Exponential Rise of Dynamical Complexity in Quantum Computing through Projections

Nature Commun. **5**, 5173 (2014) &
arXiv:1411.0316 [quant-ph]

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with

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Introduction

- Measurement in Quantum Control
- Quantum Zeno Effect/Dynamics

Measurement can make a big difference!

Trivial Controls

⇩ **measurement**

Universal Quantum Computation

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Quantum Control

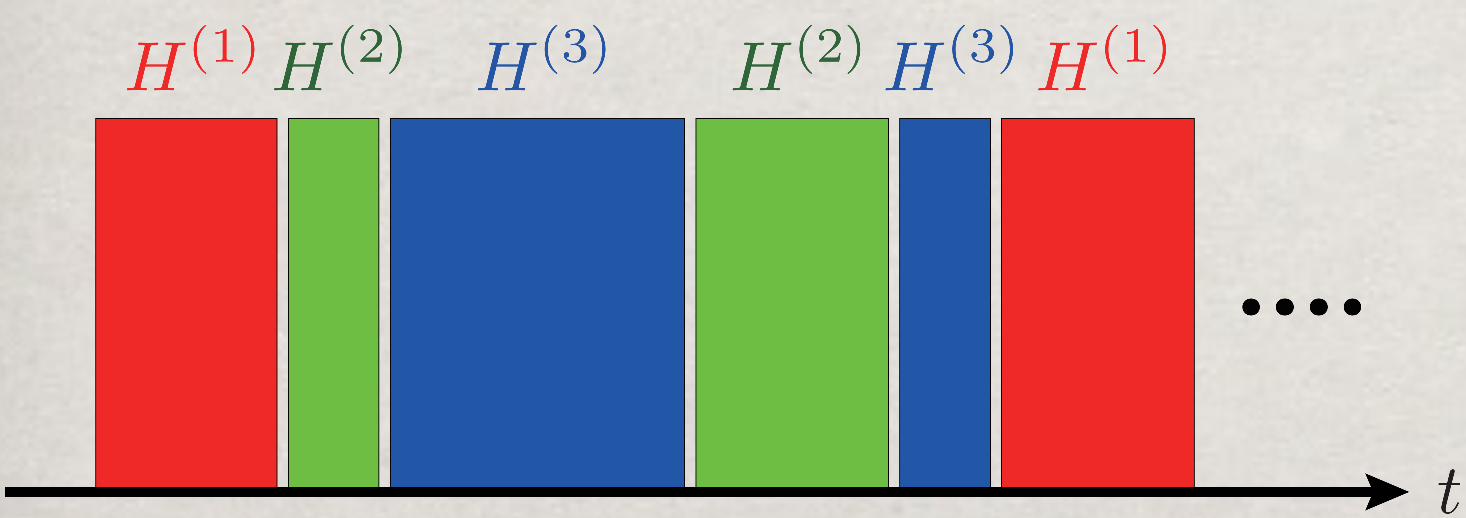
Given a set of Hamiltonians $\{H^{(k)}\}$,

$$H(t) = H^{(0)} + \sum_{k=1}^{m} f_k(t) H^{(k)}$$

→ any unitary $U(t)$

?

control pulses $f_k(t)$



"fully controllable"

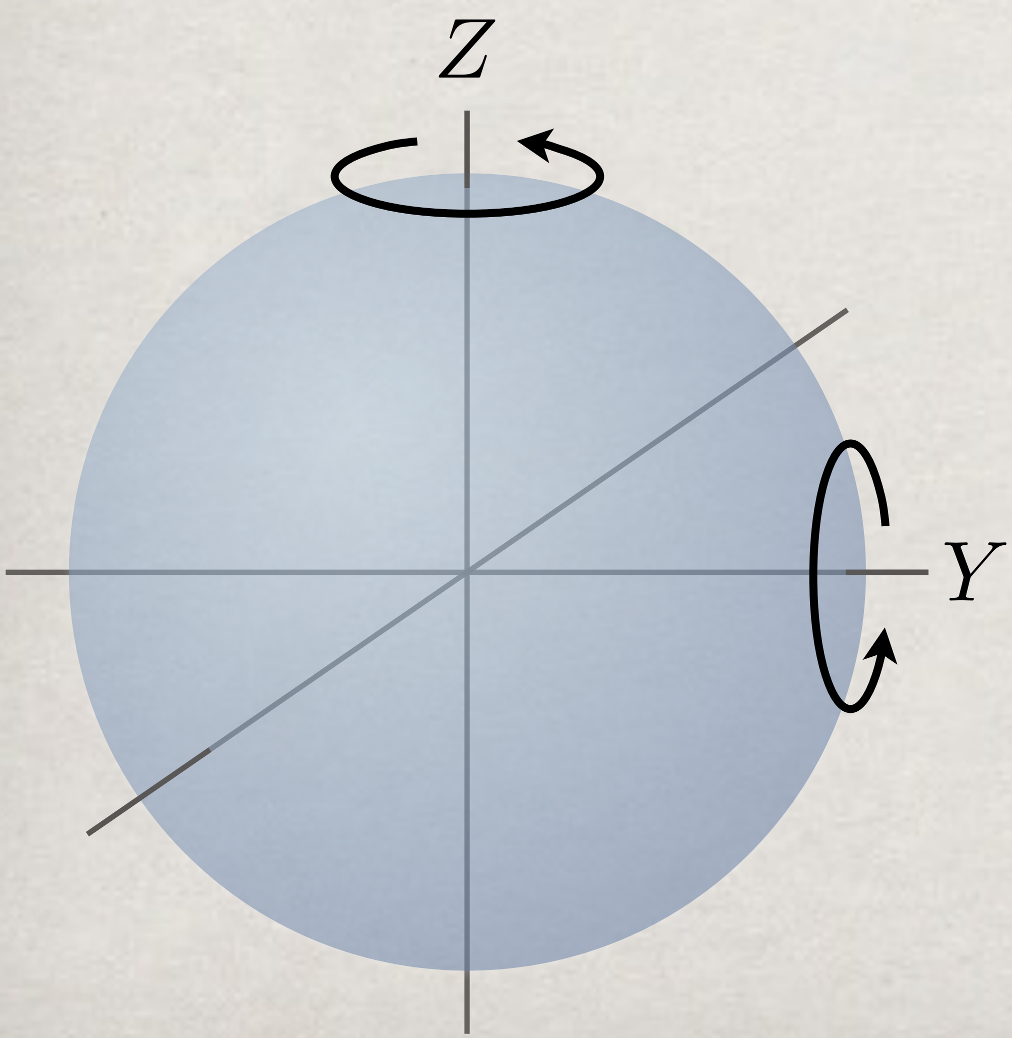
||

univ. quant. comp.

Single Qubit

Any unitary can be realized with $\{Y, Z\}$:

$$U = e^{-\frac{i}{2}\varphi Z} e^{-\frac{i}{2}\theta Y} e^{-\frac{i}{2}\chi Z}$$

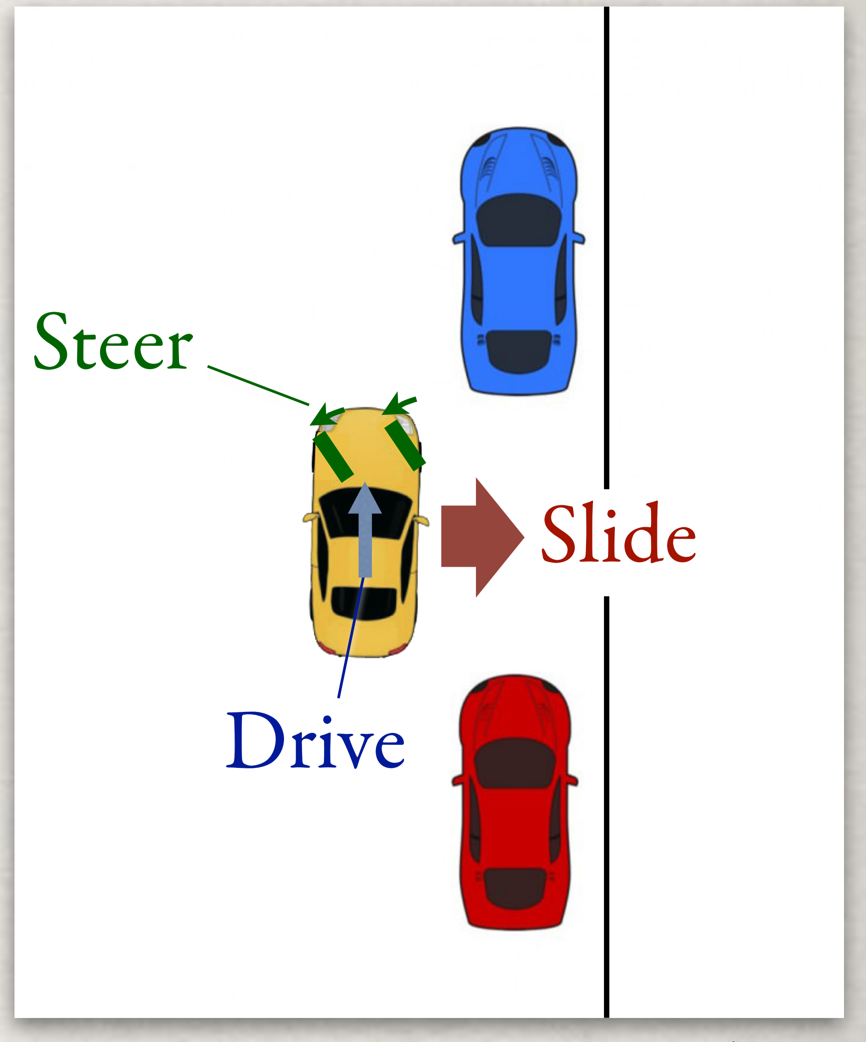


Y and Z are enough for the full controllability.

