Diario di un viaggio in corso
dalle biglie di vetro
verso una teoria invariante conforme.

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Marble (or glass) pebbles

Marble and glass are very hard substances that are nearly incompressible.

Some interesting questions (in common with other granular materials) arise when you put together under pressure a large number of pebbles.

In the real world friction is important! Angle of contact$=90^0 \pm \delta$, with a large δ for most of the materials.

Marble and glass pebbles: friction is small and δ may become small.

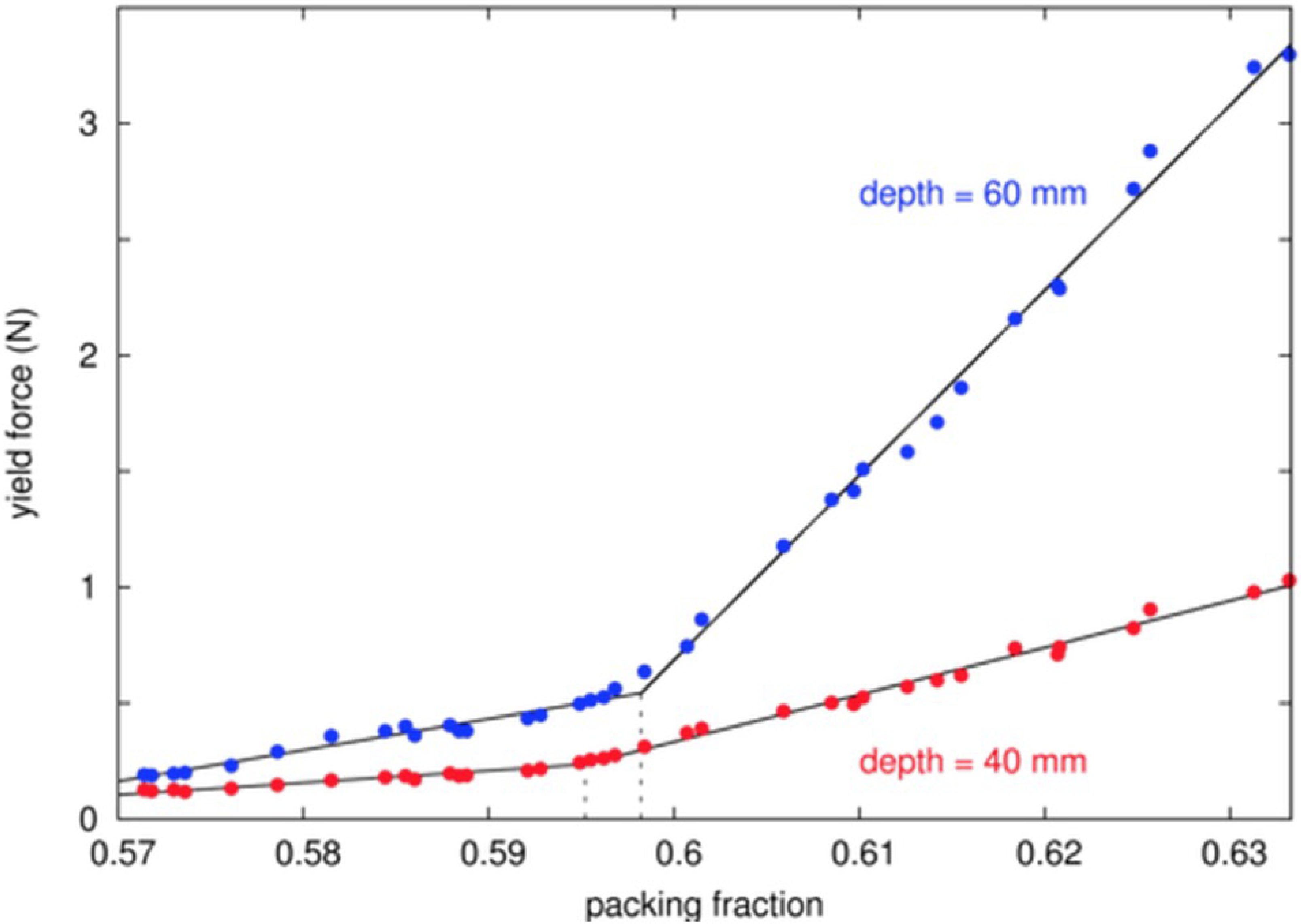
Some questions:

- ¿How the properties of the material depend on the density? ¿Which is the effect of slow compactification?
- ¿Which are the properties of the network of contacts?
- ¿Which are the properties of the distance distribution?
- ¿Which are the properties of the forces among pebbles?

I will simplify the problem by considering spherical objectes.

You can shake spheres and observe how the density decrease with decreasing the shaking (shaking mocks the temperatures).

A nice shaking machine for plastic spheres!



Force needed to push a cylindrical rod into a packing of glass spheres with a given volume fraction and at two different depths. The rod diameter is 6.3 mm the sphere diameter 265 µm. From [2].

