Sensitivity and complexity of non-equilibrium steady states of quantum spin chains

Ugo Marzolino

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In collaboration with: Tomaž Prosen, *University of Ljubljana*

U. Marzolino, T. Prosen, *Phys. Rev. A* **90**, 062130 (2014)
U. Marzolino, T. Prosen, soon on arXiv

Outline

Part I

Background

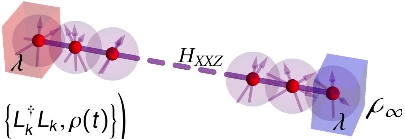
Boundary driven/dissipated quantum spin chains

- T. Prosen, *Phys. Rev. Lett.* **107**, 137201 (2011); T. Prosen, *J. Phys. A* **48**, 373001 (2015)

Steady state of L-spin chains

descrived by a Matrix Product Operator

σ_j^s: Pauli matrices of the j-th spin



$$\frac{d}{dt}\rho_t = -i\left[H_{\mathrm{XXZ}}, \rho_t\right] + \lambda \sum_{k=1}^{4} \left(L_k \rho(t) L_k^\dagger - \frac{1}{2}\left\{ L_k^\dagger L_k, \rho(t) \right\} \right)$$

$$H_{\mathrm{XXZ}} = \sum_{j=1}^{L-1} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta\, \sigma_j^z \sigma_{j+1}^z \right), \quad L_{1,2} = \sqrt{\frac{1 \pm \mu}{2}}\sigma_1^\pm, \quad L_{3,4} = \sqrt{\frac{1 \mp \mu}{2}}\sigma_L^\pm$$

Steady state: $\rho_\infty \propto \sum_{\substack{\{s_1,\dots,s_L\} \\ \in \{0,z,+,-\}^L}} \underbrace{\langle 0| \otimes \langle 0| \prod_{j=1}^{L} \mathbb{A}_{s_j} |0\rangle \otimes |0\rangle}_{\text{auxiliary space}} \bigotimes_{j=1}^{L} \sigma_j^{s_j}$

Auxiliary space: $\mathbb{C}^{2\left(\left\lfloor \frac{L}{2} \right\rfloor + 1\right)} = \mathrm{span}\left\{ |k\rangle \otimes |l\rangle \right\}$ with $k, l = 0, 1, \cdots, \left\lfloor \frac{L}{2} \right\rfloor$

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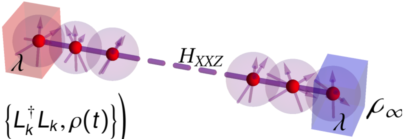

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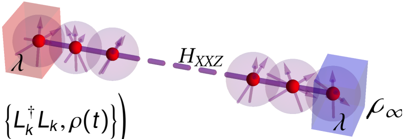

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Part II

Sensitivity of non-equilibrium steady states

Fisher information

Fisher information

Quantification of the state (ρ_Δ) sensitivity with respect to its parameters

Fidelity, overlap between two infinitesimally close states:

$$\mathcal{F}(\rho_\Delta, \rho_{\Delta+\delta\Delta}) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho_\Delta}\,\rho_{\Delta+\delta\Delta}\,\sqrt{\rho_\Delta}}\right)^2$$

Bures distance between two infinitesimally close states:

$$D(\rho_\Delta, \rho_{\Delta+\delta\Delta}) = 2\left(1 - \sqrt{\mathcal{F}(\rho_\Delta, \rho_{\Delta+\delta\Delta})}\right)$$

Fisher information or fidelity susceptibility: $F_\Delta = \lim_{\delta\Delta\to 0} \frac{4\,D(\rho_\Delta, \rho_{\Delta+\delta\Delta})}{(\delta\Delta)^2}$

Applications of superextensive Fisher information: $\longrightarrow F_\Delta \gg O(L) \longleftarrow$