Parallel Parking

E. Nelson, *Tensor Analysis* (Princeton Univ. Press, 1967).

Our controls: {Steer, Drive}



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Drive → −Steer → −Drive → Steer

→ −Drive → −Steer → Drive → Steer ⋯

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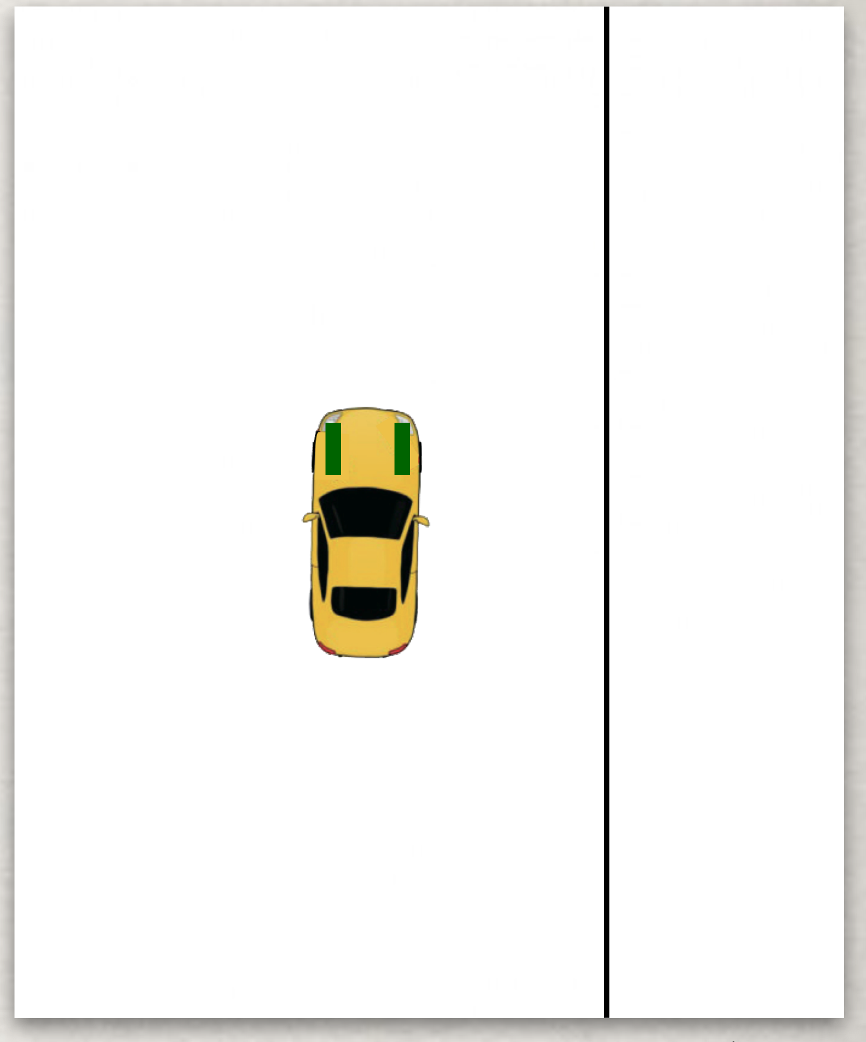

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Drive ⟶ −Steer ⟶ −Drive ⟶ Steer

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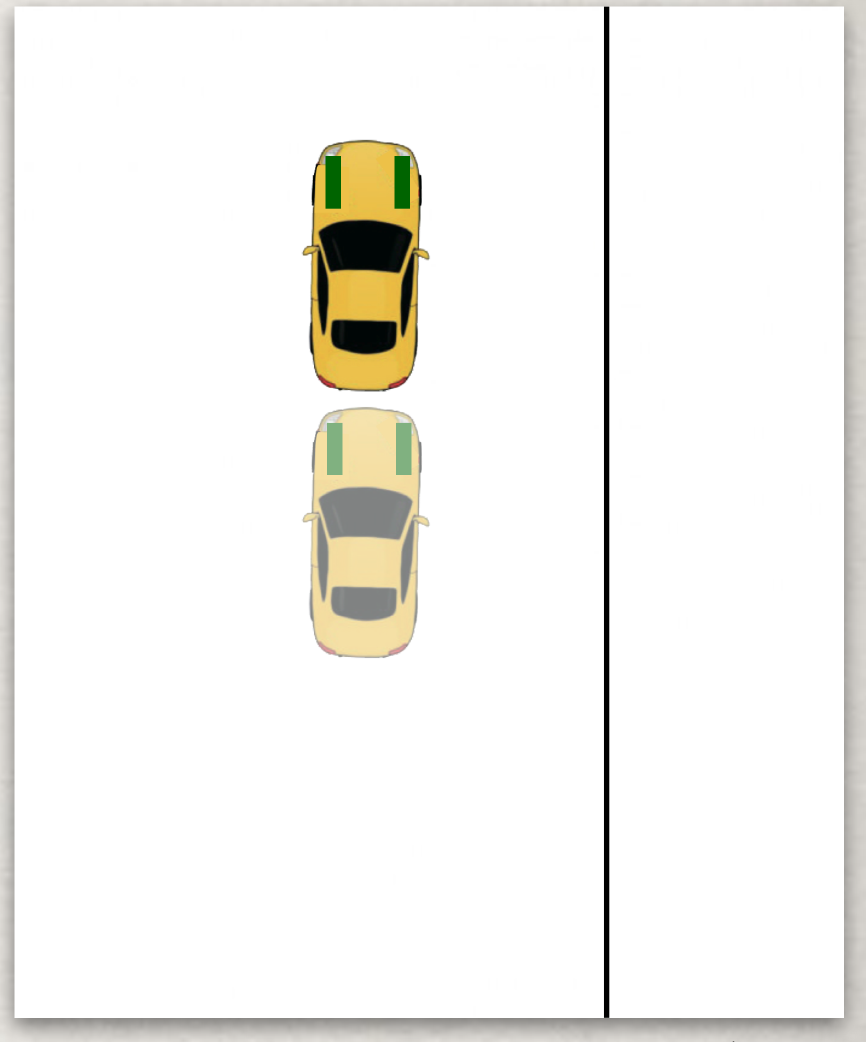

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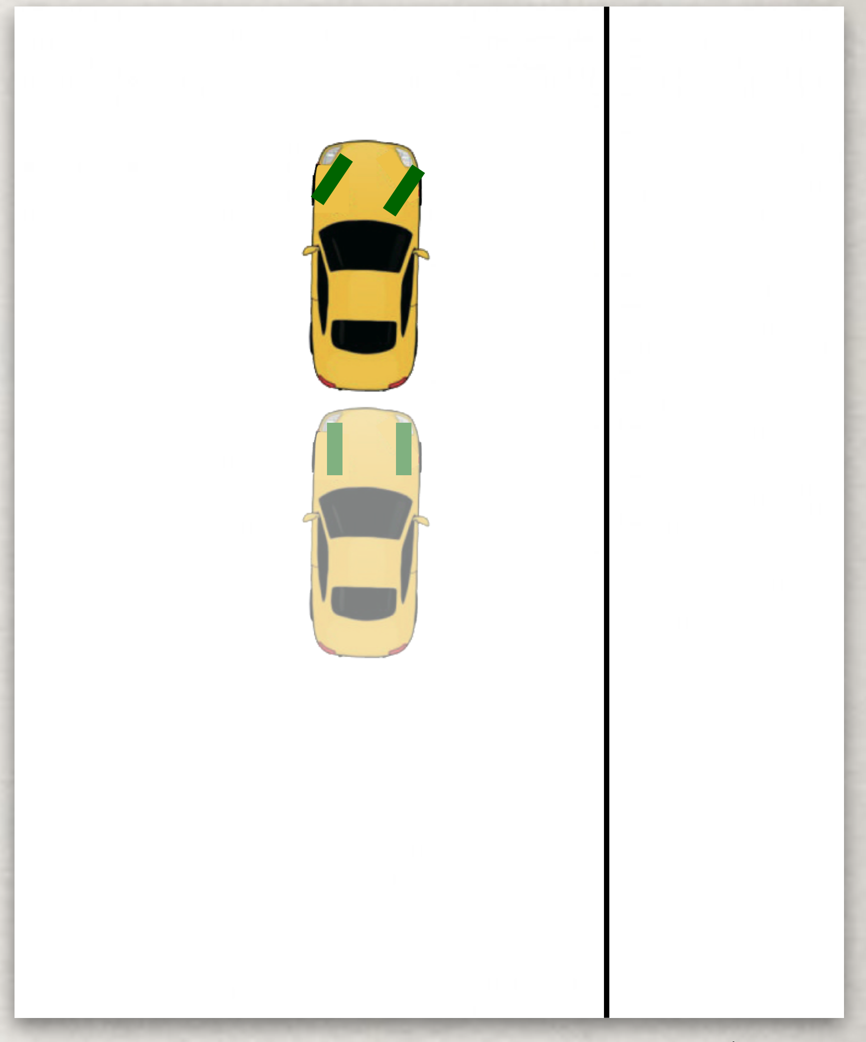

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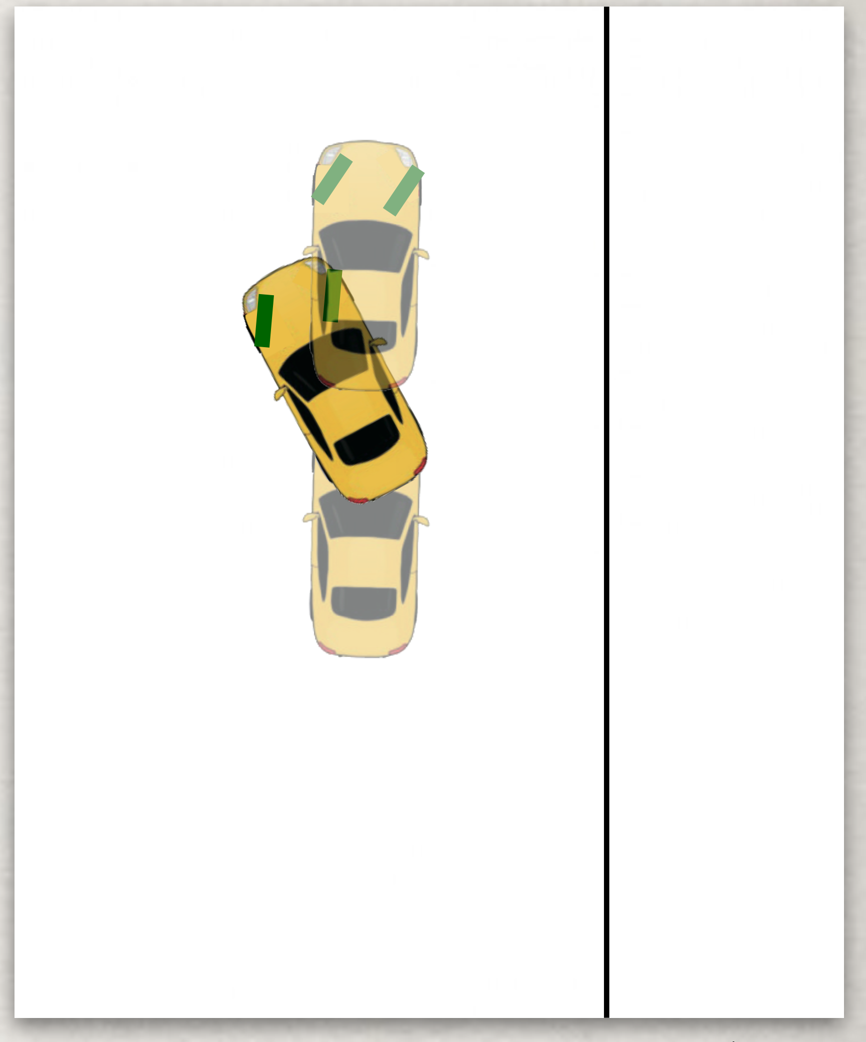