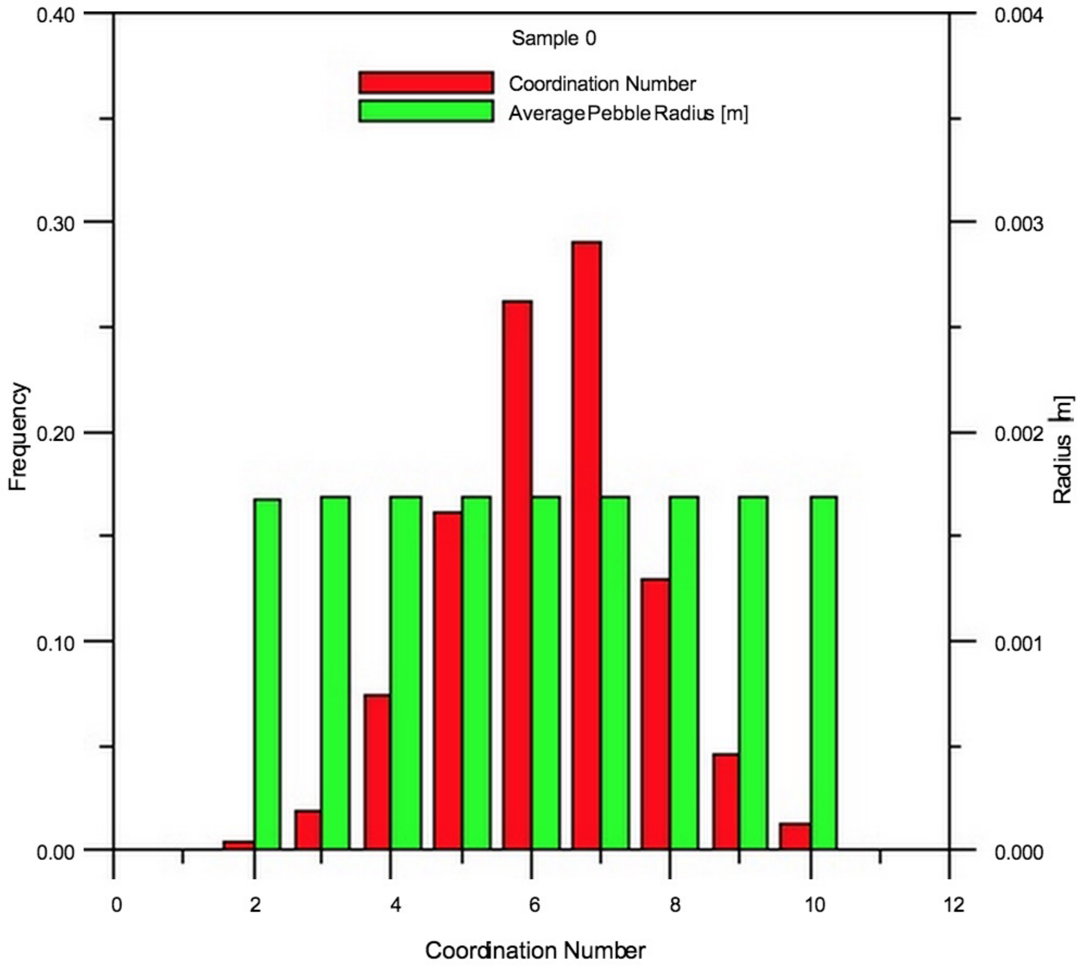
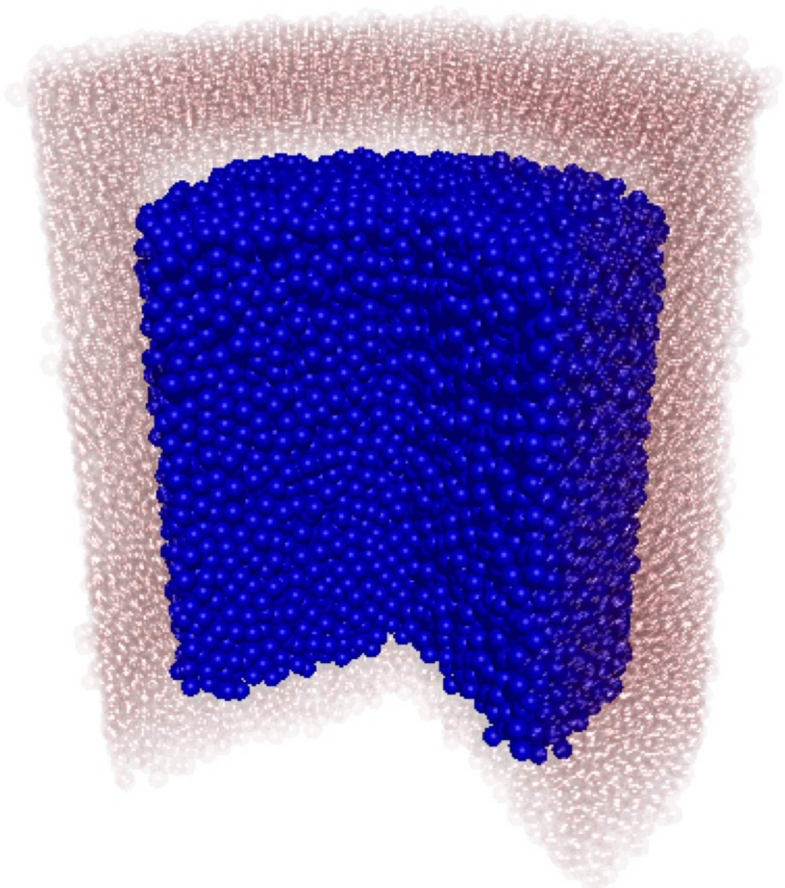
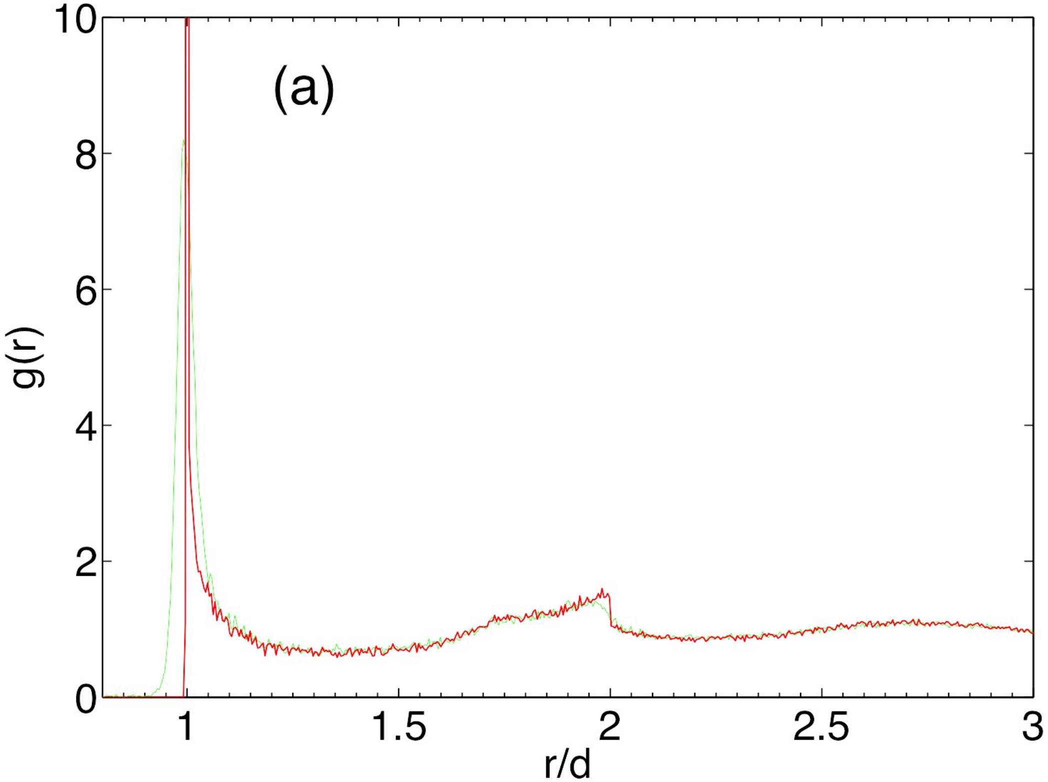


