Accidental Degeneracy in the Quantum Tridimensional Isotropic Harmonic Oscillator: a Clarification via Ladder Operators

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The Schrödinger equation of the quantum Tridimensional Isotropic Harmonic Oscillator (TIHO) is

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{m\omega^2 r^2}{2} \right] \psi(x, y, z) = E \psi(x, y, z). \tag{1}
\]

If we rewrite the eigenfunctions in the separated form
\[
\psi(x, y, z) = \phi(x) \eta(y) \chi(z),
\]
the equation (1) becomes
分开的薛定谔方程

\[
\frac{1}{\phi(x)} \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2} \right] \phi(x) + \\
+ \frac{1}{\eta(y)} \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dy^2} + \frac{m\omega^2 y^2}{2} \right] \eta(y) + \\
+ \frac{1}{\chi(z)} \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + \frac{m\omega^2 z^2}{2} \right] \chi(z) = E,
\]

这表示

\[
\frac{H_x \phi(x)}{\phi(x)} + \frac{H_y \eta(y)}{\eta(y)} + \frac{H_z \chi(z)}{\chi(z)} = E.
\]
In the last equation each term in the left hand side has to be a constant because the terms depend on the different variables \(x, y, z\) and their sum is the constant \(E\). Then we can write

\[
\frac{H_x \phi(x)}{\phi(x)} = E_x, \quad \frac{H_y \eta(y)}{\eta(y)} = E_y, \quad \frac{H_z \chi(z)}{\chi(z)} = E_z,
\]

which are three eigenvalue problems of unidimensional harmonic oscillator.
We then obtain the eigenvalues of the quantum TIHO

\[ E_n = \hbar \omega \left( n_x + n_y + n_z + \frac{3}{2} \right) = \hbar \omega \left( n + \frac{3}{2} \right), \]

with \( n_x + n_y + n_z = n \), and the corresponding eigenfunctions denoted by

\[ \psi_n(x, y, z) = \phi_{n_x}(x) \eta_{n_y}(y) \chi_{n_z}(z) \quad \text{or} \quad |\psi_n\rangle = |n_x, n_y, n_z\rangle. \]
If we write the Schrödinger equation (1)

\[ \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{m\omega^2 r^2}{2} \right] \psi(x, y, z) = E \psi(x, y, z) \]

in spherical polar coordinates and separate the eigenfunction as \( \psi(x, y, z) \equiv R(r) Y_l, m(\theta, \varphi) \), the radial equation for the function \( R(r) \equiv R_{n,l}(r) \) is

\[ \frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[ \frac{2mE_n}{\hbar^2} - \frac{m^2\omega^2 r^2}{\hbar^2} - \frac{l(l + 1)}{r^2} \right] R(r) = 0. \]
If we solve the radial equation by power series expansion, we obtain the same eigenvalues

\[ E_n = \hbar \omega \left( n_x + n_y + n_z + \frac{3}{2} \right) = \hbar \omega \left( n + \frac{3}{2} \right) \]

and the eigenfunctions in polar coordinates

\[ \psi_{nlm}(r) = R_{nl}(r) Y_{lm}(\theta, \phi) = N r^l e^{-\alpha r^2/2} L_{n/2-l-1/2}^{(l+1/2)}(\alpha r^2) Y_{lm}(\theta, \phi), \]
Parity of the quantum numbers

where

1) the nonnegative quantum numbers $n, l$ have the same parity, with $l \leq n$;

2) $\mathcal{L}^{(a)}_s(x)$ denotes the associated Laguerre polynomials whose expression for the real parameter $a > -1$ and the nonnegative integer parameter $s$ is

$$\mathcal{L}^{(a)}_s(x) = \frac{e^x}{x^a} \frac{d^s}{dx^s} (x^{s+a} e^{-x}).$$
It is worth noting that the total degeneracy $d_n$ of an eigenvalue $E_n$ is

$$d_n = \frac{(n+1)(n+2)}{2}$$

and that the number $d_n$ is coincident with the total number $\mathcal{N}$ of spherical harmonics corresponding to all quantum numbers $l$ having the same parity of $n$, with $l \leq n$. 
Comparison with the spherical harmonics