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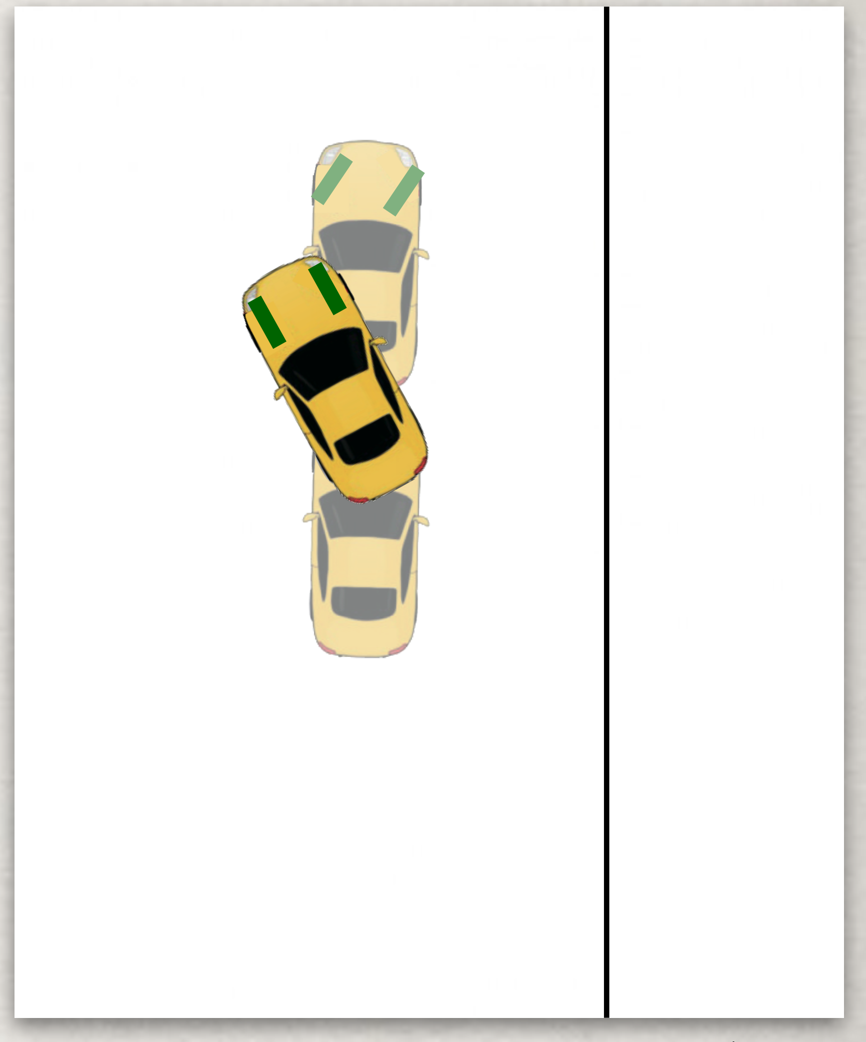

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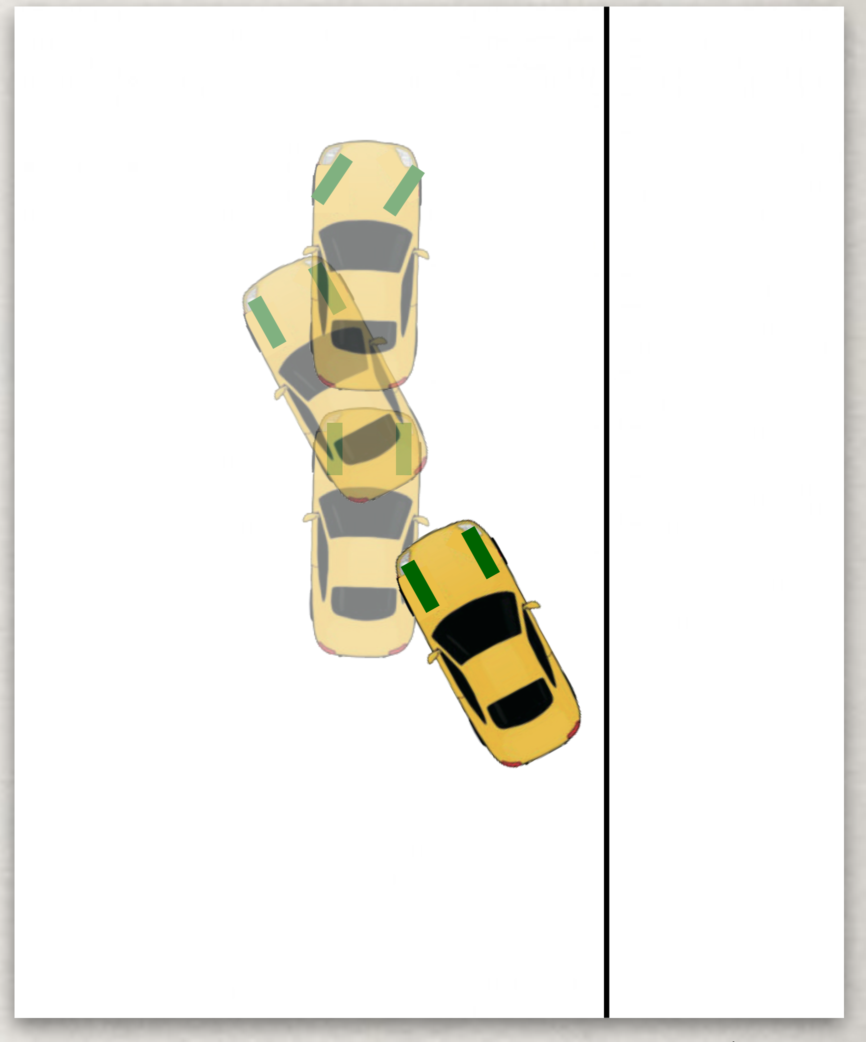

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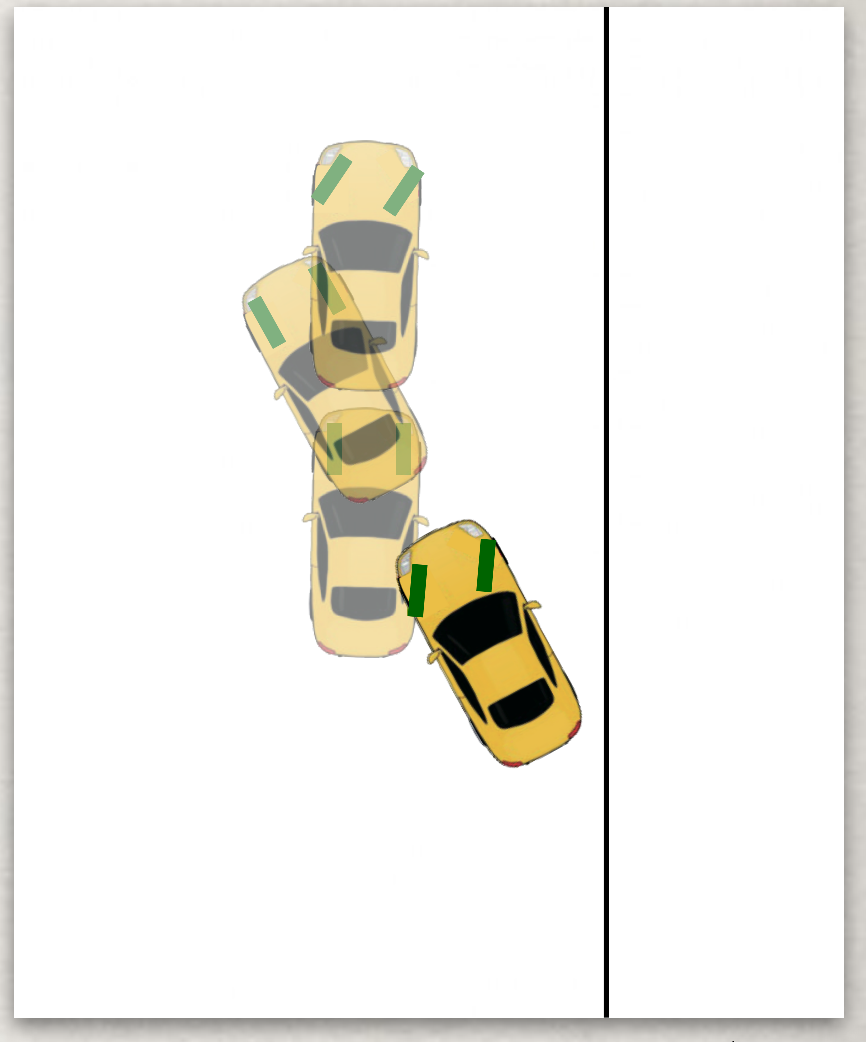

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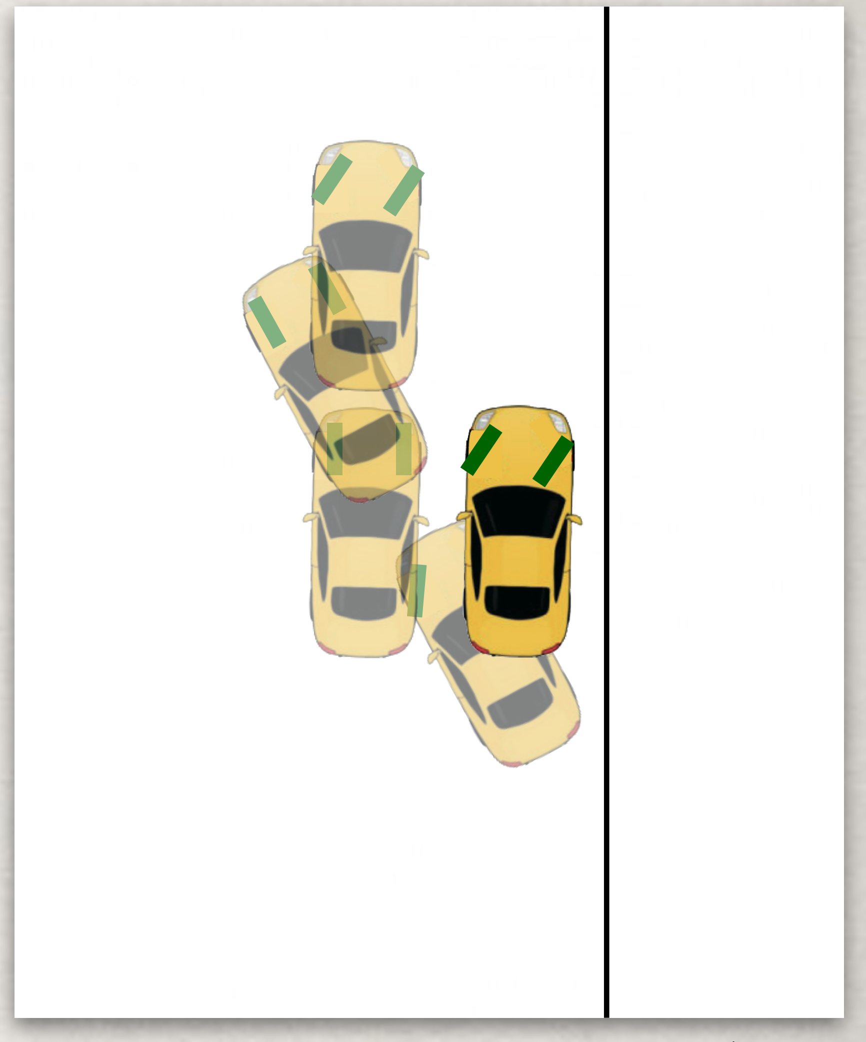