Fig. 4.6. Coordination number (red bars) and pebble average radius distribution (green bars) in sample S0.

First results by X-ray tomography (spherical pebbles have different radii).

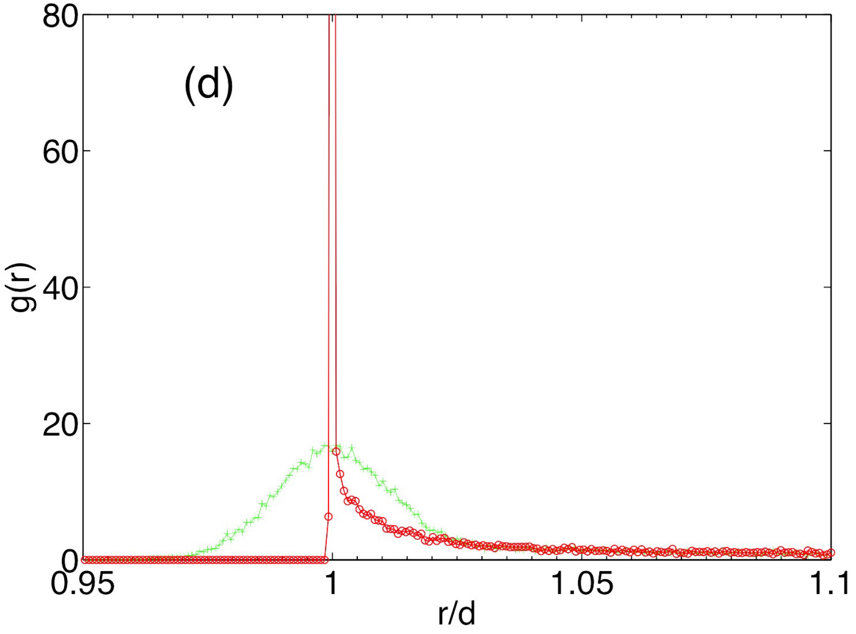


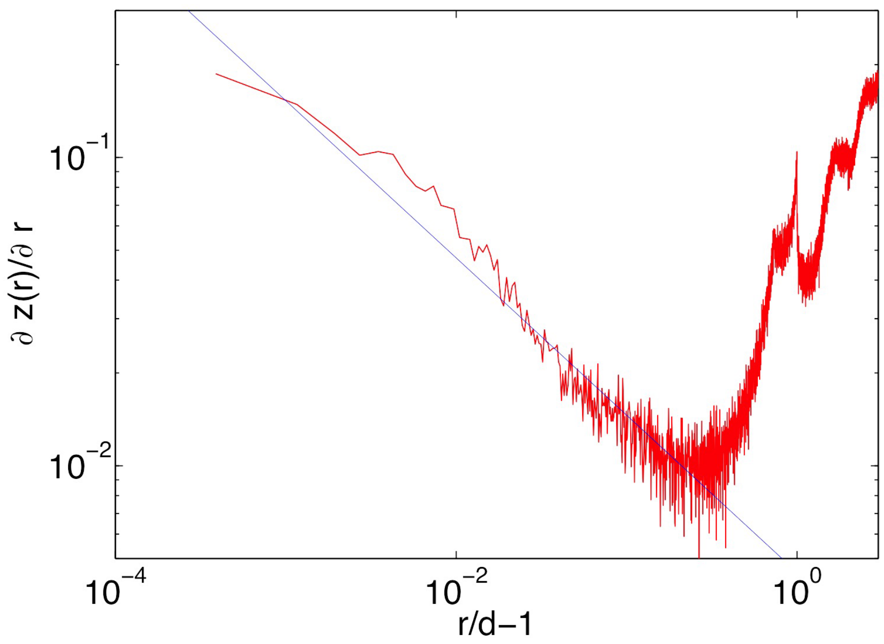


On the left 150.000 glass beads whose position has been reconstructed by tomography (Delaney, Di Matteo and Aste).

The spheres have all the same radius. The error in the position can be reduced by imposing the non overlapping condition (i.e. post-processing of the data).

On the right the radial correlation function $g(r)$.