In fact, for even $n = 2m$ we have $l = 2k$ and then

$$\mathcal{N} = \sum_{k=0}^{m} [2(2k) + 1] = 2m^2 + 3m + 1 = \frac{(2m + 1)(2m + 2)}{2} = d_{2m};$$

while for odd $n = 2m + 1$ we have $l = 2k + 1$ and then

$$\mathcal{N} = \sum_{k=0}^{m} [2(2k+1) + 1] = 2m^2 + 5m + 3 = \frac{(2m + 2)(2m + 3)}{2} = d_{2m+1}.$$
Degeneracy: example with $n = 1$

For $n = 1$ we have the $d_1 = (1 + 1)(1 + 2)/2 = 3$ eigenfunctions denoted by $|n_x, n_y, n_z\rangle$ given by

$|1, 0, 0\rangle, \quad |0, 1, 0\rangle, \quad |0, 0, 1\rangle$

or the three eigenfunctions denoted by $|\psi_{n,l,m}\rangle$ given by

$|\psi_{1,1,-1}\rangle, \quad |\psi_{1,1,0}\rangle, \quad |\psi_{1,1,1}\rangle,$

corresponding to the same eigenvalue

$$E_1 = \frac{5}{2} \hbar \omega.$$
Degeneracy: example with \( n = 2 \)

For \( n = 1 \) we have the \( d_2 = (2 + 1)(2 + 2)/2 = 6 \)
eigenfunctions denoted by \( |n_x, n_y, n_z\rangle \) given by

\[
|2, 0, 0\rangle, \quad |0, 2, 0\rangle, \quad |0, 0, 2\rangle, \quad |1, 1, 0\rangle, \quad |1, 0, 1\rangle, \quad |0, 1, 1\rangle,
\]
or the six eigenfunctions denoted by \( |\psi_{n,l,m}\rangle \) given by

\[
|\psi_{2,2,2}\rangle, \quad |\psi_{2,2,1}\rangle, \quad |\psi_{2,2,0}\rangle, \quad |\psi_{2,2,-1}\rangle, \quad |\psi_{2,2,-2}\rangle, \quad |\psi_{2,0,0}\rangle,
\]
corresponding to the same eigenvalue

\[
E_2 = \frac{7}{2} \hbar \omega.
\]
Natural and accidental degeneracy

From the eigenfunctions $|\psi_{n,l,m}\rangle$ corresponding to the eigenvalue $E_2$ we see that the energy $E_2$ is independent from both the quantum numbers $l, m$.

The independence from the quantum number $m$ is called \textit{natural degeneracy} and is due, as it is well known, to the rotational invariance of the central hamiltonian (conservation of the angular momentum).

The independence from the quantum number $l$ is called \textit{accidental degeneracy} and is due to a second conservation law of the hamiltonian.
Our aim is now to explain each degeneracy in terms of suitable *ladder operators*.

For the natural degeneracy, the corresponding ladder operator has to leave unchanged the quantum numbers $n, l$ shifting the quantum number $m$ by a unity. As it is well known, the *ladder operators* for the quantum number $m$ are the operators $L_+, L_-$. Also for the accidental degeneracy, we would like to give its clarification in terms of other *ladder operators* which leave unchanged the quantum number $n$ (and then the energy) shifting the quantum number $l$. 
In order to obtain a ladder operator, we use an important general theorem of linear algebra which could be called quantum shift theorem.

**Theorem:** given a hermitian operator $B$ and an eigenvector $|\bar{\mu}\rangle$ of $B$ corresponding to the eigenvalue $\bar{\mu}$, if another (not necessarily hermitian) operator $\mathcal{O}$ satisfies the relation

$$[B, \mathcal{O}]|\bar{\mu}\rangle = \mu \mathcal{O}|\bar{\mu}\rangle,$$

for a real number $\mu$, then it follows that either $\mathcal{O}|\bar{\mu}\rangle$ is the null vector, or it is still an eigenvector of $B$, corresponding to the eigenvalue $\bar{\mu} + \mu$. 
For the unidimensional harmonic oscillator we have

\[ H = \hbar \omega \left( N + \frac{1}{2} \right), \quad \text{with } N = a^\dagger a. \]

Since \([N, a^\dagger] = a^\dagger = (+1)a^\dagger\), that is \(\mu = 1\), it follows that for each eigenvector \(|n\rangle\) of \(N\), the vector \(a^\dagger|n\rangle\) is the eigenvector of \(N\) corresponding to the shifted eigenvalue \(n + \mu = n + 1\).