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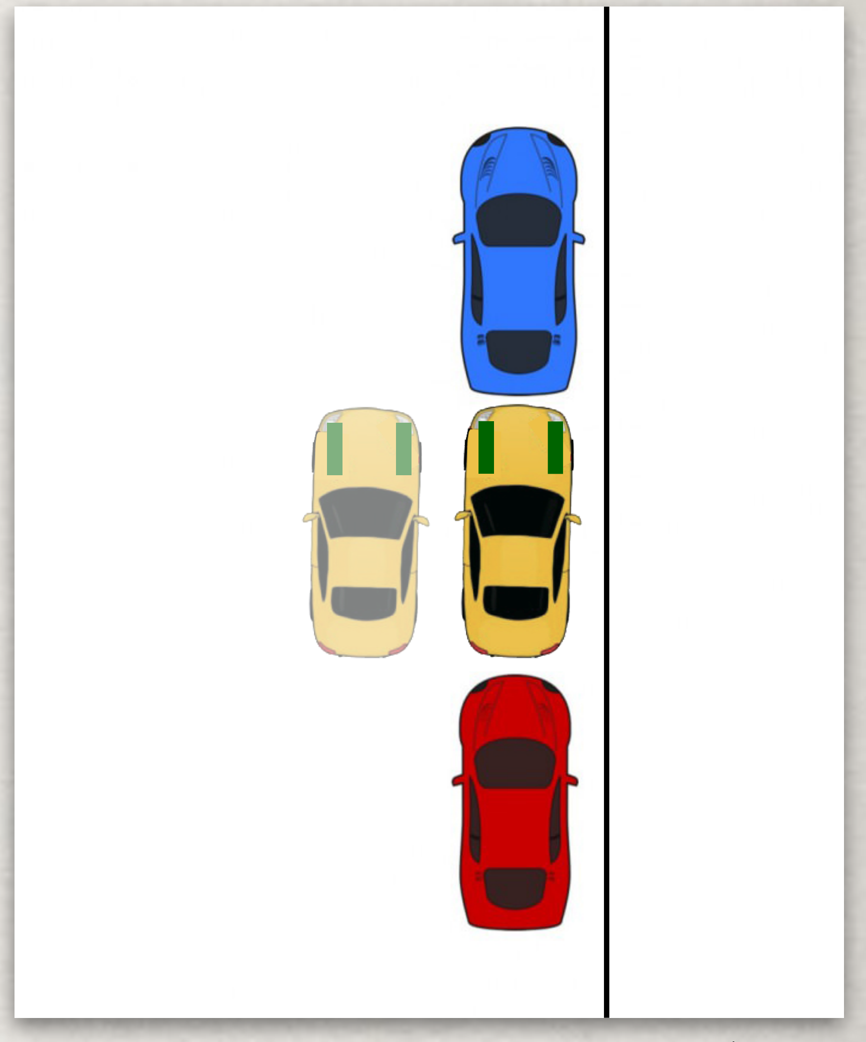

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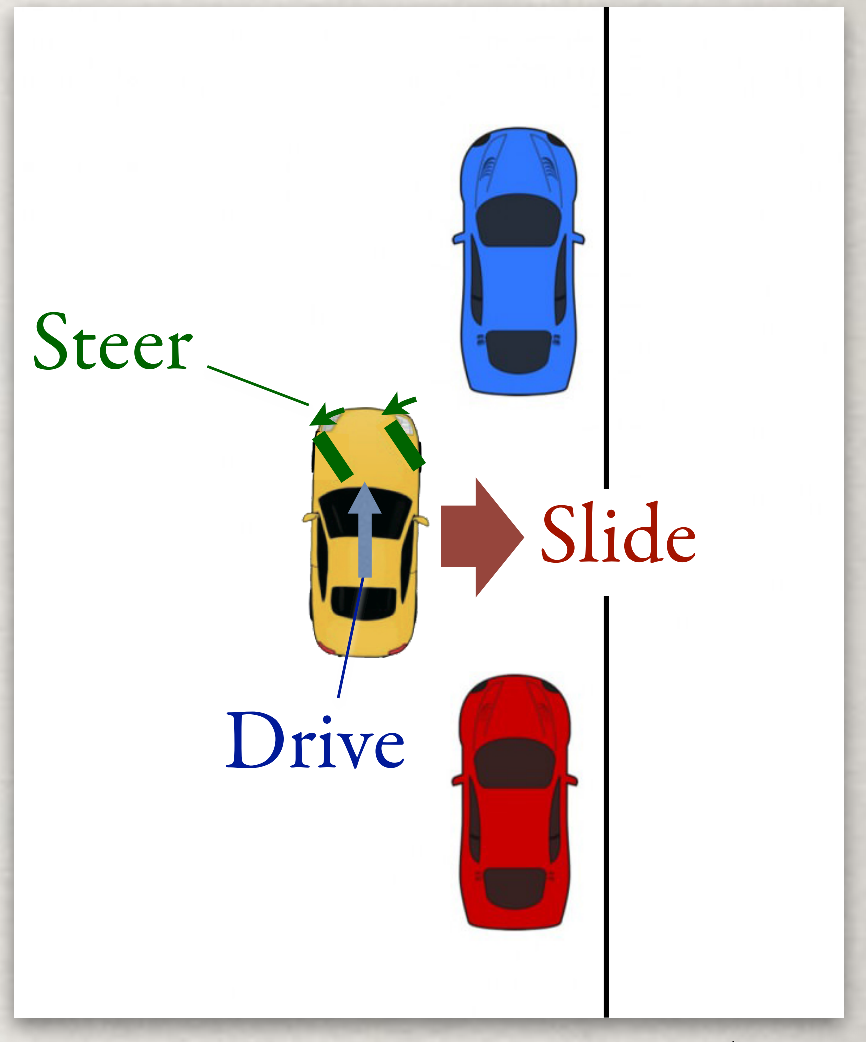

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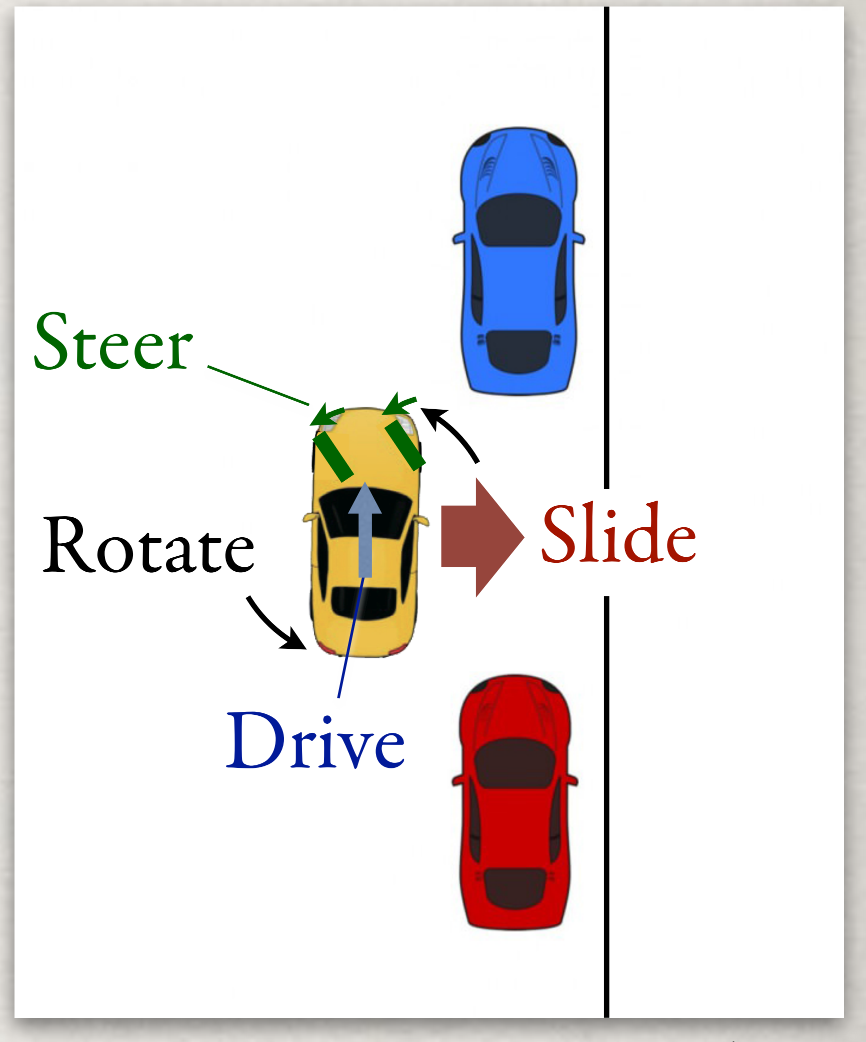
Our controls: $\{\text{Steer}, \text{Drive}\}$

$$[\text{Steer}, \text{Drive}] = -\text{Slide} + \text{Rotate}$$

$$[[\text{Steer}, \text{Drive}], \text{Drive}] = -\text{Slide}$$



imgur.com

Drive → −Steer → −Drive → Steer → −Drive → −Steer → Drive → Steer ⋯

- $e^{\sqrt{t}S}e^{-\sqrt{t}D}e^{-\sqrt{t}S}e^{\sqrt{t}D} = e^{-t[S,D]} + O(t^{3/2})$
- $e^{\sqrt[4]{t}S}e^{\sqrt[4]{t}D}e^{-\sqrt[4]{t}S}e^{-\sqrt{t}D}e^{\sqrt[4]{t}S}e^{-\sqrt[4]{t}D}e^{-\sqrt{t}S}e^{\sqrt[4]{t}D} = e^{-t[[S,D],D]} + O(t^{5/4})$

Lie Algebra and Controllability

$\mathfrak{Lie}\{H^{(0)}, \ldots, H^{(m)}\}$

Lie algebra of Hamiltonians

generated by $H^{(k)}$ and their iterated commutators

$$i[H^{(j)}, H^{(k)}], \quad i[i[H^{(j)}, H^{(k)}], H^{(\ell)}], \quad \ldots$$

fully controllable if

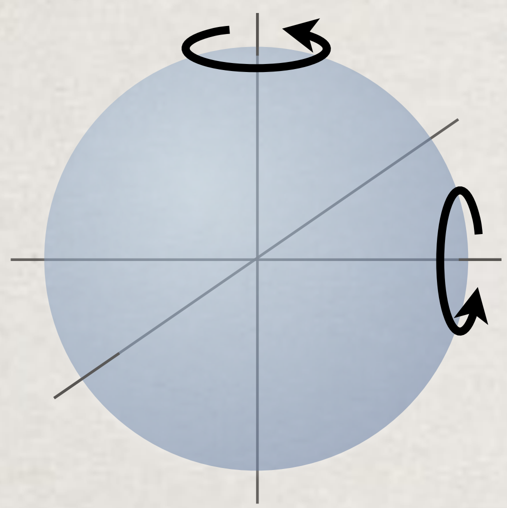
$$\mathfrak{Lie}\{H^{(0)}, \ldots, H^{(m)}\} = \mathfrak{su}(d) \ \ldots \text{ full algebra}$$

dim. of system

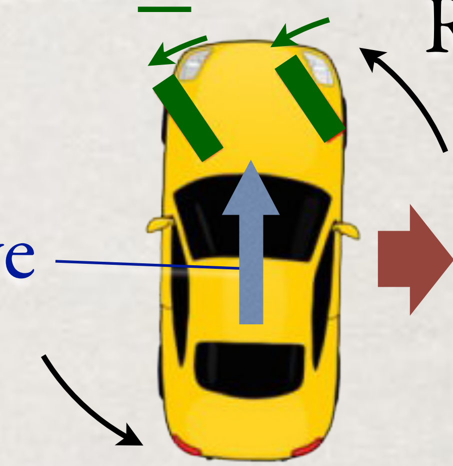
D. D'Alessandro, *Introduction to Quantum Control and Dynamics* (Chapman & Hall, 2008).