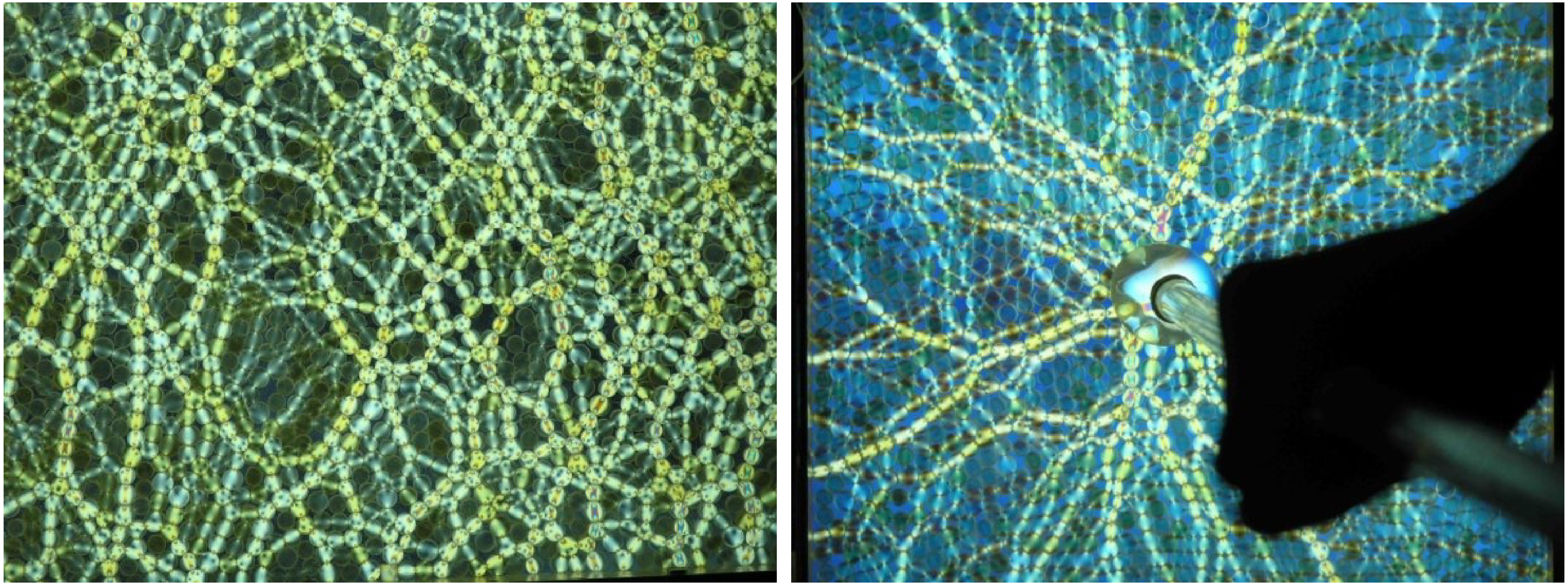




The radial correlation function $g(r)$ is singular when $r/d \to 1$ for $r/d > 1$. It behaves as

$$(r/d-1)^{-\gamma} \qquad \gamma \approx 0.5$$

The radial correlation function $g(r)$ is proportional to probability of finding two particles at distance r.



You can analise the force network: there are-arc like lines with strong forces.
Photo-elastic disks (Bob Behringer)

Zero friction spherical pebbles are hard spheres.

We can study analytically the $\delta \to 0$ limit. We can (in principle) compute the Cbluequilibrium thermodynamics at given temperature and pressure.

The Hamiltonian is

$$H(\{x\}) = \sum_{i,k=1,N} U(x_i - x_k) \;\; U(x) = \infty \text{ for } |x| < D \qquad U(x) = 0 \text{ for } |x| > D\,.$$

We can also consider harmonic spheres:

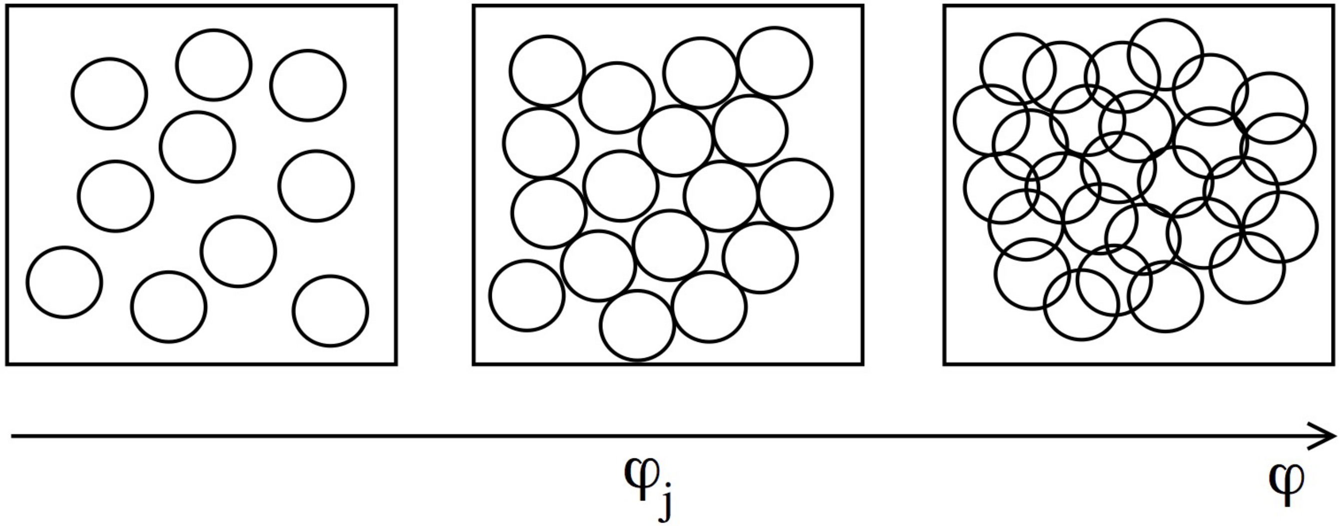
$$U(x) = (D - |x|)^2 \text{ for } |x| < D\,.$$