- Precision metrology: variance of estimation of $\Delta \longrightarrow \mathrm{Var}(\Delta) \geqslant \frac{1}{F_\Delta}$
  - M. G. A. Paris, *Inf. J. Quantum Inf.* **7**, 125 (2009)
- Detection of phase transitions: sudden change of thermal (classical PT), ground (quantum PT), or steady (non-equilibrium PT) states
  - S.-J. Gu, *Int. J. Mod. Phys. B* **24**, 4371 (2010)
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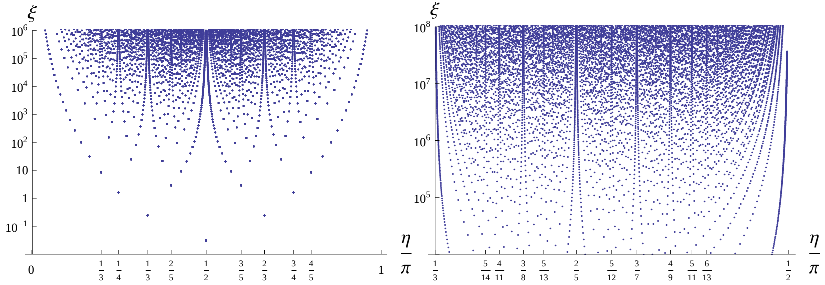
Regimes of superextensive Fisher information

Easy-plane interactions $|\Delta| < 1$

"Rational" anisotropy $\frac{\eta}{\pi} = \frac{\arccos\Delta}{\pi} = \frac{q}{p} \in \mathbb{Q}$ small dissipation $\lambda < \mathcal{O}\left(\frac{1}{\sqrt{L}}\right)$

$$F_\Delta = \lambda^2\mu^2\left(\xi L + \mathcal{O}(L^0)\right), \qquad \xi \text{ unbounded for } |p| \to \infty$$



Fisher information with respect to to the anisotropy

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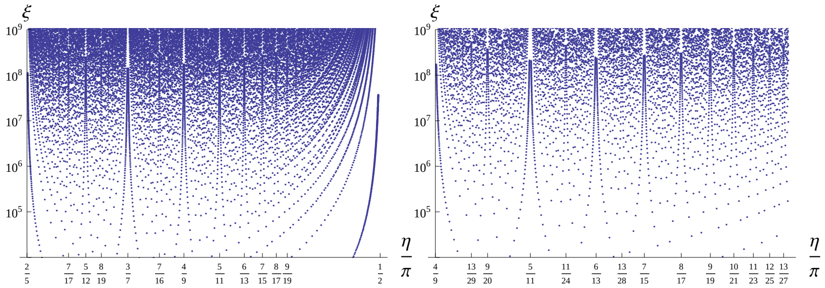
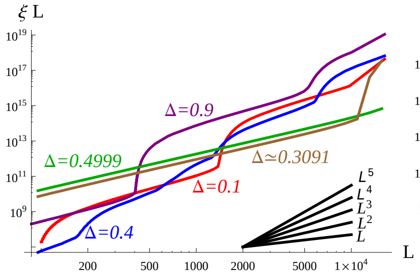


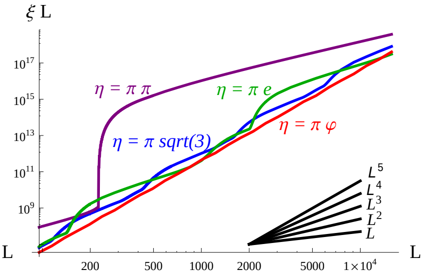
Fisher information with respect to to the anisotropy

Easy-plane interactions $|\Delta| < 1$

"Irrational" anisotropy $\frac{\eta}{\pi} = \frac{\arccos \Delta}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$ small dissipation $\lambda < \mathcal{O}\left(\frac{1}{\sqrt{L}}\right)$

$$F_\Delta = \lambda^2 \mu^2 \left(\xi L + \mathcal{O}(L^0)\right) = \mathcal{O}(L^\alpha), \qquad \alpha < 4$$





Conclusions on the sensitivity

Isotropic interactions $|\Delta| = 1$

Small dissipation $\lambda < \mathcal{O}\left(\frac{1}{L}\right)$ small anisotropy $\eta = \arccos\Delta < \mathcal{O}\left(\frac{1}{L}\right)$

$$F_\Delta = \frac{\lambda^2\mu^2}{96} L(L-1)(L-2)\left(3L - 7 - \frac{\eta^2}{30}(L-3)(261L - 799) + \mathcal{O}(\eta^4)\right) = \mathcal{O}(L^\alpha), \qquad \alpha < 2$$

Interpretation of the results

- Precision measurement of the anisotropy for $|\Delta| \leqslant 1$
- Quantum non-equilibrium phase transition at $|\Delta| = 1$ and critical phase for $|\Delta| < 1$, stronger than the corresponding quantum phase transition at equilibrium

