Electromagnetic fields and Green function in elliptical vacuum chamber

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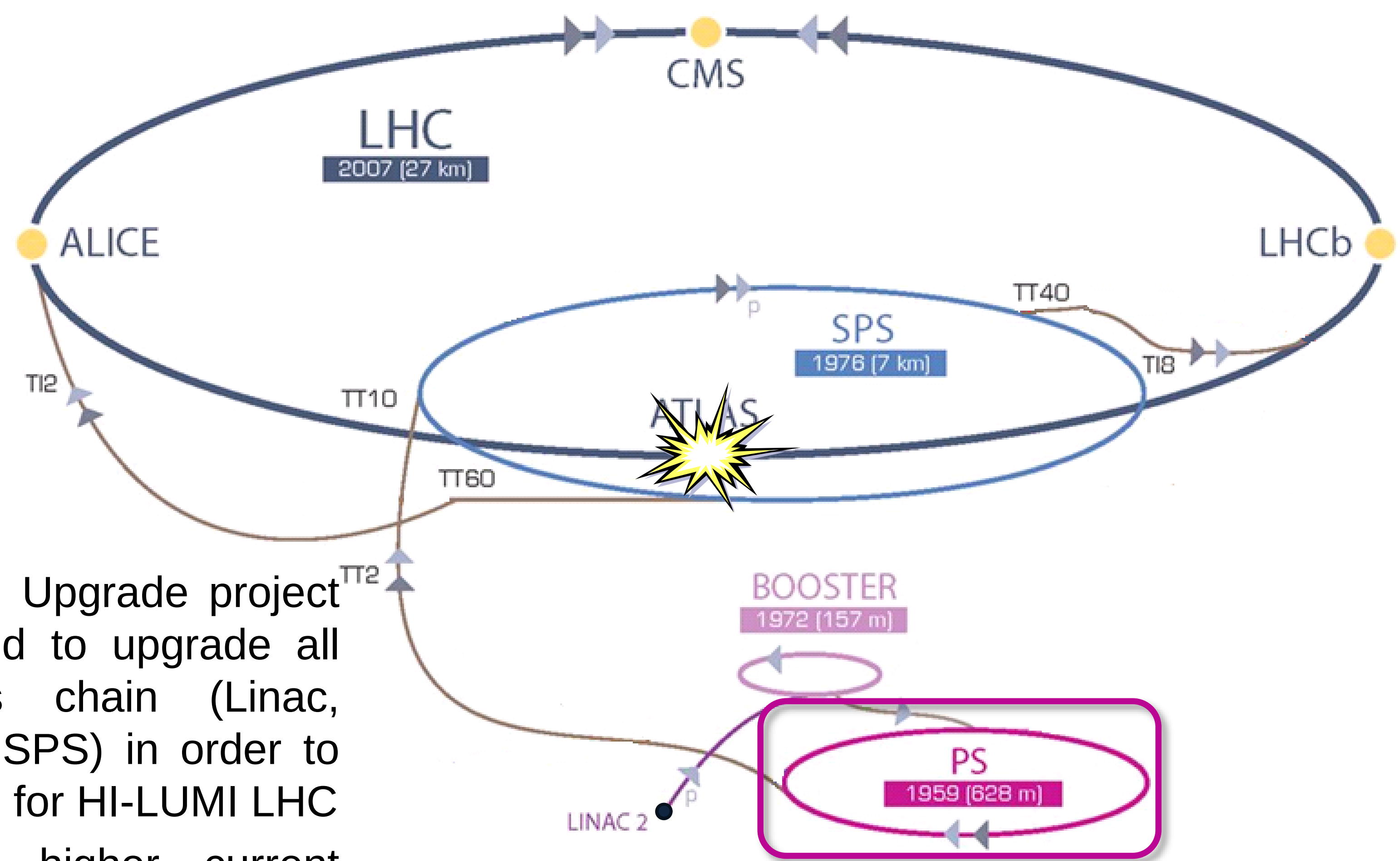
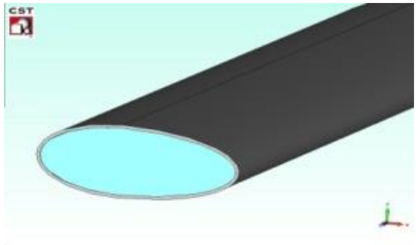
Large Hadron Collider (LHC) injectors chain