Analogously, since \([N, a] = -a = (-1)a\), that is \(\mu = -1\), it follows that for each eigenvector \(|n\rangle\) of \(N\), the vector \(a|n\rangle\) is either the null vector or is the eigenvector of \(N\) corresponding to the shifted eigenvalue \(n + \mu = n - 1\).
By virtue of these properties, the non-hermitian operators $a^\dagger$, $a$ are called **ladder operators** of the unidimensional harmonic oscillator.
By denoting with \( |l, m\rangle \) the simultaneous eigenvectors of the hermitian operators \( L^2, L_z \), we have that the specification of the quantum number \( m = -l, -l + 1, -l + 2, \ldots, l - 2, l - 1, l \) removes the degeneracy of the spectrum of \( L^2 \).

Since \([L^2, L_{\pm}] = 0\) and \([L_z, L_{\pm}] = \pm \hbar L_{\pm}\), that is \( \mu = \pm \hbar \), then if \( L_{\pm}|l, m\rangle \) is not the null vector, it follows

\[
L^2 |l, m\rangle = \hbar^2 l(l + 1)|l, m\rangle,
\]

\[
L^2 \left[ L_{\pm}|l, m\rangle \right] = \hbar^2 l(l + 1) \left[ L_{\pm}|l, m\rangle \right],
\]

that is \( |l, m\rangle \) and \( L_{\pm}|l, m\rangle \) are eigenvectors of \( L^2 \) corresponding to the same eigenvalue \( \hbar^2 l(l + 1) \).
and then

$$L_z \left[ L_\pm |l, m\rangle \right] = \hbar (m \pm 1) \left[ L_\pm |l, m\rangle \right],$$

that is, by virtue of the theorem with $\mu = \pm \hbar$, the vectors $L_\pm |l, m\rangle$ are eigenvectors of $L_z$ corresponding to the shifted eigenvalues $m \pm 1$.

For these properties, the operators $L_\pm$ are called ladder operators of the angular momentum.
Statement of the problem

By denoting the eigenvectors of the hamiltonian $\mathcal{H}$ of the TIHO with $|\psi_{n,l,m}\rangle \equiv |n, l, m\rangle$, we look for a raising operator, denoted by $\mathcal{M}_+$, such that, in analogy with the degenerate case of the angular momentum, it yields

1) $\mathcal{H}|n, l, m\rangle = E_n|n, l, m\rangle,$

2) $\mathcal{H} \left[ \mathcal{M}_+ |n, l, m\rangle \right] = E_n \left[ \mathcal{M}_+ |n, l, m\rangle \right],$

that is $|n, l, m\rangle$ and $\mathcal{M}_+ |n, l, m\rangle$ have the same energy $E_n$, and

3) $L^2 \left( \mathcal{M}_+ |n, l, m\rangle \right) = \hbar^2 [l(l + 1) + (4l + 6)] \left( \mathcal{M}_+ |n, l, m\rangle \right),$
Statement of the problem

because the difference between two consecutive eigenvalues of $L^2$ (having the same parity) is

$$(l + 2)(l + 3) - l(l + 1) = 4l + 6.$$  

As a consequence, the raising operator $M_+$ has to satisfy, by virtue of the quantum shift theorem, the two commutation rules

$$[\mathcal{H}, M_+] = 0$$  

and

$$[L^2, M_+]|n, l, \bar{m}\rangle = \hbar^2 (4l + 6) M_+|n, l, \bar{m}\rangle,$$

for some eigenvectors $|n, l, \bar{m}\rangle$ of $L^2$, with $\mu = 4l + 6$.  