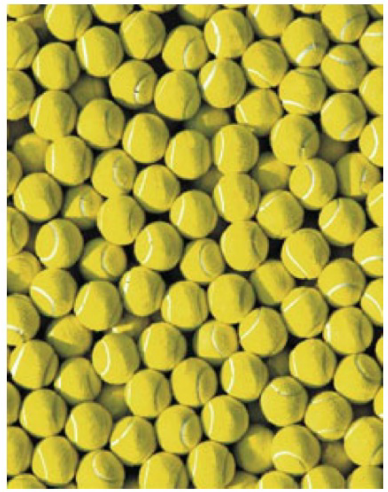
Jamming is the point where particles are in contact and they cannot move.
Infinite pressure limit of hard spheres or
zero temperature limit of soft spheres at maximal density where the energy is zero.

The jamming transition

A transition that is observed in everyday experience

An *athermal* assembly of repulsive particles
Transition from a loose, floppy state to a mechanically rigid state
Above jamming a mechanically stable network of particles in contact is formed



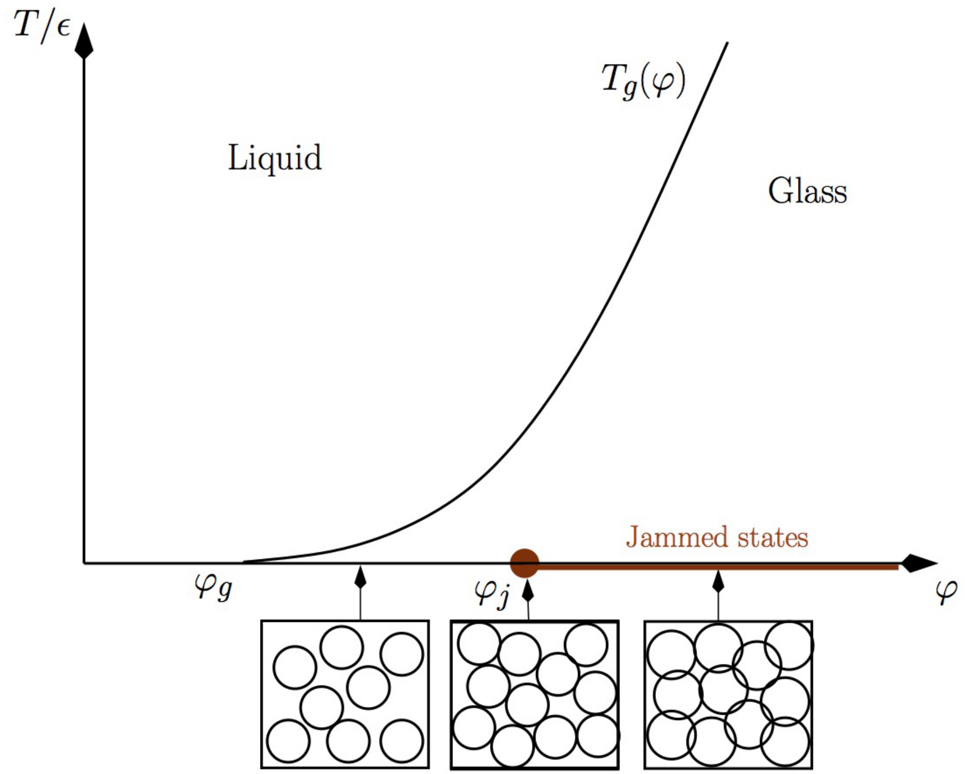


For hard spheres, φ_j is also known as *random close packing*: $\varphi_j(d = 3) \approx 0.64$

[Bernal, Mason, Nature 188, 910 (1960)]
[O'Hern, Langer, Liu, Nagel, PRL 88, 075507 (2002)]

Glass/jamming phase diagram

- Statistical mechanics:
  introduce temperature T
  and eventually send $T \to 0$
- The soft sphere model:
  $v(r) = \epsilon(1 - r/\sigma)^2\theta(r - \sigma)$
- Two control parameters:
  T/ϵ and $\varphi = V_\sigma N/V$
- $T/\epsilon = 0$ & $\varphi < \varphi_j \leftrightarrow$ hard spheres



Jamming is a transition from "entropic" rigidity to "mechanical" rigidity
A theoretical description of the glass transition is difficult; and jamming happens inside the glass!

[Berthier, Jacquin, FZ, PRE 84, 051103 (2011)]
[Ikeda, Berthier, Sollich, PRL 109, 018301 (2012)]

Forces and the network of contacts. Force satisfy the equations:

$$\forall\, i \quad \sum_k (\vec{x}_k - \vec{x}_i) f_{i,k} = 0$$

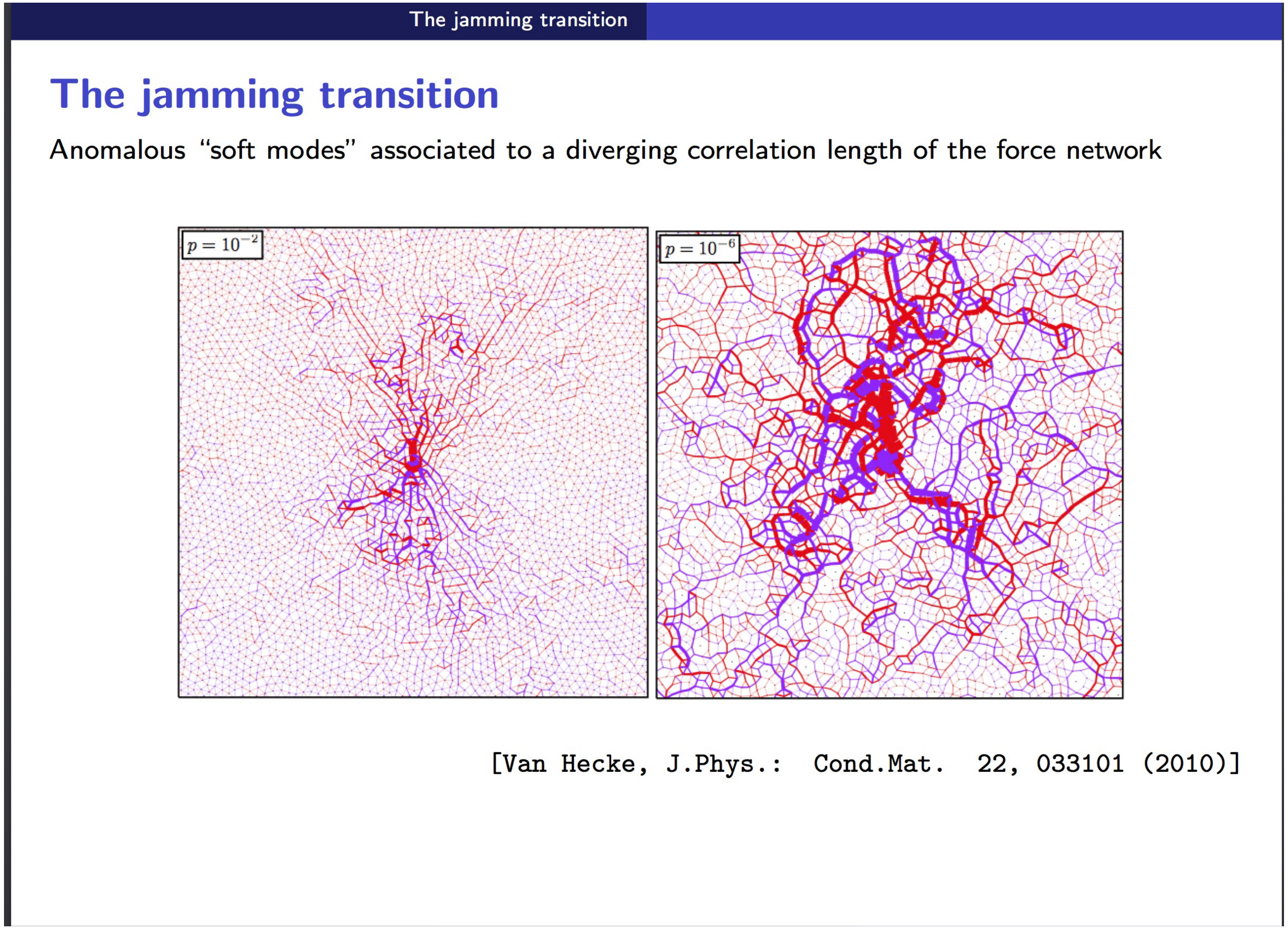
$f_{i,k}$ are scalar quantities!

The total scalar force on a given particle i is $f_i \equiv \sum_k f_{i,k}$.

If Z is the average number of contacts per particles and we have N particles we have $NZ/2$ unknown the forces for dN equation (d is the space dimension)

- $Z < 2d$ hypostatic (no solution for general x (unstable) (2 legs table)
- $Z = 2d$ isostatic (just one solution general x (stable) (3 legs table)
- $Z = 2d$ hyperstatic: many solutions solution general x (stable but the force is undefined) (4 legs table)

J. C. Maxwell, Philosophical Magazine 27, 598 (1864)



Jamming as a critical point

Let us consider the over-compressed case and we decrease the density up to jamming. The response length diverges.

A phenomenological theory

Marginal stability (Liu, Nagel Wyart).

We start from a configuration of hard spheres and we compress.

The following scenario qualitatively holds:

We are first attracted by an hypostatic configuration that has a big attraction basin.

This configuration is unstable: we escape from it (hypostatic configurations are like *saddles*).

We relax more and more up to the point where we arrive to an isostatic configuration.

Coming from soft side, we start from a configuration and we find the minimum of the energy. We decrease the energy up to the point where the energy becomes zero.

At the isostatic point there are long range correlations.

Also local quantities are singular.

$$g(r) \propto (r-D)^{-\gamma}\,, \qquad P_{forces}(f) \propto f^{\theta}\,, \qquad \overline{\Delta^2} \propto p^{-\kappa}\,.$$

where $\Delta_i^2 \equiv \lim_{t\to\infty} t^{-2} \int_0^t dt' dt'' (x_i(t') - x_i(t''))^2$ is the size of the cage of the i^{th} particle and p is the pressure (D is the diameter of the spheres).

We assumes that at very high density the system behaves like a solid: vibrations, but no diffusion, i.e. $\overline{\Delta^2} < \infty$.

A phenomenological analysis (Wyart), similar in spirit to the one done 50 years ago for the standard phase transitions, gives the following relations for the exponents:

$$\gamma = 1/(2+\theta) \qquad \kappa = (1+\theta)/(3+\theta)\,.$$

Large scale numerical simulations give:

$$\gamma \approx \theta \approx .4 \qquad \kappa \approx 1.4\,.$$

¿Which is the shape of the cages at finite pressure?

$$P_i(x_i - \overline{x_i})$$

The order parameter is the probability distribution of the form of the cages $\mathcal{P}[\mathcal{P}]$.

Gaussian cages: a computation (GP and Zamponi 2006) in all dimensions d

$$Z = 2Ad \qquad A = 0.971....$$

We were very happy $A \approx 1$; 3% discrepancy. Not bad for a mean field theory! No isostaticity.

$$\theta = \gamma = 0 \qquad \kappa = 2$$

¿Why we do not get the exact result and the correct exponents?

Many excuses:

- Real cages are not Gaussian.
- The trivial exponents mean field exponents should be obtained only in the limit $d \to \infty$.