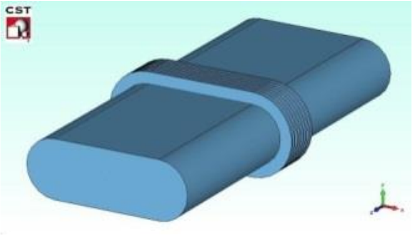


- LHC Injectors Upgrade project (LIU) is aimed to upgrade all the injectors chain (Linac, Booster, PS, SPS) in order to deliver beams for HI-LUMI LHC
- Due to the higher current circulating in the accelerators, it's important to study in details collective effects
- We are developing a longitudinal and transverse coupling impedance model for the PS

Overview of elliptical coordinates

Objective :
Calculate the Green function in elliptical coordinates



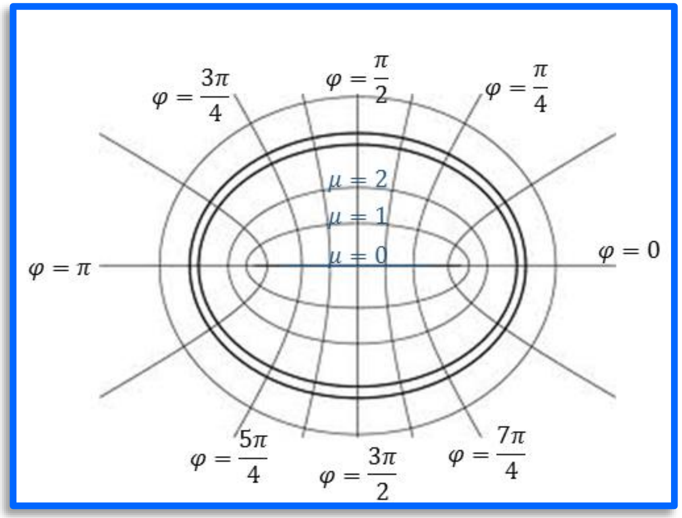


Relation between elliptical and Cartesian coordinates:

$$\begin{cases} x = F\, \cosh\mu\, \cos\varphi \\ y = F\, \sinh\mu\, \sin\varphi \end{cases}$$

$0 \leq \boldsymbol{\varphi} \leq 2\pi$ → **Angular coordinate** (hyperbolas)

$0 \leq \boldsymbol{\mu} \leq \infty$ → **Radial coordinate** (ellipses)

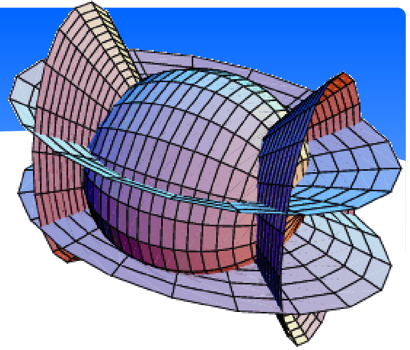


F is the focal distance of the ellipse, with a and b the semi-axis of the ellipse :

$$F = \sqrt{a^2 - b^2}$$

The polar coordinate is a special case of the elliptic coordinate in the limit $F \to 0$

Wave equation in elliptical coordinates



$$\frac{2}{F^2(cosh2\mu - cos2\varphi)}\left(\frac{d^2}{d\mu^2}+\frac{d^2}{d\varphi^2}\right)E_z - k_t^2 E_z = 0$$

Separation of coordinates:

Ordinary Mathieu equation

$$\frac{d^2V}{d\varphi^2} - (a - 2qcos2\varphi)V = 0$$

Modified Mathieu equation

$$\frac{d^2U}{d\mu^2} + (a - 2qcosh2\mu)U = 0$$

$$q = \frac{k_t^2 F^2}{4}$$

a=separation constant

Solution:
Angular (ordinary) Mathieu functions:

Solution:
Radial (modified) Mathieu functions:

EVEN solutions

$$ce_{2n}(\varphi,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2r\varphi$$

$$ce_{2n+1}(\varphi,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)\varphi$$

$$Ce_{2n}(\mu,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2r\mu$$

$$Ce_{2n+1}(\mu,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh(2r+1)\mu$$

ODD solutions

$$se_{2n+1}(\varphi,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} sin(2r+1)\varphi$$

$$se_{2n+2}(\varphi,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} sin(2r+2)\varphi$$

$$Se_{2n+1}(\mu,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} sinh(2r+1)\mu$$

$$Se_{2n+2}(\mu,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} sinh(2r+2)\mu$$

General solution of the wave equation

Spatial solution of the wave equation in elliptic coordinates:

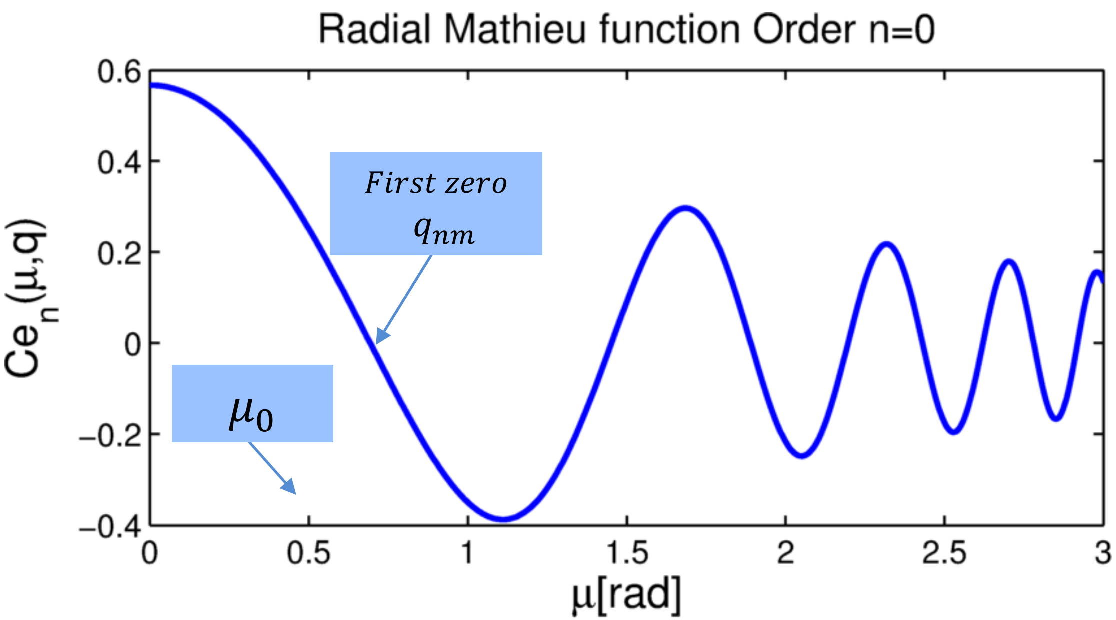
$$E_z(\mu,\varphi) = \sum_{n=0}^{\infty} \begin{cases} Ce_{2n}(\mu,q)ce_{2n}(\varphi,q) \\ Se_{2n+1}(\mu,q)se_{2n+1}(\varphi,q) \end{cases}$$

$$f_{nm} = \frac{c}{2\pi}\sqrt{\frac{q_{nm}}{F^2} + k_z^2}$$

Calculate of cut-off frequencies requires the determination of the eigenvalues q_{nm}

q_{nm} is the m-th root of the Mathieu radial function (or its derivative) and is obtained imposing the boundary conditions to the spacial solution :