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It is easy to prove that the second conservation law of the hamiltonian

\[ \mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 r^2}{2} \]

of the TIHO is the symmetric second order tensor

\[ T_{ij} = \frac{p_ip_j + m^2 \omega^2 x_ix_j}{2}, \]

whose trace is coincident with the hamiltonian.
Construction of the ladder operators

With the conserved components

\[ M_x = \frac{p_x p_y}{2m\omega} + \frac{m\omega}{2} \, xy \quad \text{and} \quad M_y = \frac{p_y^2 - p_x^2}{4m\omega} + \frac{m\omega}{4} (y^2 - x^2), \]

of the tensor \( T_{ij} \) we construct the operators \( M_{\pm} = M_x \pm iM_y \), which verify the commutation rules

\[ [L_z, M_{\pm}] = \pm 2\hbar M_{\pm} \quad (2a) \]

and

\[ [L^2, M_+] = 4\hbar M_+ L_z + 6\hbar^2 M_+ + O(r, p)L_+, \quad (2b) \]

where \( O(r, p) \) is a differential operator depending on the components of \( r, p \).
Construction of the ladder operators

From the relation (2b) it effectively follows, with $\mu = 4l + 6$

$$[L^2, M_+]|n, l, l\rangle = \hbar^2 (4l + 6)M_+|n, l, l\rangle. \quad (3)$$

At this point, the conservation of $M_+$, that is $[H, M_+] = 0$, assures that the two vectors $|n, l, l\rangle$ and $M_+|n, l, l\rangle$ have the same energy, that is they correspond to the same energy eigenvalue $E_n$.

Further, the relations (2a) and (3) give us, up to a normalization constant $C$, the operatorial action

$$M_+|n, l, l\rangle = C|n, l + 2, l + 2\rangle$$

and then the operator $M_+$ is the raising operator of $L^2$. 

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Clarification of the accidental degeneration

By iterating the action of the *raising operator* $\mathcal{M}_+$ on the eigenvectors $|n, l, l\rangle$ starting from $l = 0$ or $l = 1$, we obtain all the eigenvectors of the degenerate eigenspace of $E_n$, because we get

$$|n, 0, 0\rangle \rightarrow |n, 2, 2\rangle \rightarrow |n, 4, 4\rangle \rightarrow \ldots \rightarrow |n, n, n\rangle$$

for even $n$ and

$$|n, 1, 1\rangle \rightarrow |n, 3, 3\rangle \rightarrow |n, 5, 5\rangle \rightarrow \ldots \rightarrow |n, n, n\rangle$$

for odd $n$. 
Clarification of the accidental degeneration

In order to terminate, we want to explain shortly why the *lowering operator* of $L^2$ does not exist. Both the commutation rule

$$[L^2, M_-]|l, -l\rangle = \hbar^2 (4l + 6) M_- |l, -l\rangle$$

and the relation (2a) give us the operatorial action of $M_-$

$$M_- |n, l, -l\rangle = \tilde{C} |n, l + 2, -l - 2\rangle,$$

from which we can realize that the action of $M_-$ on $|n, l, -l\rangle$ can not give an eigenvector with a lower quantum number of the operator $L^2$, because the quantum number $m = -l - 2$ can not belong to the multiplet of $l - 2$. 
Clarification of the accidental degeneration

By iterating the action of the *lowering operator* $\mathcal{M}_-$ on the eigenvectors $|n, l, -l\rangle$ starting from $l = 0$ or $l = 1$, we then obtain

$$|n, 0, 0\rangle \rightarrow |n, 2, -2\rangle \rightarrow |n, 4, -4\rangle \rightarrow \ldots \rightarrow |n, n, -n\rangle$$

for even $n$ and

$$|n, 1, -1\rangle \rightarrow |n, 3, -3\rangle \rightarrow |n, 5, -5\rangle \rightarrow \ldots \rightarrow |n, n, -n\rangle$$

for odd $n$. 
THANK YOU VERY MUCH FOR YOUR ATTENTION