Perspectives

- Non-perturbative computation
- Inference of information on the Liouvillian spectal gap and on correlation functions

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Part III

Computational complexity of non-equilibrium steady states

Computational complexity

Computational classes

defined by means of models of computation, known as Turing machines

- S. Arora, B. Barak, *Computational Complexity A Modern Approach* (Cambridge, Univ. Press, 2009)
- D.-Z. Du and K. Ko, *Theory of Computational Complexity 2nd ed.* (John Wiley & Sons, Inc., 2014)
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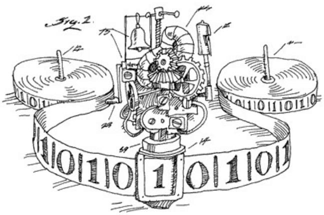

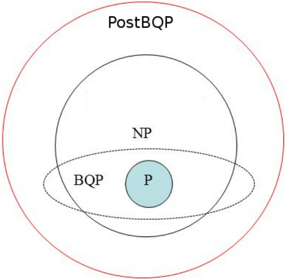

n: input size

- P: poly(n) deterministic instructions
- NP: poly(n) non-deterministic (multivalued) instructions
- BQP: poly(n) quantum instructions, high success probability
- PostBQP: poly(n) quantum instructions and general linear operations, high success probability

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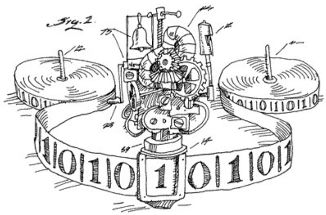

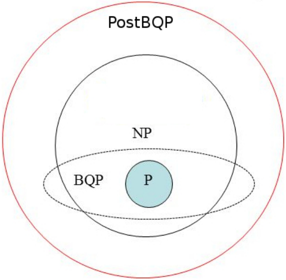

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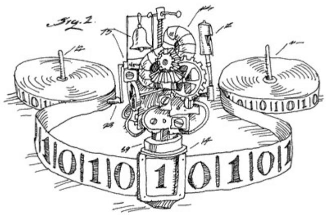

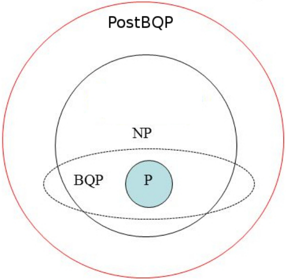

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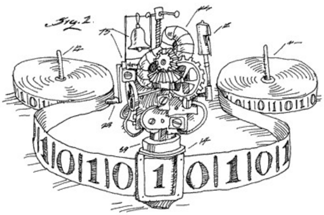

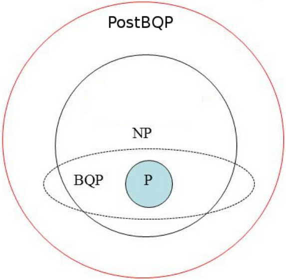

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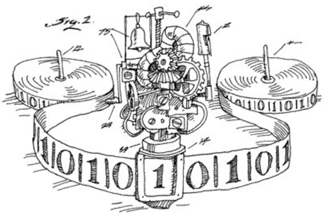

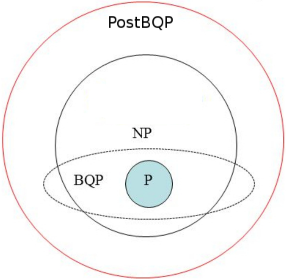

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Applications of computational complexity

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Classifications of computational problems based on computational resources as a characterization of condensed matter physics

Hardness of

- finding the ground state energy
  - J. Kempe, A. Kitaev, O. Regev, *SIAM J. Comput.* **35**, 1070 (2006)
  - R. Oliveira and B. Terhal, *Quantum Inf. Comput.* **8**, 1170 (2008)
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  - J. D. Whitfield, P. J. Love, and A. Aspuru-Guzik, *Phys. Chem. Chem. Phys.* **15**, 397 (2013)
- approximating the ground state energy
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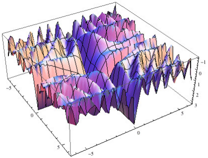
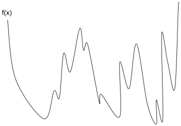
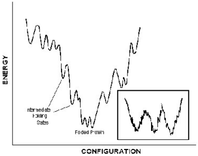
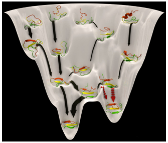
  - A. M. Childs, D. Gosset, Z. Webb, *arXiv:1503.07083* (2015)