$Ce_{2n}(q,\mu_0) = 0$ (TM modes)
$Se'_{2n+1}(q,\mu_0) = 0$ (TE modes)



Green function in free space

$$\nabla_t^2 E_z^s - \frac{k_0^2}{\beta^2\gamma^2} E_z^s = G\delta(P_0)$$

Green function in cylindrical coordinates: $E_z^s = G K_0\left(\frac{k_0 r}{\beta\gamma}\right)$ $\quad G = -jZ_0 \frac{Qk_0}{2\pi\beta^2\gamma^2}$

- K_0 is expanded in separated functions of μ,φ using **Gegembauer's theorem**

$$K_0\left(\frac{k_0 F}{\beta\gamma}\sqrt{sinh^2\mu + cos^2\varphi}\right) = \sum_{n=0}^{\infty} (-1)^n \epsilon_n I_n\left(\frac{k_0 F}{2\beta\gamma} e^{-\mu}\right) K_n\left(\frac{k_0 F}{2\beta\gamma} e^{\mu}\right) \; cos2n\varphi \quad (*)$$

- Expanding $cos2n\varphi$ in terms of Mathieu's functions, and after some manipulations (e.g. orthogonality of Mathieu's functions), we get the

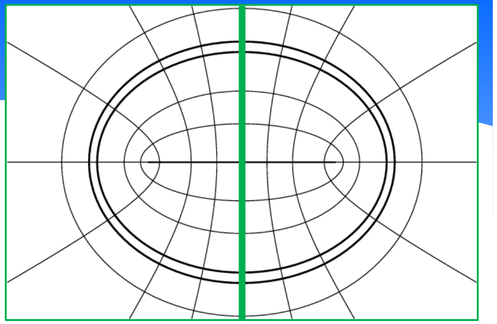
Green function in elliptical coordinates in free space:

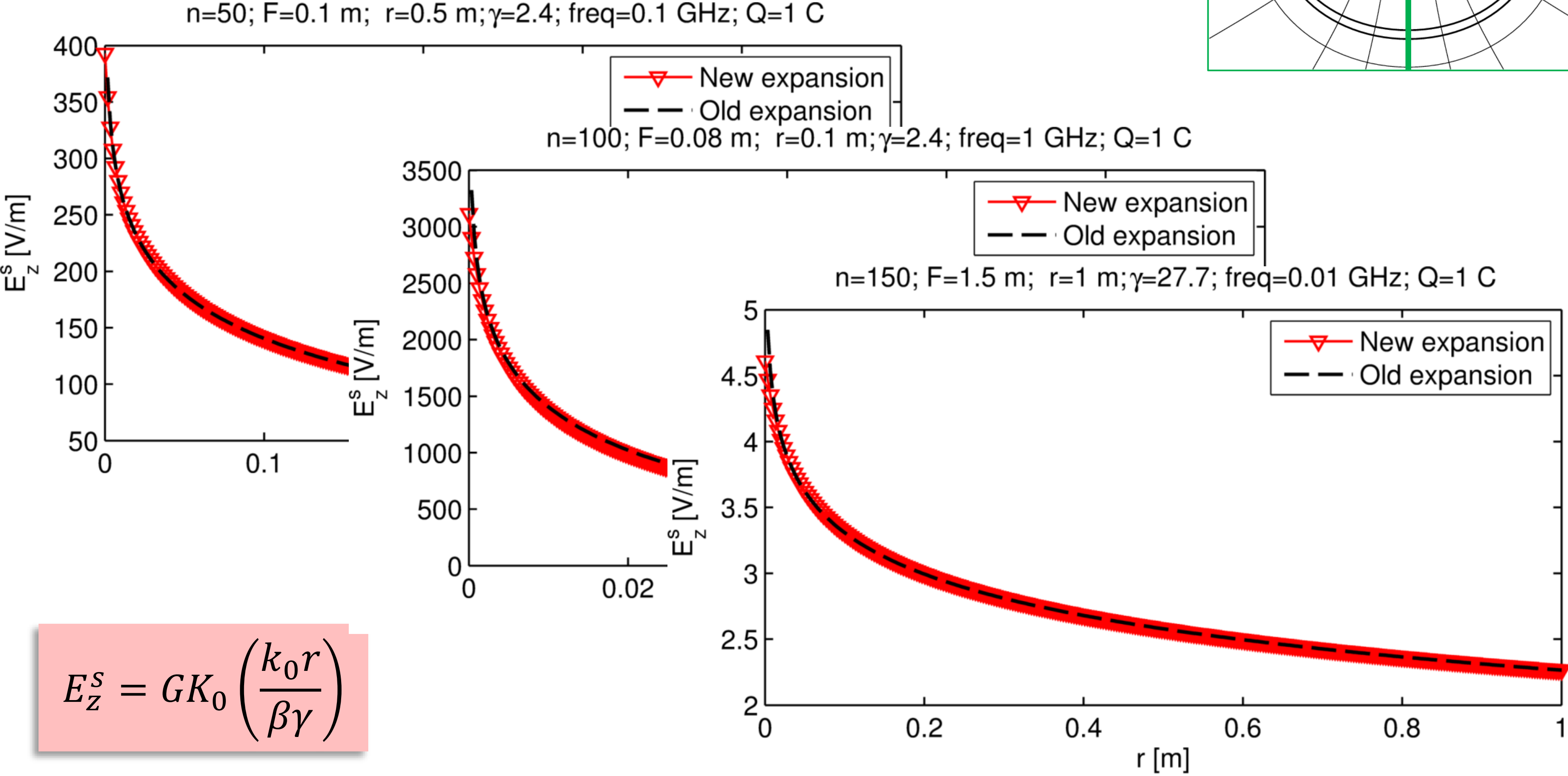
$$E_z^s = 2G \sum_{l=0}^{\infty} ce_{2l}\left(\frac{\pi}{2} - \varphi, q\right) \sum_{n=0}^{\infty} A_{2n}^{(2l)} I_n\left(\frac{k_0 F}{2\beta\gamma} e^{-\mu}\right) K_n\left(\frac{k_0 F}{2\beta\gamma} e^{\mu}\right)$$

$$E_z^s = 2\pi G \sum_{l=0}^{\infty} \frac{(-1)^l A_0^{(2l)}}{p_{2l}} ce_{2l}\left(\frac{\pi}{2} - \varphi, q\right) Fek_{2l}(\mu, -q)$$

Fek: second-kind radial Mathieu function

Green function: numerical results





$$E_z^s = G K_0\left(\frac{k_0 r}{\beta\gamma}\right)$$

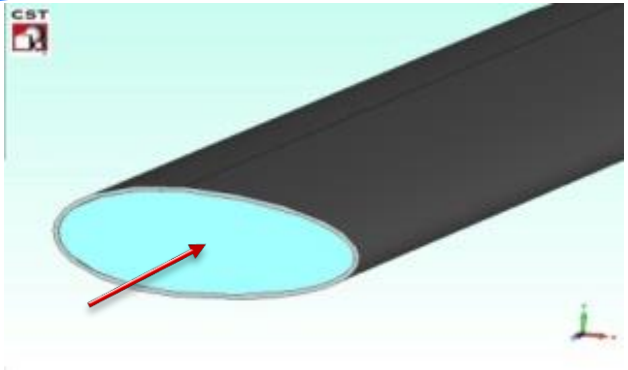
$$E_z^s = 2G\sum_{l=0}^{\infty} ce_{2l}\left(\frac{\pi}{2}-\varphi, q\right)\sum_{n=0}^{\infty} A_{2n}^{(2l)} I_n\left(\frac{k_0 F}{2\beta\gamma}e^{-\mu}\right) K_n\left(\frac{k_0 F}{2\beta\gamma}e^{\mu}\right)$$

