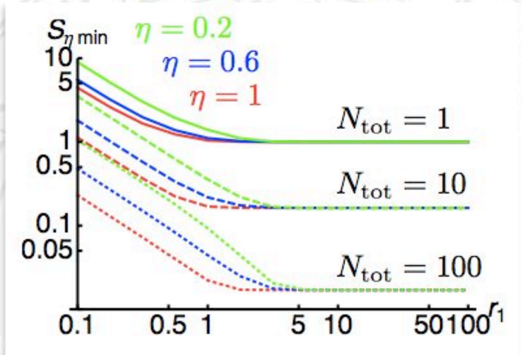

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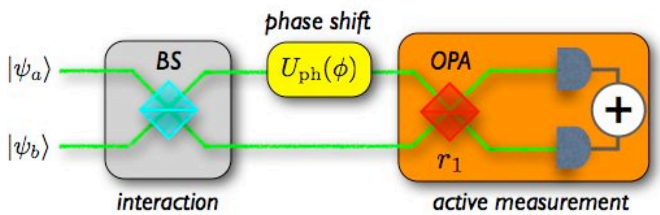

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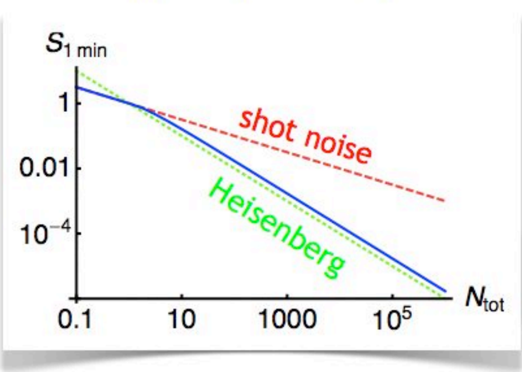

Non-unit quantum efficiency
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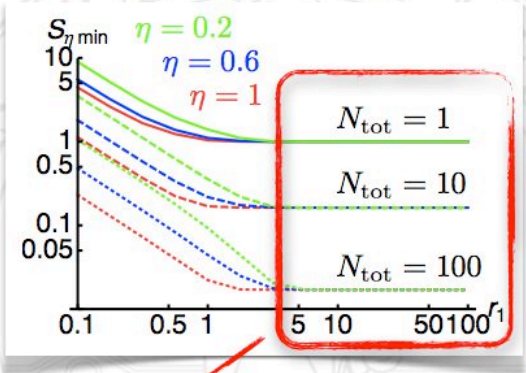
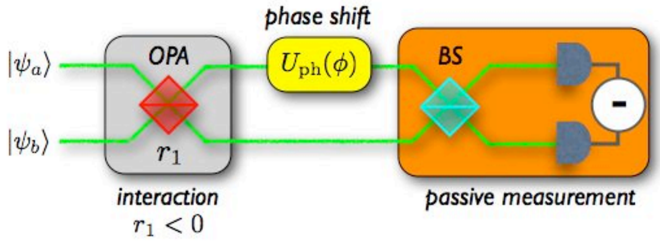


$$S \approx \left(1 + \frac{1}{\sqrt{2}}\right) \frac{1}{N_{\rm tot}}$$

Active int. & photon number detection



Unit quantum efficiency
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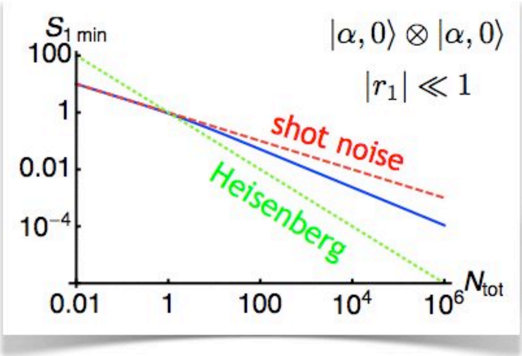
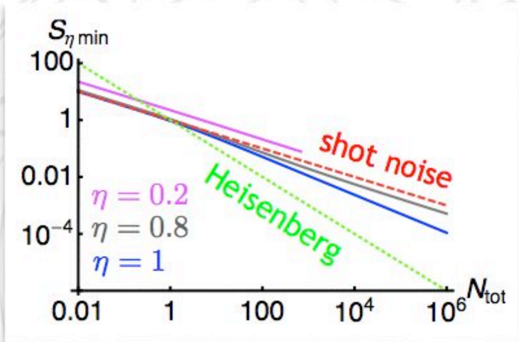