- creating pure quantum states
  - N. Schuch, M. M. Wolf, F. Verstraete, J. I. Cirac, *Phys. Rev. Lett.* **98**, 140506 (2006)

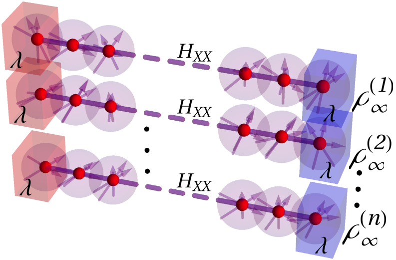
Boundary driven/dissipated quantum spin chains

- T. Prosen, *Phys. Rev. Lett.* **107**, 137201 (2011); T. Prosen, *J. Phys. A* **48**, 373001 (2015)

Steady state of L-spin chains

described by a Matrix Product Operator

$\sigma^s_{k,j}$: Pauli matrices of the j-th spin along the k-th chain



$$\frac{d}{dt}\rho_t^{(k)} = -i\left[H_{XX}^{(k)}, \rho_t^{(k)}\right] + \lambda\left(\sigma^+_{k,1}\rho_t^{(k)}\sigma^-_{k,1} - \frac{1}{2}\left\{\sigma^-_{k,1}\sigma^+_{k,1}, \rho_t^{(k)}\right\}\right) + \lambda\left(\sigma^-_{k,L}\rho_t^{(k)}\sigma^+_{k,L} - \frac{1}{2}\left\{\sigma^+_{k,L}\sigma^-_{k,L}, \rho_t^{(k)}\right\}\right)$$

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Steady state:

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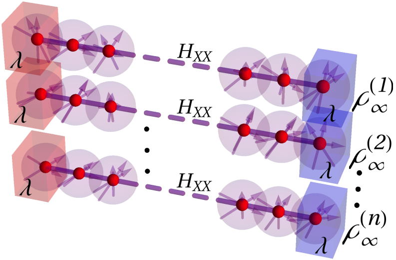

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Boundary driven/dissipated quantum spin chains

Matrix algebra in the auxiliary space

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- 4×4 matrices $\mathbb{A}_s^{(k)}$ but span a $GL(2)$ subgroup $\Longrightarrow$ single qubit operations

Encoding of single qubit evolution

The evolution of a single qubit is encoded in the auxiliary space of one spin chain

$$\text{Tr}\left((\sigma_{k,L}^{s_{k,L}})^\dagger \cdots (\sigma_{k,2}^{s_{k,2}})^\dagger (\sigma_{k,1}^{s_{k,1}})^\dagger \rho_\infty^{(k)}\right) \propto \langle 0_k|\langle 0_k|\, \mathbb{A}_{s_{k,L}}^{(k)} \cdots \mathbb{A}_{s_{k,2}}^{(k)} \mathbb{A}_{s_{k,1}}^{(k)}\, |0_k\rangle|0_k\rangle$$

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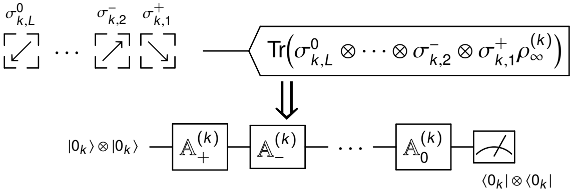

Encoding of general quantum circuits

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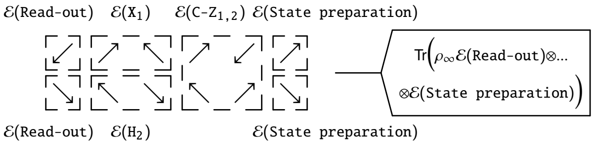
Polynomials of Pauli matrices and using many spin chains

$$\mathrm{Tr}\Big(\cdots \mathcal{E}(G_2)\,\mathcal{E}(G_1)\,\rho_\infty\Big) \propto \Big(\langle 0|\langle 0|\Big)^{\otimes n}\Big(\cdots G_2 G_1\Big)\Big(|0\rangle|0\rangle\Big)^{\otimes n}$$