$$\left(\varphi = \frac{\pi}{2}\right)$$

Green function in elliptical vacuum chamber

$$\nabla_t^2 E_z^0 - \frac{k_0^2}{\beta^2\gamma^2} E_z^0 = G\delta(P_0)$$

$\boldsymbol{E^0}$ → ***primary*** field



$$E_z^0 = E_z^s + E_z^i$$

$$\nabla_t^2 E_z^s - \frac{k_0^2}{\beta^2\gamma^2} E_z^s = G\delta(P_0)$$

$$\nabla_t^2 E_z^i - \frac{k_0^2}{\beta^2\gamma^2} E_z^i = 0$$

Direct field: satisfies the no-homogeneous equation in free space and the condition of infinite radiation

Indirect field: satisfies the homogeneous equation and boundary condition

Primary field: Indirect term

- The total field is the sum of the direct and the indirect field:

$$E_z^s + E_z^i = 2\pi G \sum_{l=0}^{\infty} \frac{(-1)^l A_0^{(2l)}}{p_{2l}} ce_{2l}\left(\frac{\pi}{2} - \varphi, q\right) Fek_{2l}(\mu, -q) + 2\pi G \sum_{l=0}^{\infty} M_{2l} ce_{2l}\left(\frac{\pi}{2} - \varphi, q\right) Ce_{2l}(\mu, -q)$$

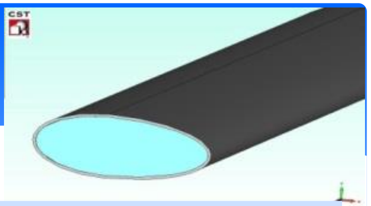
- Applying boundary conditions on the elliptic surface $\mu = \mu_1$ for every value of φ:

$$2\pi G \sum_{l=0}^{\infty} \frac{(-1)^l A_0^{(2l)}}{p_{2l}} ce_{2l}\left(\frac{\pi}{2} - \varphi, q\right) Fek_{2l}(\mu_1, -q) + 2\pi G \sum_{l=0}^{\infty} M_{2l} ce_{2l}\left(\frac{\pi}{2} - \varphi, q\right) Ce_{2l}(\mu_1, -q) = 0$$

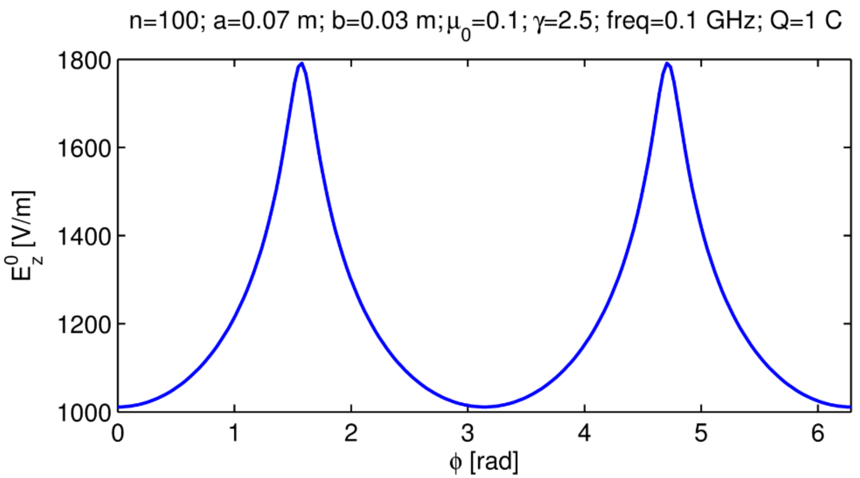
- Being Mathieu's function a complete set of orthogonal functions