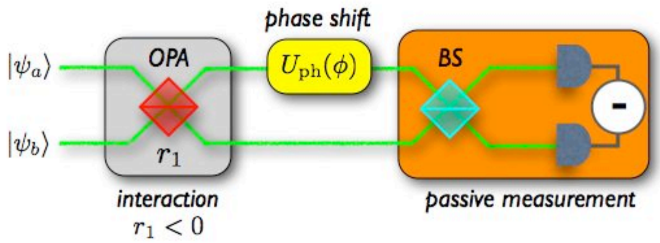
$$S \approx \frac{1}{N_{\rm tot}^{2/3}}$$

Non-unit quantum efficiency
(after optimization)



$$S \approx \frac{1}{N_{\rm tot}^{\epsilon(\eta)}}$$

Active int. & photon number detection



Unit quantum efficiency
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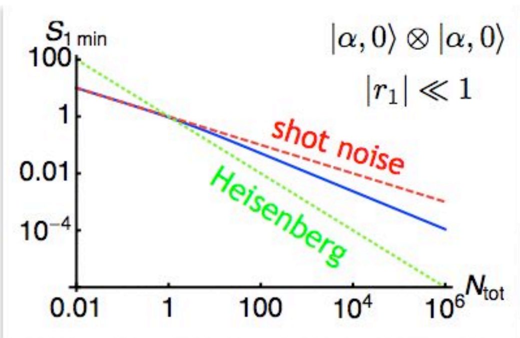

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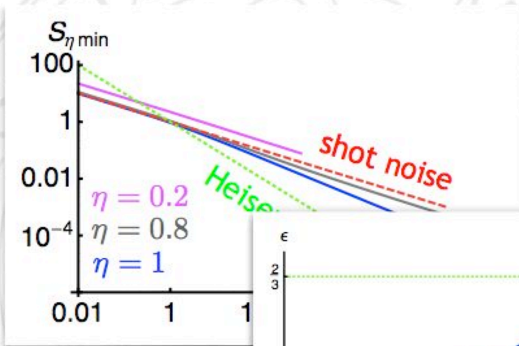
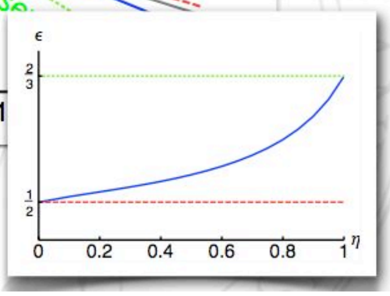
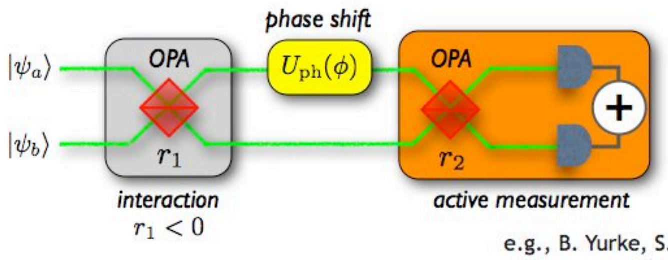
Non-unit quantum efficiency
(after optimization)





