# Topology and Quantum States: Electron-Monopole System

11 Settembre 2015

# Contents

# Contents

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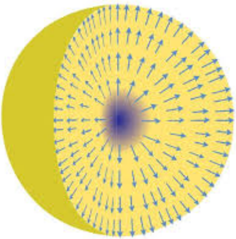
# Topological obstruction in the electron-monopole system





- The configuration space is $\mathbb{R}^3_0 = \mathbb{R}^3 - \{0\} \simeq S^2 \times \mathbb{R}_+$
- There is a topological obstruction : $\mathbf{H}^2(S^2) = \mathbb{Z}$
- It is not possible to define a global potential

$$dF = 0 \qquad \Rightarrow \qquad F \neq dA$$

- No global Lagrangian
- No canonical variables for quantization $\Rightarrow$ Dirac solution with a string singularity

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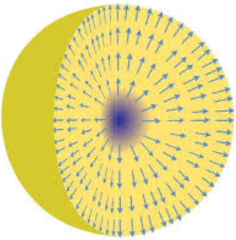

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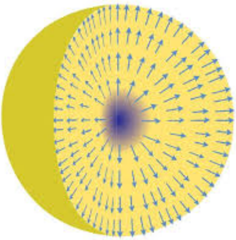

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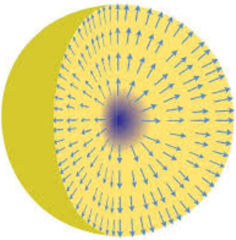

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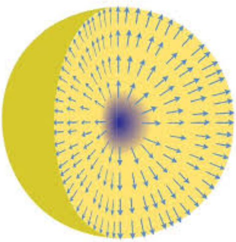

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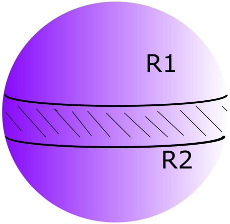
# QUANTUM DESCRIPTION : WU-YANG APPROACH






- Considering two charts which cover the whole space
- In each chart a potential exists

$$A^1_\varphi = \frac{g}{r\sin\theta}(1-\cos\theta)$$
$$A^2_\varphi = \frac{g}{r\sin\theta}(1+\cos\theta)$$

- It is possible to solve Schroedinger equation in each region: the two solutions are linked by a gauge transformation in the overlap
- Replacing wave-functions with sections of vector-bundle

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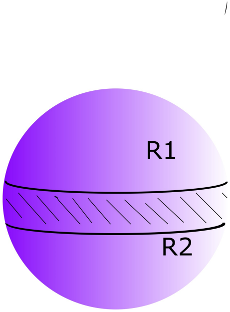


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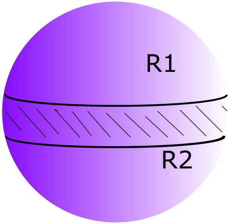


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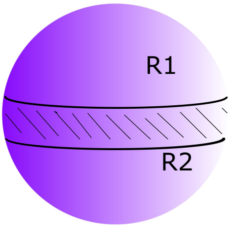


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# BALACHANDRAN'S APPROACH TO ELECTRON-MONOPOLE SYSTEM

$$\begin{array}{ccc} S^3 & \xrightarrow{\phi} & S^2 \times S^1 \\ {\scriptstyle\pi}\downarrow & & \\ S^2 & & \end{array}$$

Extension to $U(1)$ - principal bundle

$$\pi \ : \ S^3 \ \to \ S^2$$

$$\hat{x}_j \sigma^j = s\sigma^3 s^{-1} \qquad s^{-1}ds = i\sigma_j \theta^j$$

- Existence of a global potential in the enlarged space $F = d\theta^3$
- Definition of a global Lagrangian $\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{4}mr^2 Tr\dot{\hat{x}}^2 + inTr\sigma_3 s^{-1}\dot{s}$
- Primary constraint $\quad x_k L_k = n$
- Hamiltonian $\quad H = \frac{p_r^2}{2m} + \frac{L^2 - n^2}{2mr^2}$

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# BALACHANDRAN'S APPROACH : CANONICAL QUANTIZATION



- Hilbert space $\mathcal{L}^2(S^3 \times \mathbb{R}_+, d\mu)$
- Selection of the subspace satisfying the constraint

$$(x_k L_k - n)\psi^{(n)} = 0$$

- The eigenvalue equation for the Hamiltonian operator is

$$\frac{1}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\psi\right) + \frac{1}{2mr^2}\left(L_a L^a - n^2\right)\psi^{(n)} = E\psi^{(n)} \quad (1)$$

- Solutions are $\psi^{(n)} = \frac{u(r)^{(n)}_{jm}}{r} Y^{(n)}_{jm}(s)$ where

$$L_z Y^{(n)}_{jm} = m Y^{(n)}_{jm} \qquad L^2 Y^{(n)}_{jm} = j(j+1) Y^{(n)}_{jm} \quad (2)$$

and

$$\lim_{r \to 0^+} r^{-\frac{1-\sqrt{(2l+1)^2-n^2}}{2}} u^{(n)}_{jm}(r) = 1$$

- The functions $Y^{(n)}_{jm}(s)$ are the monopole harmonics.