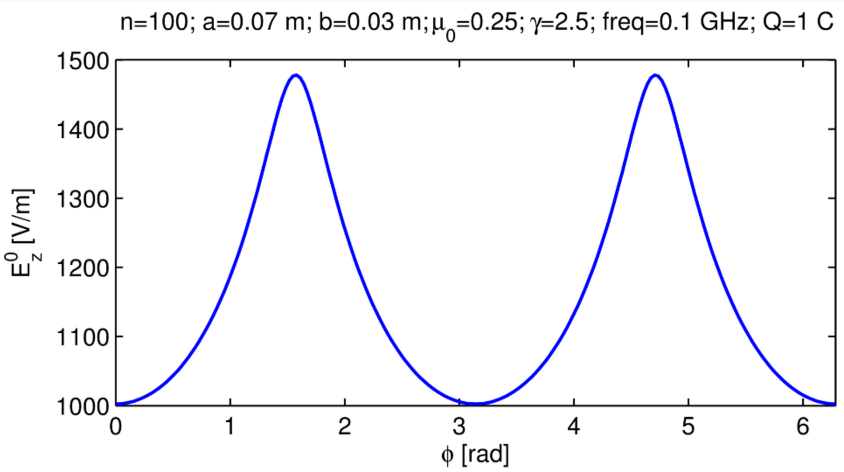
$$M_{2l} = -\frac{(-1)^l A_0^{(2l)}}{p_{2l}} \frac{Fek_{2l}(\mu_1, -q)}{Ce_{2l}(\mu_1, -q)}$$

Primary field in the PS beam chamber



$$E_z^0 = 2\pi G \sum_{l=0}^{\infty} \left(\frac{A_0^{(2l)}}{p_{2l}} \right)^2 ce_{2l}(\varphi, -q) \left[Fek_{2l}(\mu, -q) - \frac{Fek_{2l}(\mu_1, -q) Ce_{2l}(\mu, -q)}{Ce_{2l}(\mu_1, -q)} \right]$$

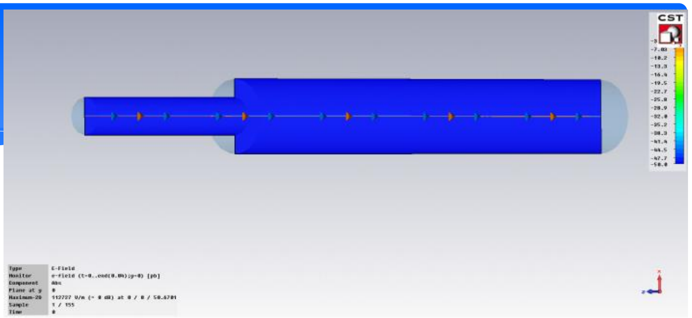




- For comparison we report here the Green function in a circular beam pipe

$$E_z^0 = G \left[K_0\left(\frac{k_0 r}{\beta\gamma}\right) - \frac{I_0\left(\frac{k_0 r}{\beta\gamma}\right) K_0\left(\frac{k_0 r_0}{\beta\gamma}\right)}{I_0\left(\frac{k_0 r_0}{\beta\gamma}\right)} \right]$$

Radiation process in step transition



$$E^{tot} = E^0 + E^{rad}$$

E^0 → ***primary*** field

+

E^{rad} → ***radiated*** field

The longitudinal coupling impedance depends only on the radiated field, because it is this field which is scattered by the discontinuity