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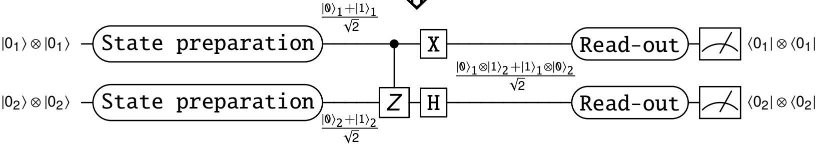

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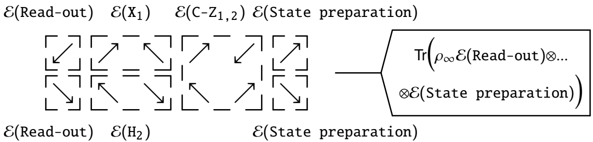

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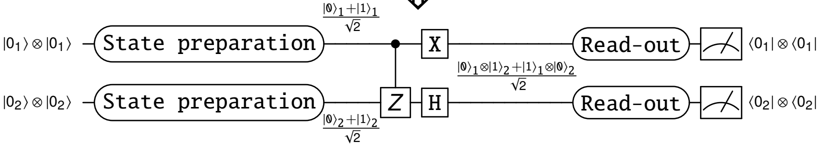

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Read-out and complexity

Measure of the computational result

of the encoded circuit through estimations of transition amplitudes

Any PostBQP circuit can be transformed such that the final state is

$$\Big(\alpha\,|\mathtt{T}\rangle + \beta\,|\mathtt{F}\rangle\Big) \otimes |\mathtt{x}\rangle \qquad\qquad \alpha, \beta :\ \mathrm{Tr}\Big(\cdots \mathcal{E}(\mathtt{G}_2)\,\mathcal{E}(\mathtt{G}_1)\,\rho_\infty\Big)$$

- with the true (false) computational result: $\mathtt{T}$ $(\mathtt{F})$
- with a known reference state: $|\mathtt{x}\rangle$
- and success probability exponentially close to 1: $|\alpha|^2 \geqslant 1 - 2^{-\mathrm{poly}(n)}$

Complexity analysis

- Physical measurements performed on a finite number (at most 4) physical spins for each elementary gate
- The number of spin chains equals that of logical qubits n
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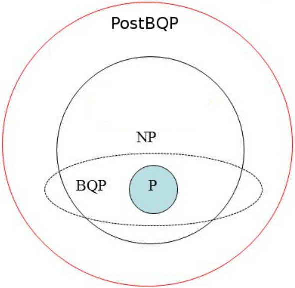
Huge complexity in estimating expectations of local operators



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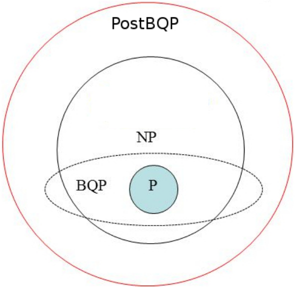

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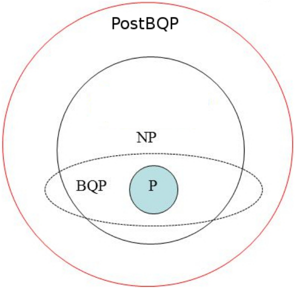

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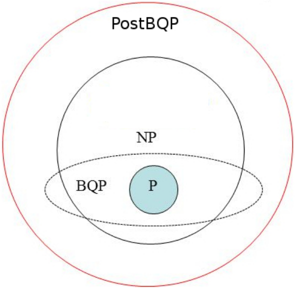

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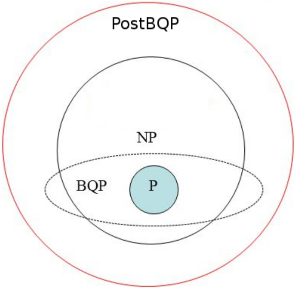

Part IV

Conclusions

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Characterizations of non-equilibrium steady states of quantum spin chains

using tools from quantum information and computation.

Sensitivity of non-equilibrium steady states:

- precision measurement of the anisotropy
- identification of a quantum non-equilibrium phase transition with of a critical phase

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- new characterization of PostBQP

U. Marzolino, T. Prosen, *Phys. Rev. A* **90**, 062130 (2014)
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