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# A GEOMETRIC HAMILTONIAN OPERATOR



- $U(1)$-principal bundle extension
- Consider a non euclidean metric tensor on the enlarged space

$$g = dr \otimes dr + r^2 \left(\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2\right) + k\theta^3 \otimes \theta^3$$

- The Laplace-Beltrami operator associated with this metric tensor is

$$\Delta\psi = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(X_1^2 + X_2^2\right) + \frac{1}{k}X_3^2\right]\psi$$

- One recovers the initial description by reducing to the Hilbert subspace of equivariant functions

$$\psi(pg) = \rho(g)^{-1}\psi(p) \qquad \Rightarrow \qquad X_3\psi = in\psi$$

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# THE HILBERT SPACE OF SQUARE-INTEGRABLE DIFFERENTIAL FORMS



- Using geometric Hamiltonian operators allows to extend its action from functions to differential forms
- The scalar product on the space $\Lambda^*(M)$ is

$$\langle\alpha|\beta\rangle = \int_M (\alpha, \beta)\,\omega$$
$$(\alpha|\beta)\,\omega = \alpha \wedge *\beta$$

  where $* : \Lambda^k(M) \rightarrow \Lambda^{m-k}(M)$ is the Hodge dual operator
- Square-integrable differential forms are a Hilbert space which contains more information about the topology of the carrier space (e.g. : de Rham cohomology)
- The use of this space allows to write Dirac-type operators as scalar differential operators

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# TENSOR HARMONICS AS DIFFERENTIAL FORMS



The Hilbert space of square-integrable differential forms can be used as vector space for groups and algebras representations.

## $su(2)$ Representations j integer

- Differential forms $\alpha$ which satisfy the conditions $i_{X_3}\alpha = 0$ and $X_3\alpha = 0$ can be used to build representations with $j$ integer.
- These forms are of the kind $\alpha = \alpha_+\theta^+ + \alpha_-\theta^-$ with the coefficients that obey $X_3\alpha_\pm = \mp i\alpha_\pm$.
- The eigenvalue equations which one has to solve are $Y_3\alpha = im\alpha$ and $Y^2\alpha = -j(j+1)\alpha$.

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# Tensor harmonics as differential forms



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An orthonormal basis for the representation $j = 1$ is given by the three differential forms

$$\alpha_1 = i\frac{1}{\sqrt{\pi}}\sqrt{\frac{3}{8\pi}}\left[v^2(\bar{v}d\bar{u} - \bar{u}d\bar{v}) + \bar{u}^2(udv - vdu)\right]$$

$$\alpha_0 = i\frac{1}{\sqrt{\pi}}\sqrt{\frac{3}{4\pi}}\left[-vu(\bar{v}d\bar{u} - \bar{u}d\bar{v}) + \bar{v}\bar{u}(udv - vdu)\right]$$

$$\alpha_{-1} = i\frac{1}{\sqrt{\pi}}\sqrt{\frac{3}{8\pi}}\left[u^2(\bar{v}d\bar{u} - \bar{u}d\bar{v}) + \bar{v}^2(udv - vdu)\right]$$

## $su(2)$ Representations j half-integer

- Differential forms satisfying $i_{X_3}\alpha = 0$ and $X_3\alpha = \frac{i}{2}\alpha$ can be used as a vector space for the representation j half-integer.
- The coefficients satisfy the equations $X_3\alpha_+ = -\frac{i}{2}\alpha_+$ and $X_3\alpha_- = i\frac{3}{2}\alpha_-$.

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## $su(2)$ Representations j half-integer

- Differential forms satisfying $i_{X_3}\alpha = 0$ and $X_3\alpha = \frac{i}{2}\alpha$ can be used as a vector space for the representation j half-integer.
- The coefficients satisfy the equations $X_3\alpha_+ = -\frac{i}{2}\alpha_+$ and $X_3\alpha_- = i\frac{3}{2}\alpha_-$.

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$$\alpha_{\frac{1}{2}} = \frac{1}{2\pi} v\theta^+$$
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# DIRAC OPEATOR AS A SCALAR DIFFERENTIAL OPERATOR

- Laplace-Beltrami operator is $\Delta = (d + \delta)^2$. Therefore its square-root, the Dirac operator, can be written as

$$D = d + \delta$$

- Kaehler showed that it is possible to represent a Clifford algebra on the exterior algebra of a manifold $\Lambda^*(M)$ through the definition of a $\vee$-product.
- One can decompose this representation into the sum of irreducible ones by means of orthonormal projecctors $P_j$. Their ranges are vector spaces on which the exterior algebra acts irreducibly. They are called algebraic spinors (Graf).
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## Example: Dirac Operator on $\mathbb{R}^2$

- Let us consider the space $\mathbb{R}^2$ equipped with the euclidean metric

$$g = dx \otimes dx + dy \otimes dy$$

- Projectors for the clifford albegra $\Lambda^*(\mathbb{R}^2, \vee)$ are characterised by two parameters

$$P = \frac{1}{2} + \rho dx + \xi dx \wedge dy$$

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# THANK YOU FOR YOUR ATTENTION