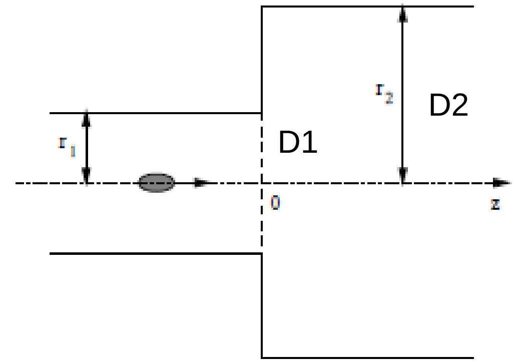
$$Z_L(k) = -\frac{1}{q}\left[\int_{-\infty}^{0} E_{1z}^{rad}(r,z)e^{j\frac{k}{\beta}z}\,dz + \int_{0}^{\infty} E_{2z}^{rad}(0,z)e^{j\frac{k}{\beta}z}\,dz\right]$$

Matching system for radiated field calculation

Mode Matching

$$E^0_{1,t} + E^{rad}_{1,t} = E^0_{2,t} + E^{rad}_{2,t} \text{ in D1}$$

$$0 = E^0_{2,t} + E^{rad}_{2,t} \text{ in D2-D1}$$



Modal expansion coefficient

TM modes: $ce_{2n}Ce_{2n}$

$$\boldsymbol{E}^0_{1,t} + \left\{ \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \alpha_{1_{2n,m}} \boldsymbol{e}^{(t)}_{\boldsymbol{2n}}\left(q^{TM}_{1_{2n,m}}\right) + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \beta_{1_{2n+2,m}} \boldsymbol{e}^{(t)}_{\boldsymbol{2n}}\left(q^{TE}_{1_{2n+2,m}}\right) \right\}$$

$$= \boldsymbol{E}^0_{2,t} + \left\{ \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \alpha_{2_{2n,m}} \boldsymbol{e}^{(t)}_{\boldsymbol{2n}}\left(q^{TM}_{2_{2n,m}}\right) + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \beta_{2_{2n+2,m}} \boldsymbol{e}^{(t)}_{\boldsymbol{2n}}\left(q^{TE}_{2_{2n+2,m}}\right) \right\}$$

We scalar multiply both members for a convenient set of **TM** modal function and integrate in D_1 e D_2, respectively.

We scalar multiply both members for a convenient set of **TE** modal function and integrate in D_1 e D_2, respectively.

Matching equation #1

Matching equation #2

Mode matching system for impedance calculation

$$\begin{cases} F + T_1\alpha_1 + S_1\beta_1 = D_1\alpha_2 \\ S_2\beta_1 = D_2\beta_2 \\ V + D_3\alpha_1 = P_3\alpha_2 \\ W + D_4\beta_1 = P_4\alpha_2 + Q_4\beta_2 \end{cases}$$

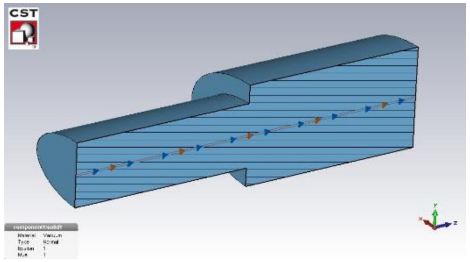
$\alpha_1, \alpha_2, \beta_1, \beta_2$ → Unknown of dimension **(n x m)x1**

$T_1, S_1, S_2, P_3, P_4, Q_4$ → Tensor of dimension **(n x m)x(r x s)**

F, V, W → Known terms of dimension **(r x s)x1**

D_1, D_2, D_3, D_4 →Diagonal matrix of dimension **(r x s)x(r x s)**

- **Determining a set of modal amplitudes** associated with the field expansions in the two regions



$$Z_L(k) = -\frac{1}{jq}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\alpha_{1_{2n}}\frac{ce_{2n}(q_1)Ce_{2n}(q_1)}{\sqrt{k_0^2 - \frac{4q_1}{F^2}}} + \alpha_{2_{2n}}\frac{ce_{2n}(q_2)Ce_{2n}(q_2)}{\sqrt{k_0^2 - \frac{4q_2}{F^2}}}\right]$$

Thank you for your attention