$$S \approx \frac{1}{N_{\rm tot}^{\epsilon(\eta)}}$$

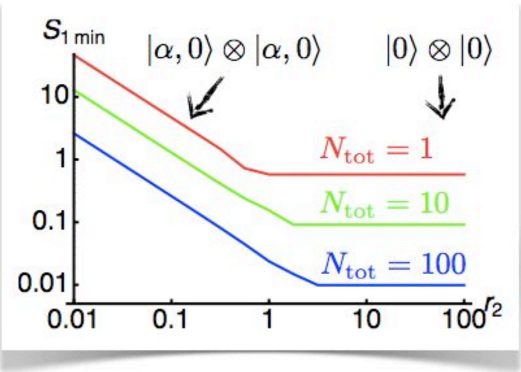
Active int. & photon number detection

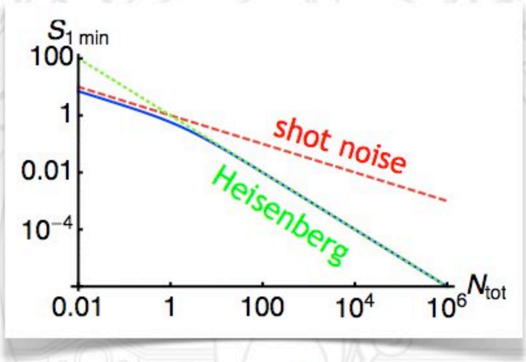


e.g., B. Yurke, S. L. McCall, and J. R. Klauder, Phys Rev. A **33**, 4033–4054 (1986)

W. N. Plick, *J. P. Dowling, and G. S. Agarwal,* New J. Phys. **12**, 083014 (2010)

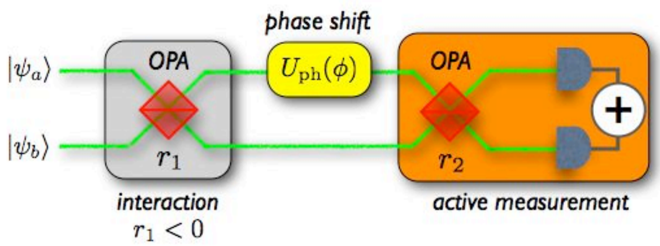
Unit quantum efficiency
(after optimization)





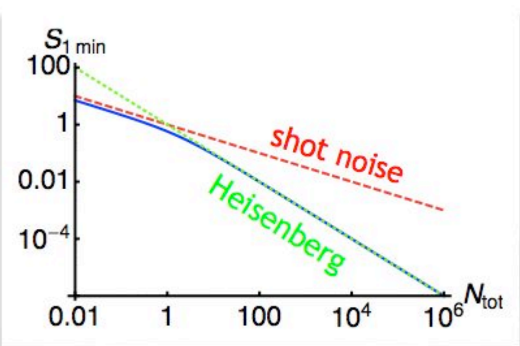
$$S \approx \frac{1}{\sqrt{N_{\rm tot}(N_{\rm tot}+2)}}$$

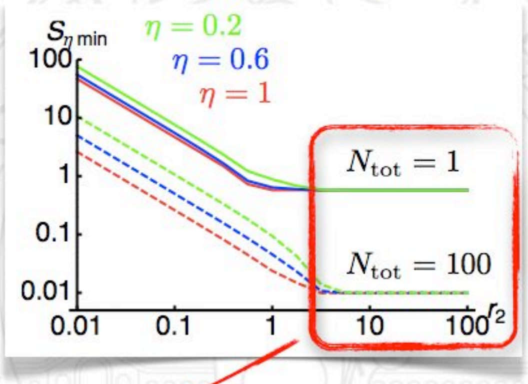
Active int. & photon number detection



Unit quantum efficiency
(after optimization)