Examples

single qubit



car parking



Controls: $\{Y, Z\}$

$$\Rightarrow \begin{cases} i[Y, Z] = -2X \\ i[X, Y] = -2Z \\ i[X, Z] = 2Y \end{cases}$$

$\mathfrak{Lie}\{Y, Z\} = \mathfrak{su}(2)$

full algebra

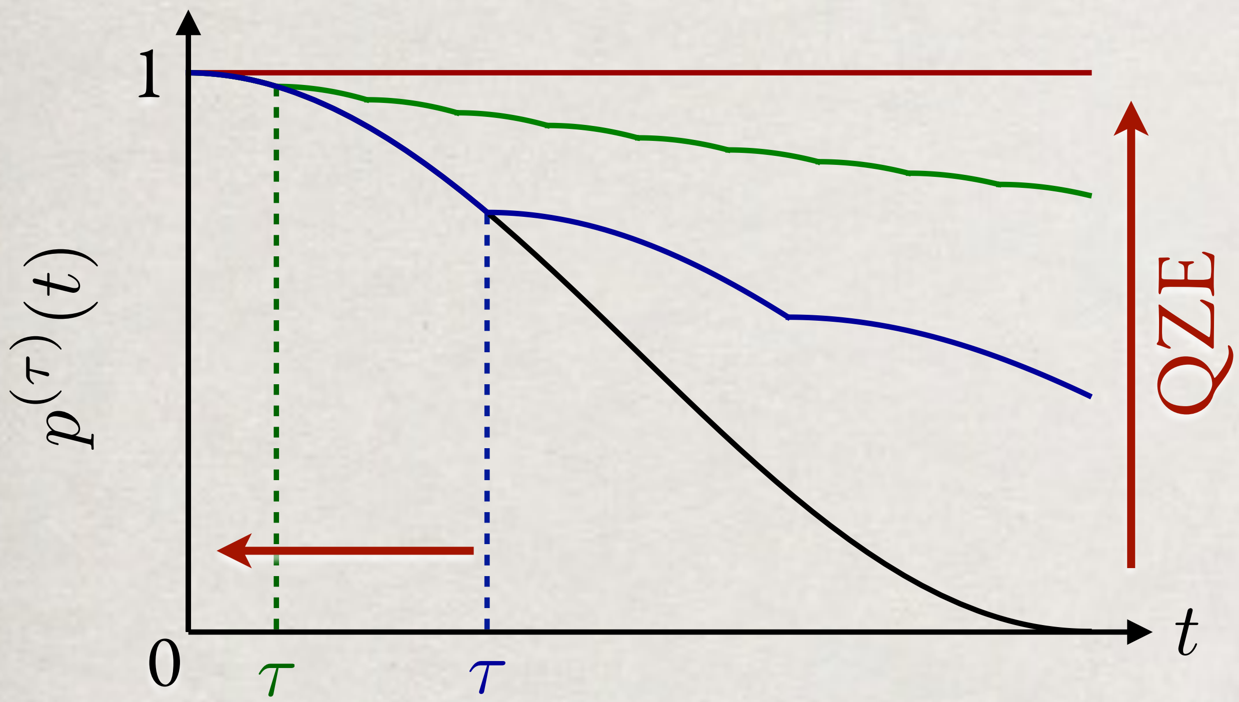
Controls: $\{\text{Steer}, \text{Drive}\}$

$$\Rightarrow \begin{cases} [\text{Steer}, \text{Drive}] = -\text{Slide} + \text{Rotate} \\ [[\text{Steer}, \text{Drive}], \text{Steer}] = \text{Drive} \\ [[\text{Steer}, \text{Drive}], \text{Drive}] = -\text{Slide} \end{cases}$$

$\mathfrak{Lie}\{\text{Steer}, \text{Drive}\} \cdots \dim = 4$

full algebra

Quantum Zeno Effect (QZE)

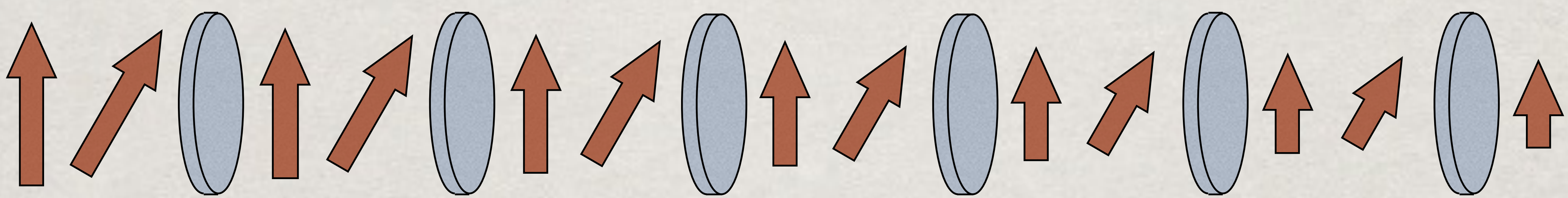


$$p(t) \sim 1 - \frac{t^2}{\tau_Z^2}$$

$$p_N(t) = [p(t/N)]^N$$

$$\sim \left(1 - \frac{t^2}{N^2 \tau_Z^2}\right)^N \xrightarrow{N\to\infty} 1$$

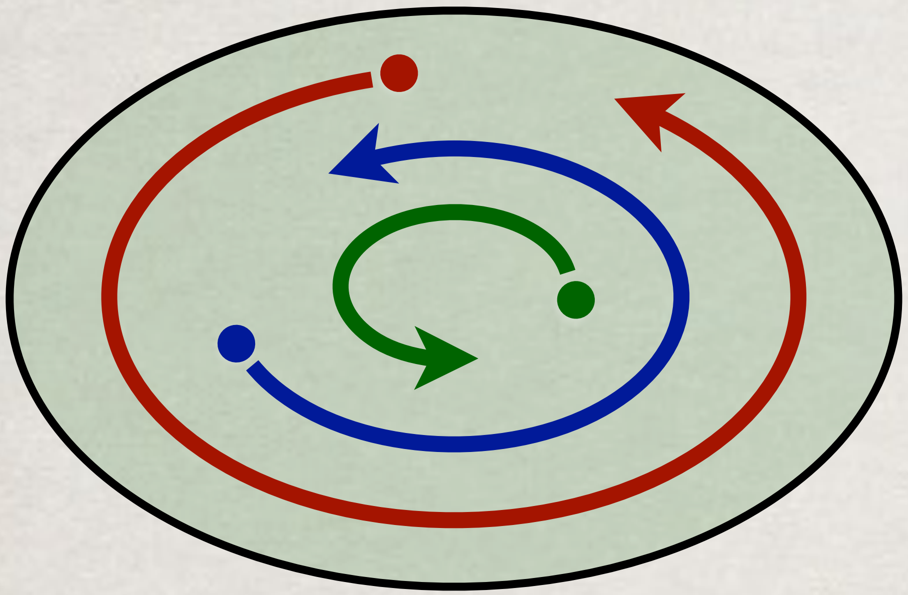
Example: Precession



$$p(t) = \cos^2 \omega t \quad \Rightarrow \quad p_N(t) = [\cos^2(\omega t/N)]^N \sim \left(1 - \frac{\omega^2 t^2}{N^2}\right)^N \xrightarrow{N\to\infty} 1$$

P. Facchi and S. Pascazio, JPA **41**, 493001 (2008).

Quantum Zeno Dynamics (QZD)



$$U(t) = e^{-iHt}$$

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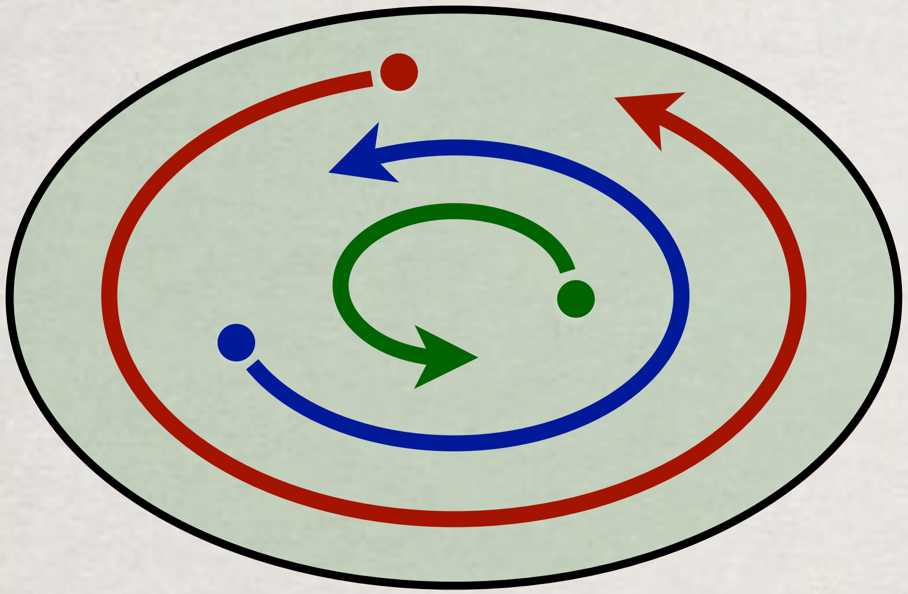
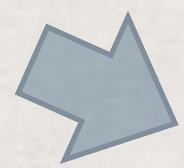
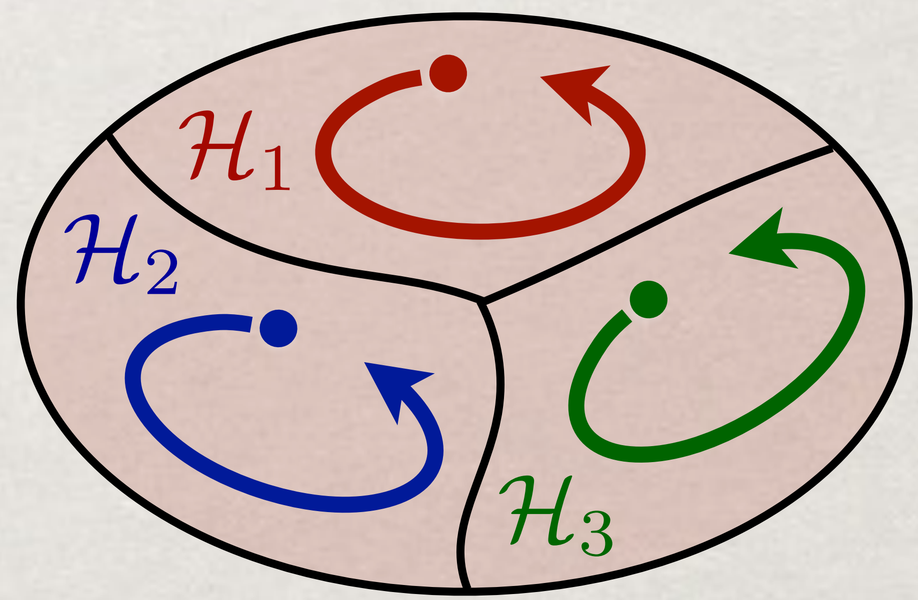
Quantum Zeno Dynamics (QZD)



Frequently check in which subspace $\mathcal{H}_n \ldots$

$$U(t) = e^{-iHt}$$





H is projected to $P_n H P_n$.