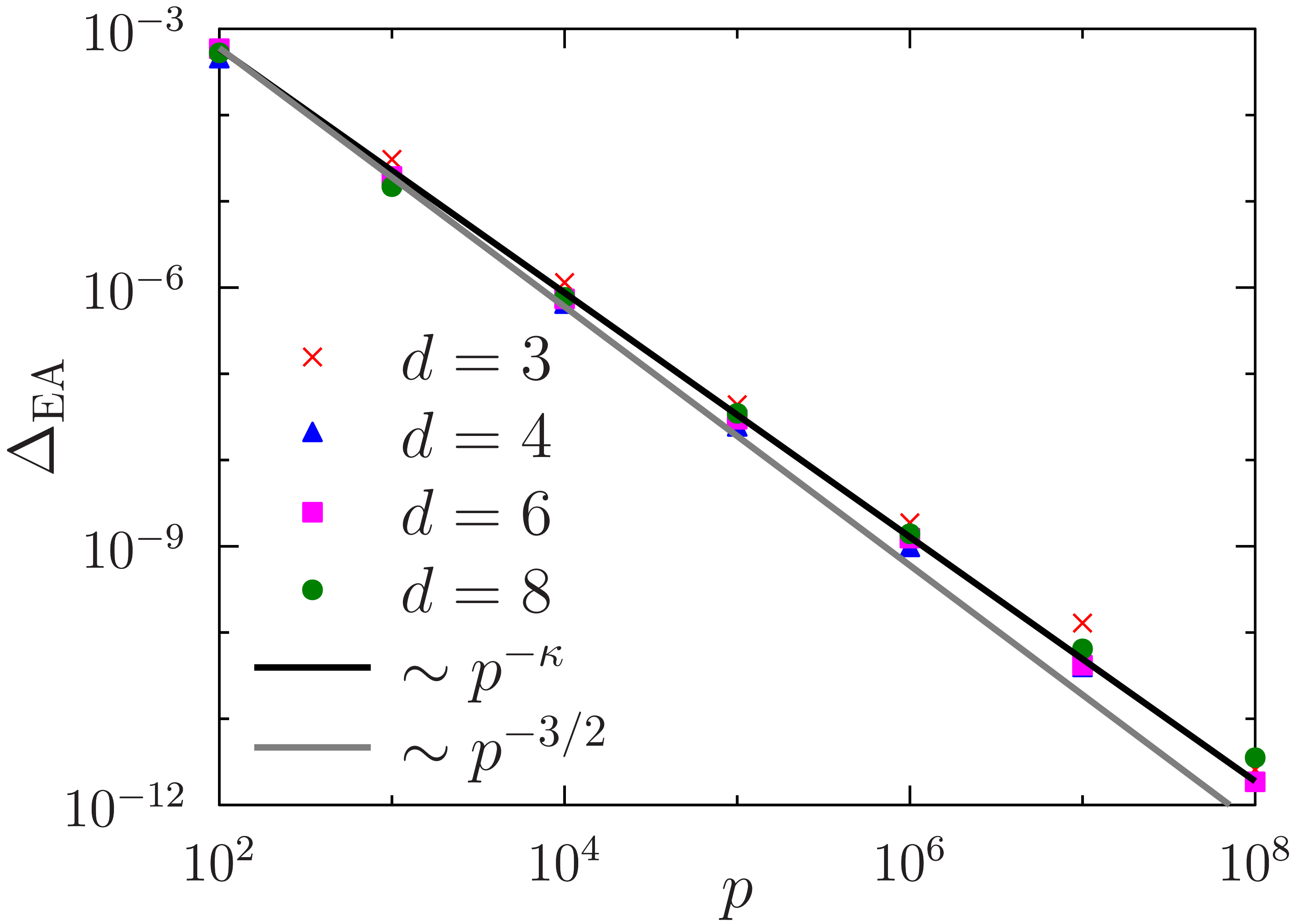
Thise excuses faded away after it was discovered that:

- Real cages must become Gaussian when $d \to \infty$ (Kurchan, GP, Zamponi).
- The exponents are practically constant in high dimensions (up to 13) (Charbonneau, Corwin, GP, Zamponi).

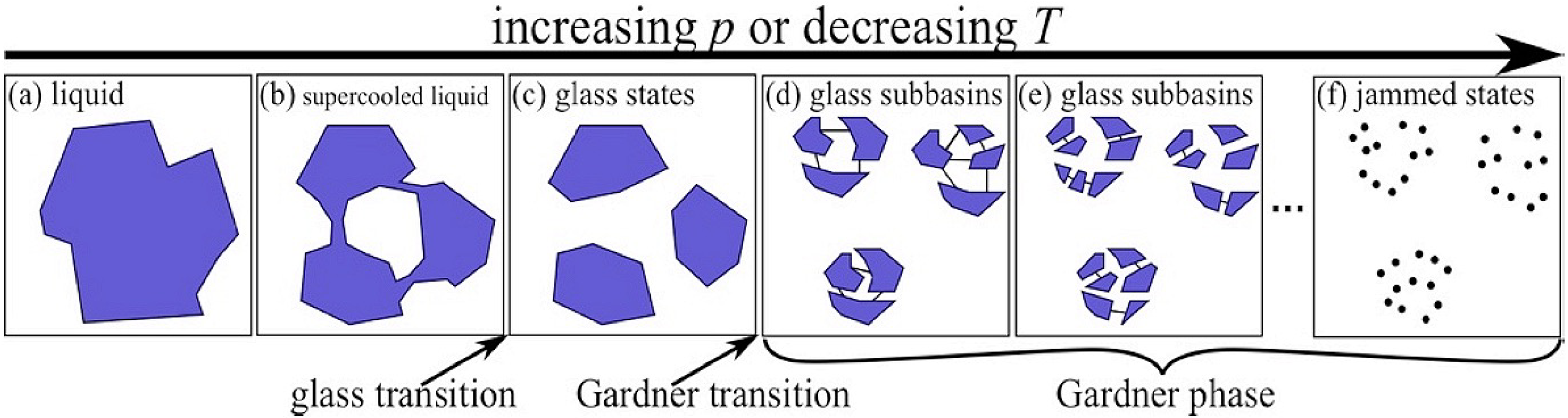
The correct mean field theory must give exponents similar to the ones numerically observed.