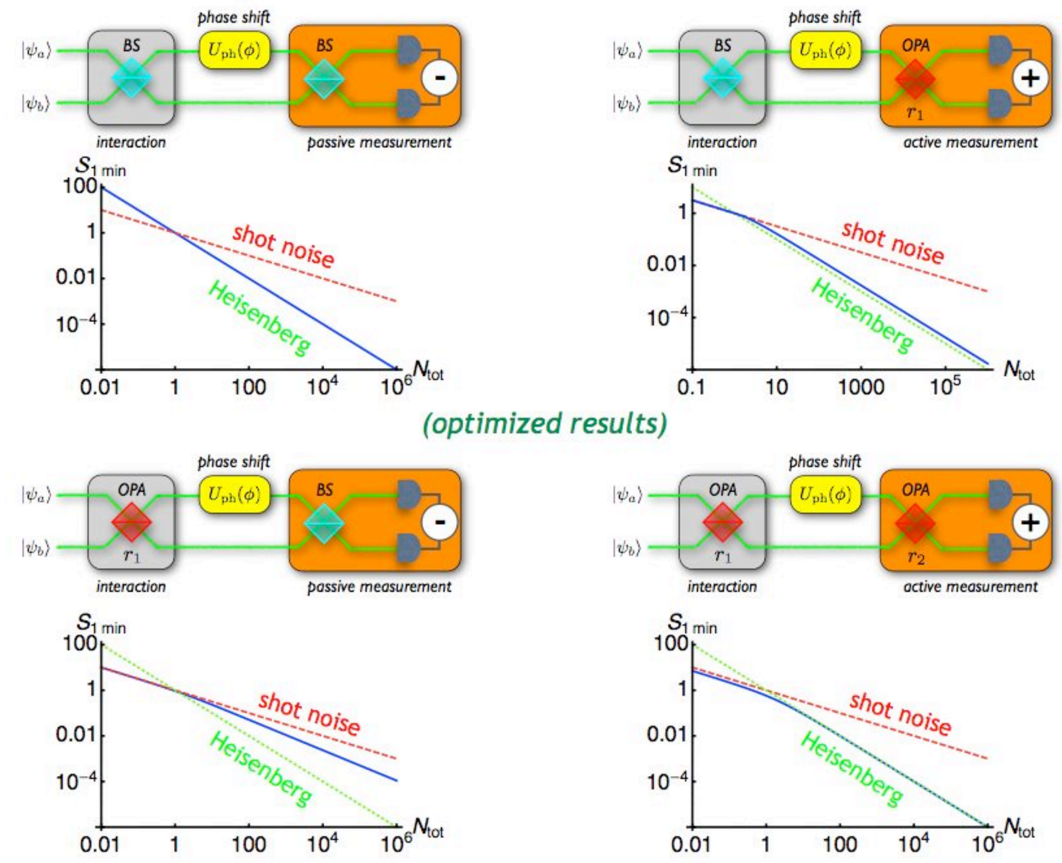
Non-unit quantum efficiency
(after optimization)