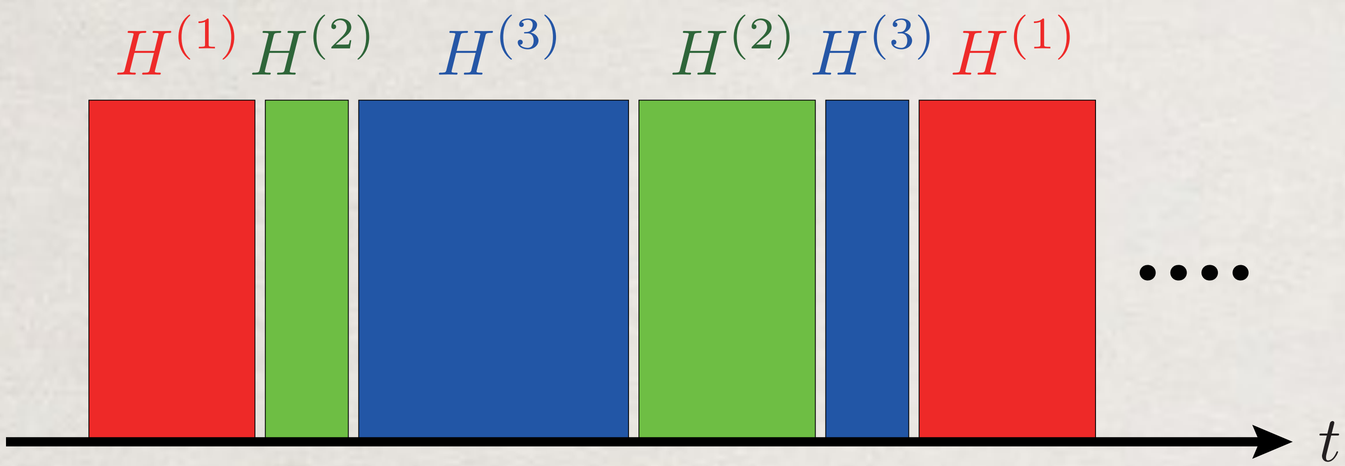
$$U_Z^{(n)}(t) = \lim_{N\to\infty} (P_n e^{-iHt/N} P_n)^N$$

$$= e^{-iP_n H P_n t} P_n$$

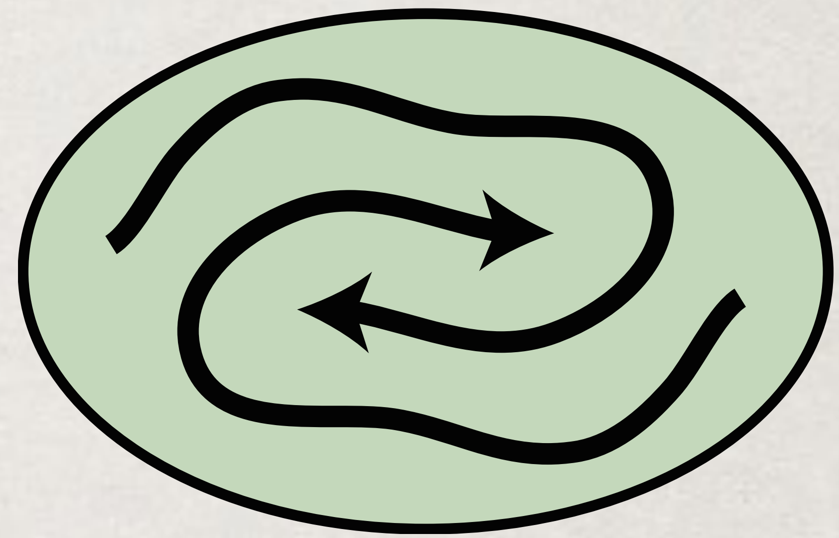
P. Facchi and S. Pascazio, JPA **41**, 493001 (2008).

QZD in Quantum Control

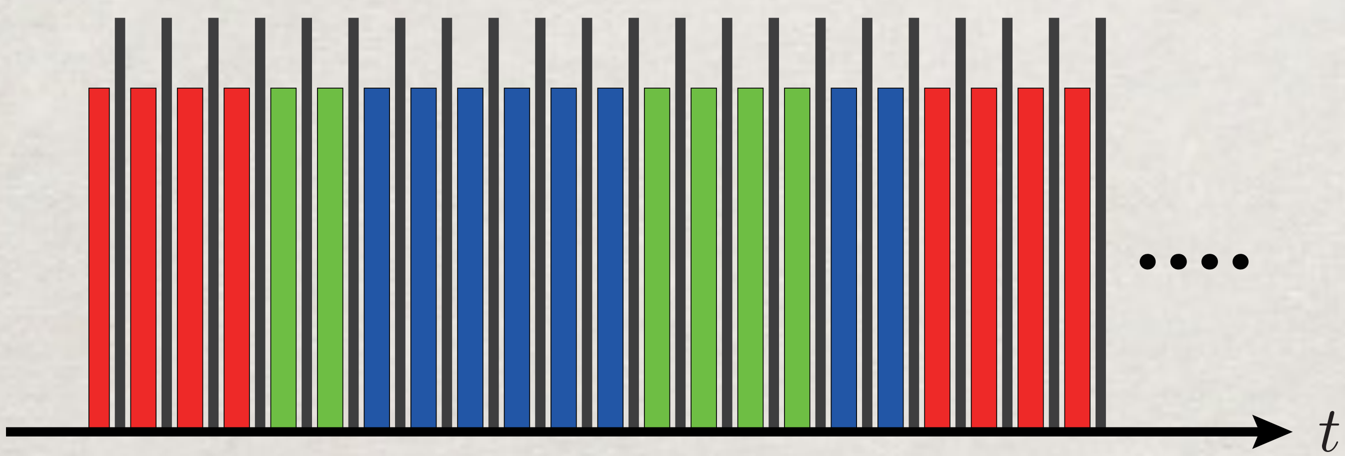
control pulses



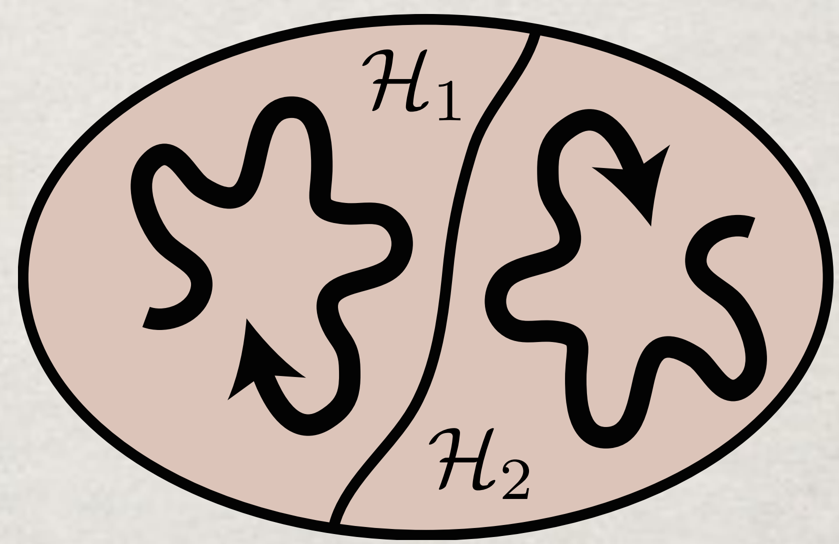
no measurement



unitary control



frequent measurements



Zeno dynamics

$$H^{(k)} \to \bar{H}^{(k)} = P_n H^{(k)} P_n$$

Control Hamiltonians $H^{(k)}$ are projected.

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Non-Commutativity by Projection

$$[H^{(1)}, H^{(2)}] = 0 \Rightarrow [PH^{(1)}P, PH^{(2)}P] \neq 0$$

Projection allows more complex control,

even universal quantum computation!

Nature Commun. **5**, 5173 (2014).

Two Qubits

$$\begin{cases} H^{(1)} = X_1 X_2 \\ H^{(2)} = Z_1 Z_2 \end{cases}$$

$$[H^{(1)}, H^{(2)}] = 0$$

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$$\begin{cases} H^{(1)} = X_1 X_2 \\ H^{(2)} = Z_1 Z_2 \end{cases}$$

$$[H^{(1)}, H^{(2)}] = 0$$

$$P_1 = \frac{1 + (X_1 + Y_1 + Z_1)/\sqrt{3}}{2}$$

$$\begin{cases} \bar{H}^{(1)} = P_1 H^{(1)} P_1 = \underline{P_1 X_2}/\sqrt{3} \\ \bar{H}^{(2)} = P_1 H^{(2)} P_1 = \underline{P_1 Z_2}/\sqrt{3} \end{cases}$$

$$[\bar{H}^{(1)}, \bar{H}^{(2)}] = -2i\underline{P_1 Y_2}/3$$

full algebra on the second qubit!

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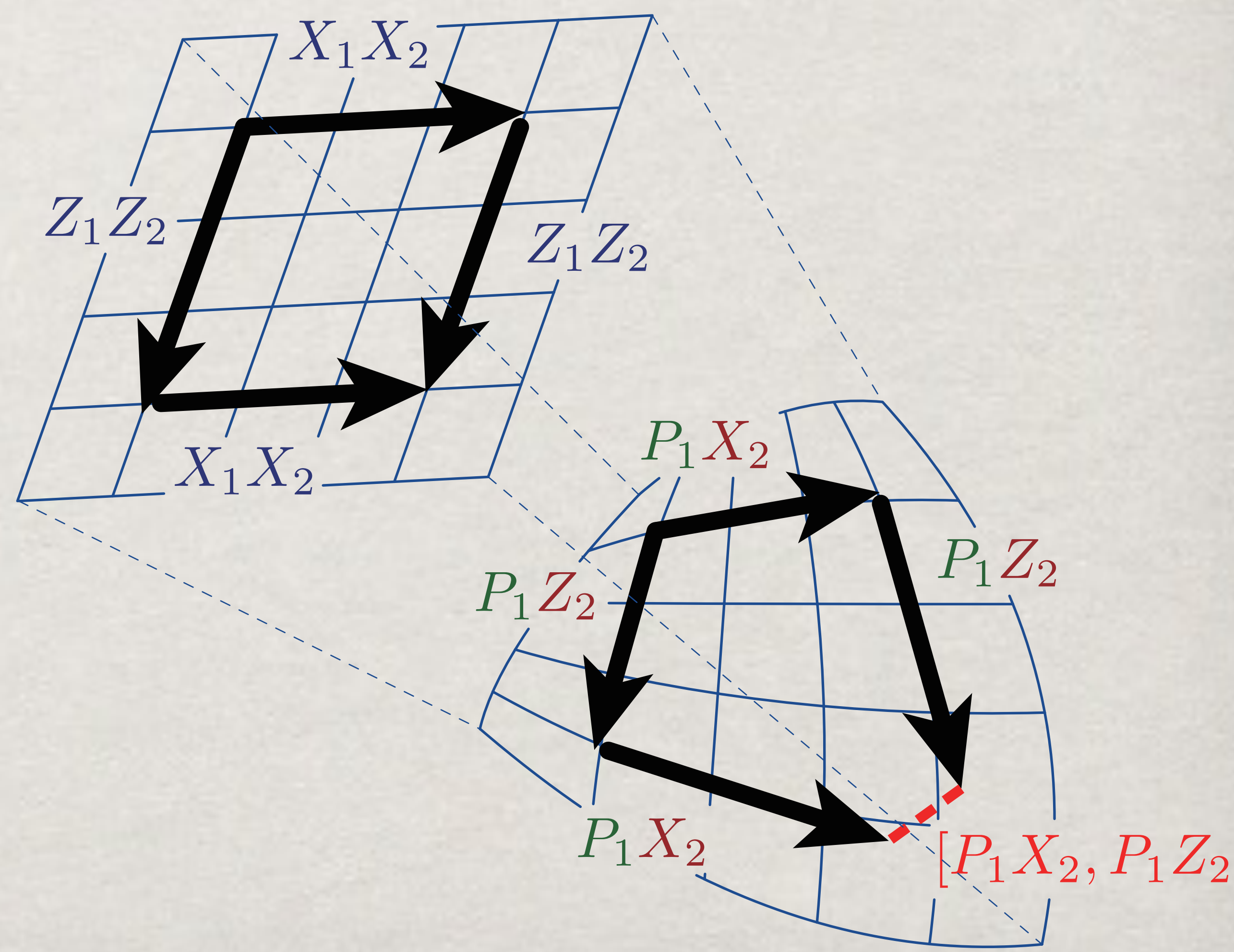