It would be very strange if the exponents would be trivial only for dimensions grater then 26.

$\gamma \approx \theta \approx .4 \quad \kappa \approx 1.4$ in all dimensions $d \geq 2$.



The Garner transition



At finite high pressure cages breaks up in smaller cages that are not too far one form the others (Kurchan, GP, Urbani, Zamponi)

At infinite process each jammed state is surrounded by other jammed states that form a fractal in configuration space (Charbonneau, Kurchan, GP, Urbani, Zamponi)

This cage breaking process is described by a functional order parameter $y(\Delta)$.

$y(\Delta)$ can be measured directly by studying fluctuations dissipation relations or the response of the ground state to an external perturbation.

Continuous Replica symmetry breaking

The hierarchical organization of cages is called Continuous Replica symmetry breaking because in the very compact replica formalism it correspond to a spontaneous continuous breaking of the replica symmetry.

The formalisms was introduced in the case of the mean field theory of spin glasses (the Sherrington Kirpatrick model)

It was proved rigorously by Guerra, Talagrand, Panchenko that if gives the correct results in the case of the mean field theory of spin glasses.

The Gardner transition was discovered in generalized spin glasses by Gross, Kanter and Sompolinsky and by Gardner.

Functional order parameters.

Overlap in presence of a perturbation x^* minimum

$$\Delta(x^*, x) = N^{-1} \sum_{i=1,N} (x_i^* - x_i)^2 \qquad H(x|\epsilon) = H(x) - \epsilon\Delta(x^*, x)$$

$$P(\Delta|\epsilon) = \frac{\partial}{\partial\Delta}\left(\exp\left(-\epsilon^{-1}\int_0^\Delta d\Delta' y(\Delta')\right)\right)$$

Surprising Results

After a long computations one finds (Charbonneau, Kurchan, GP, Urbani, Zamponi):

- Isostaticity, i.e $A = 1$.
- Marginal stability.
- Apparently irrational exponents:

  $\gamma = 0.41269\ldots \qquad \theta = 0.42311\ldots \qquad \kappa = 1.41574\ldots.$

The computations uses heavily the very compact replica symmetry breaking formalism that was developed for spin glasses.

Numerical simulations tell us that there are tiny corrections to mean field theory exponents from 2 dimensions up to ∞.

Open problems: long range correlations

Up to now exponents for local quantities.

We know that we have have long range correlations.

¿Which are the values of the exponents?

¿Can we relate these new exponents to the local exponents?

We can consider large forces correlations, small forces correlations ¿Many different exponents???

Careful numerical computations should be done!

Analytic computations very difficult, but doable.

We likely end up with a new scale invariant theory.

Very open questions: Conformal invariance

¿Is the theory *both* scale invariant and conformal invariant?

Conformal invariance determines the form (in the scaling regime) of the three points correlations (if we know the two point function).

¿Can we check the conformal invariance predictions in experiments with 150.000 spheres?

¿If conformal invariance holds, can we use conformal bootstrap arguments to get some information of the values of the exponents?

Constrain satisfaction models.

Let us consider a graph (nodes connected by edges).

¿ Can we color it with q colours in such a way that each edges connects nodes of different colours?

If the graph is planar 4 colours are enough!

If we take a random graph with average coordination number z, we find that, with probability 1 when N goes to ∞, the graph can be coloured for

$$z < z_c(q)\,.$$

It cannot be coloured, with probability 1 when N goes to ∞, for

$$z > z_c(q)\,.$$

This is the 0-1 law

Continuous constraint satisfaction problems. We have M constraints:

$$\forall a \in 1....M \quad f_a(\vec{w}) > S\,; \qquad P(\vec{w}) \propto \prod_{a=1,M} \theta(f_a(\vec{w}) - S)\,,$$

where $\vec{w}$ is a N-dimensional vector of continuous variables.
In the thermodynamic limit:

$$\alpha = M/N \quad \text{and} \quad N \to \infty\,.$$

¿Which is the value of α (α_c) at fixed S (or the critical value of S (S_c) at fixed α) where the solutions disappear?
The volume of the solutions of the constraints shrinks to zero where we reach the critical point. ¿Can we define exponents and which is their value?
Jamming is a critical point of a continuous constraint satisfaction problem.

Optimal machine learning

We have a function F that gives the output as function of the input; $\vec{w}$ denotes the control parameters:

$$OUTPUT_a = F(INPUT_a|\vec{w})$$

Each input-output pair is a constraint on the $\vec{w}$.

Learning from examples: we have a large set of inputs and output and we want to find a function that always gives the correct result.

Perfect learning: all constraint are satisfied.

The critical point is where perfect leaning fails: the satisfaction problem becomes unsatisfiable.

We can define critical exponents. We conjecture that in a large class of leaning model the exponents are the same as jamming.

Jamming belongs to a wide universality class of continuous constraint satisfaction models.

Neural networks

In the case of neural networks with uncorrelated inputs analytic computations are possible.

In the case of one neuron (perceptron) one finds (Franz and GP) that in the region of negative threshold (the so called negative perceptron) the critical exponents are the same as those of jamming.

The equations are slightly different, but they belongs to the same universality class.

It should not be too difficult to check these results are correct also in the case two layered neural networks. Work is in progress (Franz, GP and Zecchina).

Open problems

- Consolidate the theory: why it works and how big are the corrections?
- Find when it can be applied: universality classes in mean field theory and beyond mean field theory.
- Compute the scaling (and conformal invariant) correlations.
- Extend the theory to the quantum case: this is relevant for the properties of very low temperature glasses.