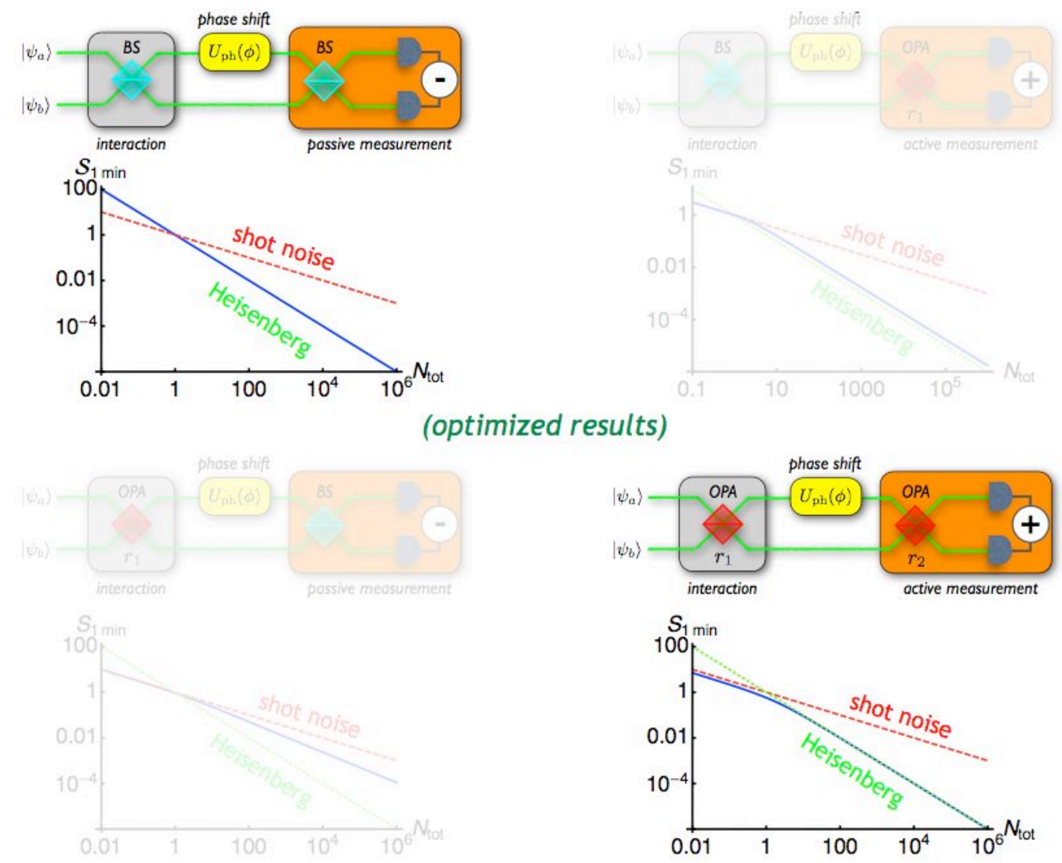


$$S \approx \frac{1}{\sqrt{N_{\rm tot}(N_{\rm tot}+2)}}$$

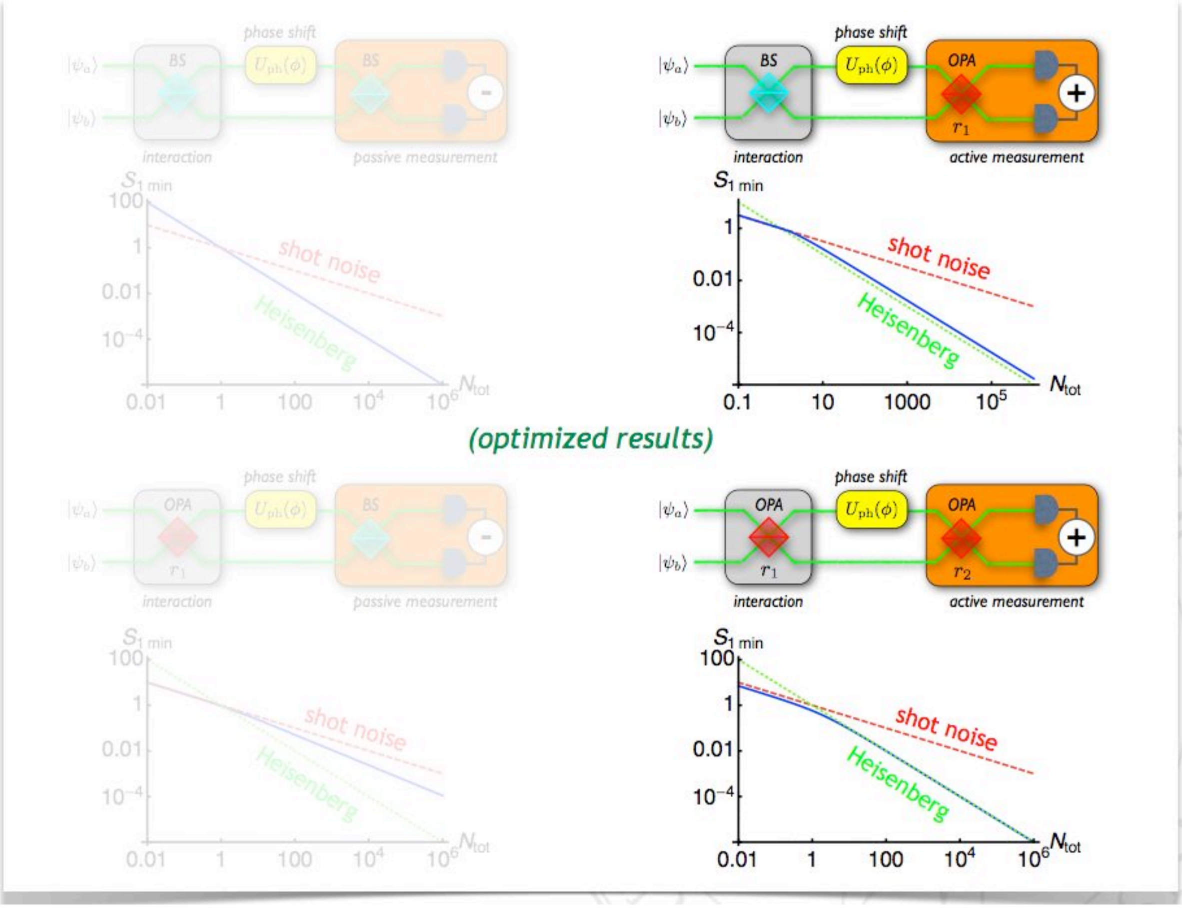
Summary



Summary: unit quantum efficiency

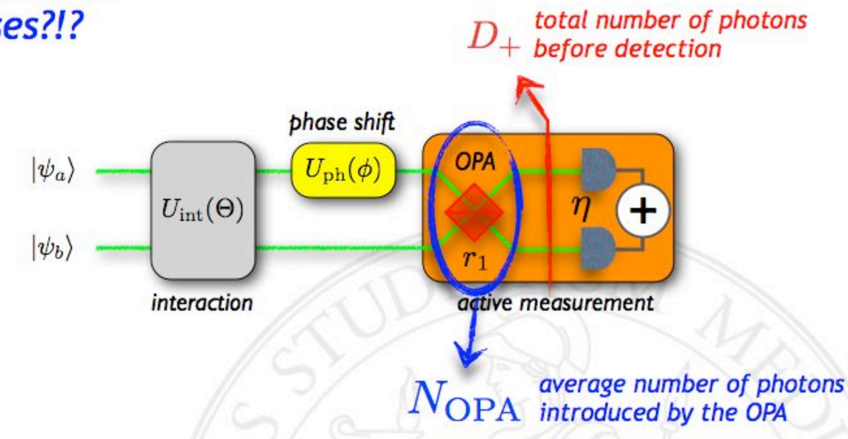


Summary: non-unit quantum efficiency



Summary: non-unit quantum efficiency

Why does active detection compensate for losses?!?



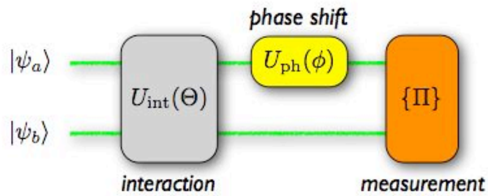
$$\mathcal{S}_\eta = \mathcal{S}_1 \sqrt{1 + \frac{1-\eta}{\eta}\,\frac{D_+}{\Delta D_+^2}} \approx \mathcal{S}_1 \quad (N_{\rm OPA} \gg 1)$$

... this is the sensitivity of the ideal case ($\eta = 1$)!

Concluding remarks

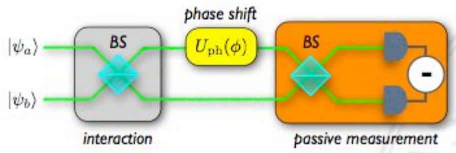
Passive & active interferometry

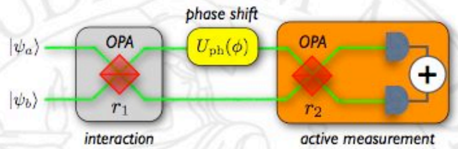
Best strategy:
passive device + squeezed-coherent states



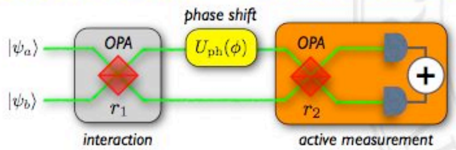
C. Sparaciari, S. Olivares and M. G. A. Paris, J. Opt. Soc. Am. B **32**, 1354 (2015)

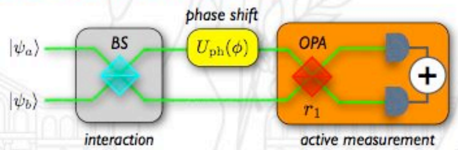
Photon number detection (ideal):
optimized input states + symmetric configuration = *Heisenberg scaling!*





Photon number detection (losses):
optimized input states + active detection = *ideal case results!*





C. Sparaciari, S. Olivares and M. G. A. Paris, *coming soon!*

Thank you for your attention!

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- *MIUR, Firb "LiCHIS" (RBFR10YQ3H)*
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- *UniMi, "UNIMI14" (15-6-3008000-609)*