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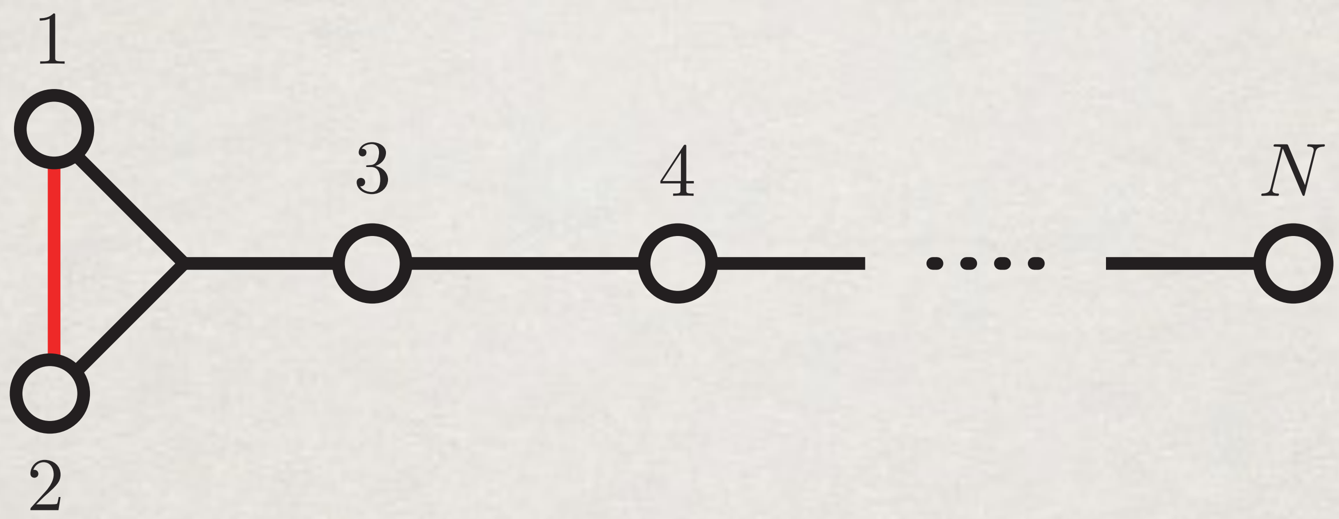
N Qubits



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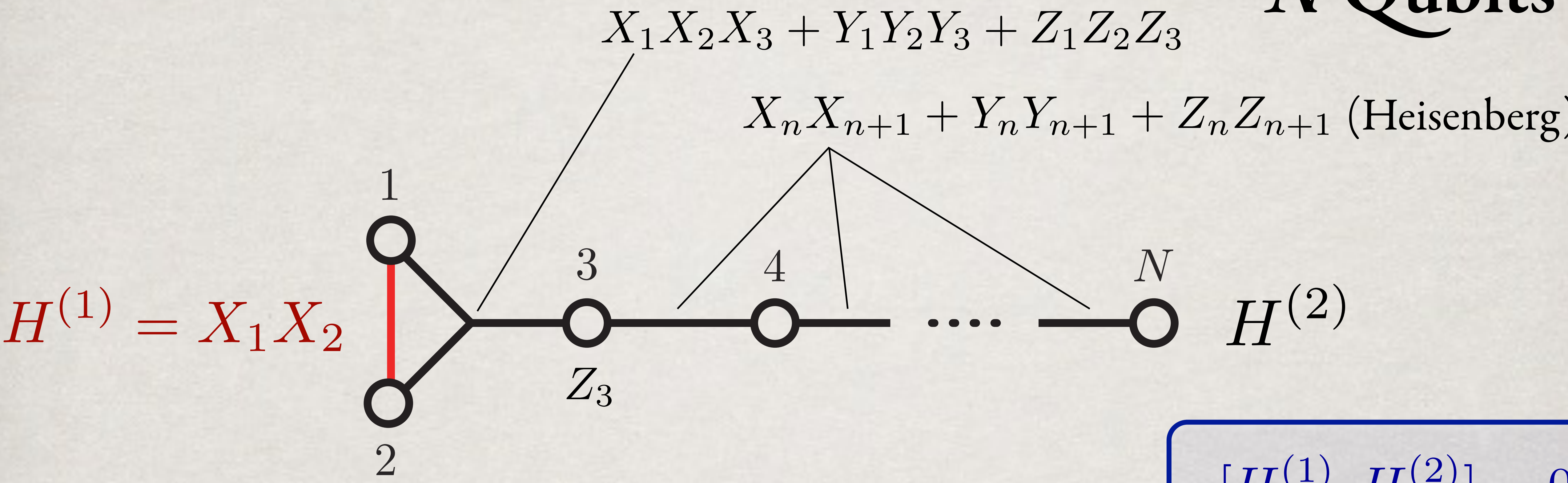

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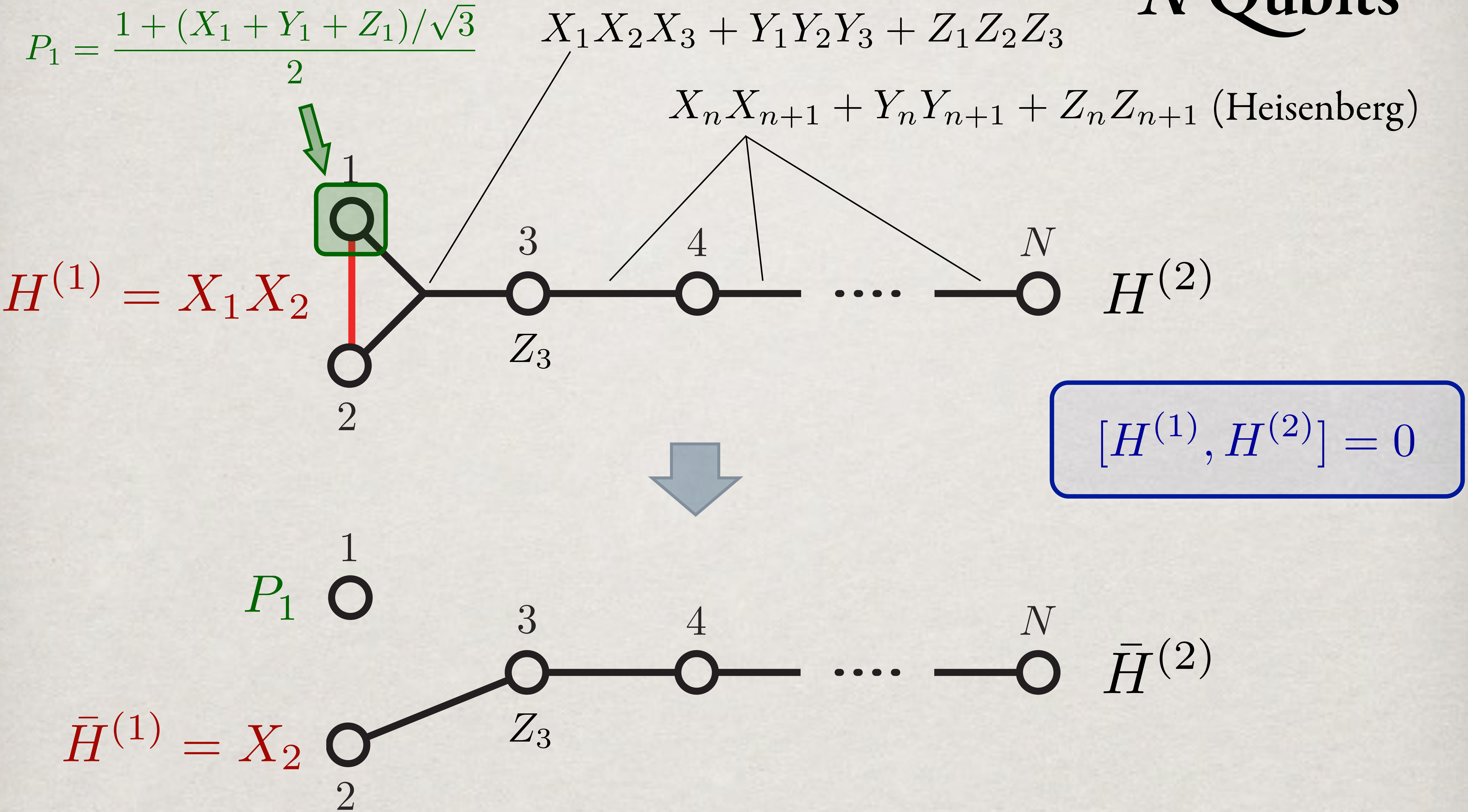

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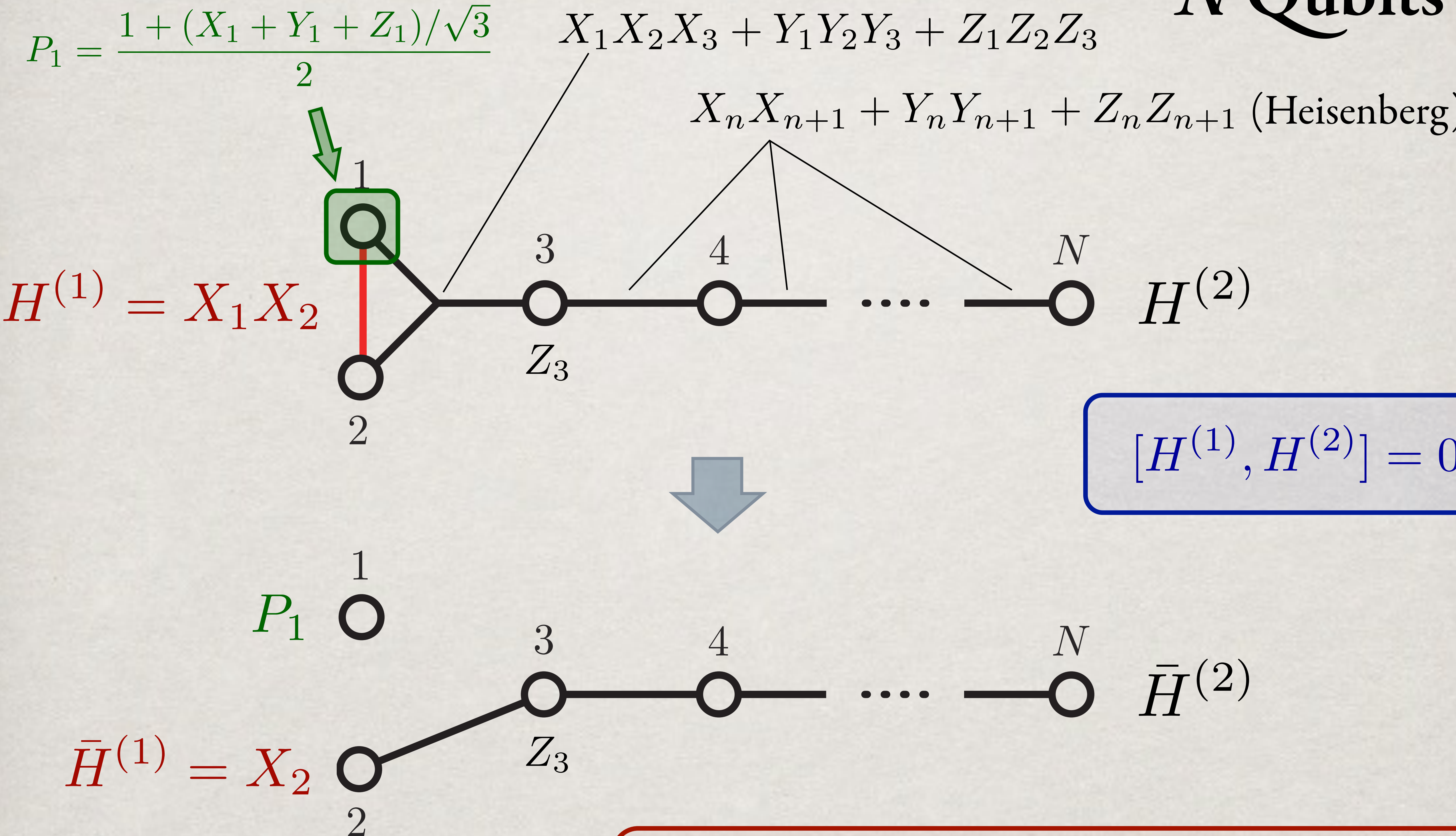

$$[H^{(1)}, H^{(2)}] = 0$$

$$\mathfrak{L}_Z = \mathfrak{Lie}\{\bar{H}^{(1)}, \bar{H}^{(2)}\} = P_1 \mathfrak{su}(2^{N-1})$$

universal quantum computation!

$\dim \mathfrak{L}_Z = 4^{N-1}$ … exponential!

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Generality

- Commuting Hamiltonians for N qubits: $[H^{(1)}, H^{(2)}] = 0$
- Projection on the 1st qubit: P_1

choose randomly

$$\Rightarrow \begin{cases} \bar{H}^{(1)} = P_1 H^{(1)} P_1 \\ \bar{H}^{(2)} = P_1 H^{(2)} P_1 \end{cases}$$

$$\mathfrak{L}_Z = \mathfrak{Lie}\{\bar{H}^{(1)}, \bar{H}^{(2)}\} = P_1 \mathfrak{su}(2^{N-1})$$

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Hamiltonian Purification

$$H^{(1)}, H^{(2)} \in \mathcal{H}_{d_E}$$

$$[H^{(1)}, H^{(2)}] = 0$$

$$\begin{cases} PH^{(1)}P = \bar{H}^{(1)} \oplus 0 \\ PH^{(2)}P = \bar{H}^{(2)} \oplus 0 \end{cases}$$

$$d_E > d$$

$$\bar{H}^{(1)}, \bar{H}^{(2)} \in \mathcal{H}_d$$

$$[\bar{H}^{(1)}, \bar{H}^{(2)}] \neq 0$$

arXiv:1411.0316 [quant-ph].

Hamiltonian Purification

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$$d_E > d$$

$$d_E^{(\mathrm{min})} = 2d - 1$$

"Hamiltonian Purification"

$$H^{(1)}, H^{(2)} \in \mathcal{H}_{d_E}$$

$$[H^{(1)}, H^{(2)}] = 0$$

commutative in extended space

arXiv:1411.0316 [quant-ph].

Hamiltonian Purification

$$\{\bar{H}^{(1)}, \ldots, \bar{H}^{(m)}\} \in \mathcal{H}_d$$

$$[\bar{H}^{(i)}, \bar{H}^{(j)}] \neq 0$$

$$PH^{(k)}P = \bar{H}^{(k)} \oplus 0$$

$$d_E > d$$

$$2d - 1 \leq d_E^{(\min)} \leq m(d-1) + 1$$

"Hamiltonian Purification"

$$\{H^{(1)}, \ldots, H^{(m)}\} \in \mathcal{H}_{d_E}$$

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commutative in extended space

arXiv:1411.0316 [quant-ph].

Ryōan-ji Temple in Kyoto

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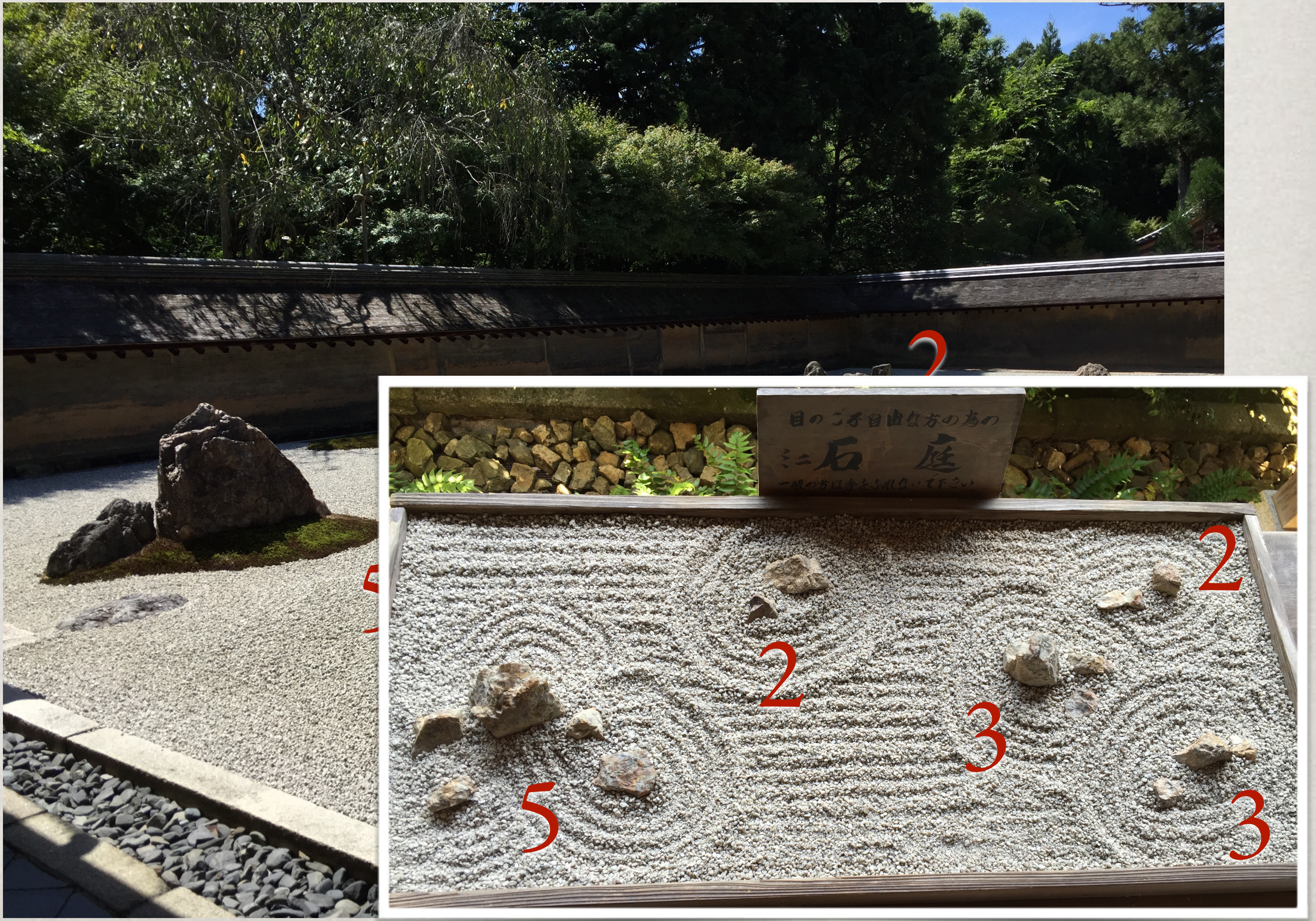


Summary

projection on a small part

$$[H^{(1)}, H^{(2)}] = 0 \Rightarrow \mathfrak{Lie}\{PH^{(1)}P, PH^{(2)}P\} = P\mathfrak{su}(2^{N-1})$$

trivial controls

Exponential Rise of Dynamical Complexity

Universal Quantum Computation

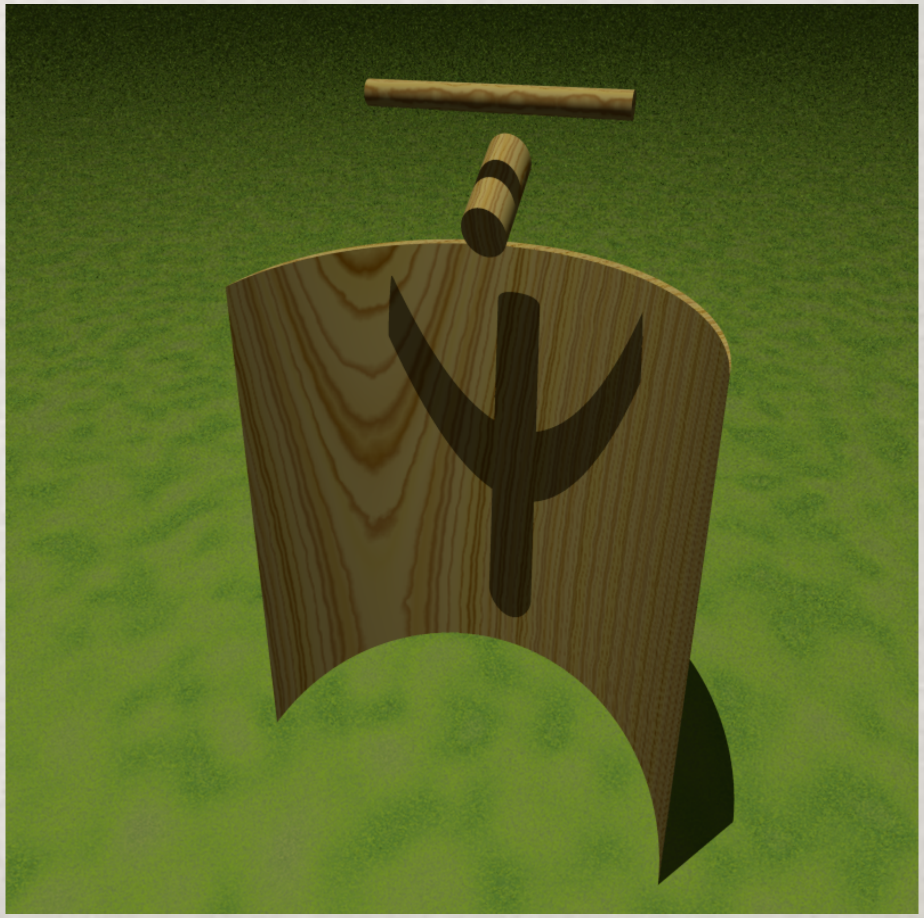
Hamiltonian Purification

$$[\bar{H}^{(i)}, \bar{H}^{(j)}] \neq 0$$

$$d_E > d \qquad \Downarrow \qquad PH^{(k)}P = \bar{H}^{(k)} \oplus 0$$

$$[H^{(i)}, H^{(j)}] = 0$$

commutative in extended space



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Thank You !!

Ryōan-ji in Kyoto