Stability of plasma tearing modes and asymptotic expansion

Brunello Tirozzi
Sapienza – ENEA – CINFAI
brunellotirozzi@gmail.com

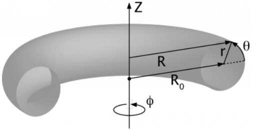
Camillo Tassi
camillo.tassi@gmail.com

Paolo Buratti
ENEA
paolo.buratti@enea.it

List of topics

Resistive $m = 1$ mode

We have considered the MHD equations to describe the plasma dynamics in a large aspect ratio tokamak. We have linearized the equations around an equilibrium state and considered a perturbation:



$$\vec{\xi} = \xi(r)\, e^{\lambda\, t + i\,\theta - i\,\phi}$$

- For $\Re(\lambda) > 0$ we have an unstable perturbation (where $\Re(\lambda)$ is the growth rate),
- $\Re(\lambda) < 0$ the perturbation is stable,
- $\Re(\lambda) = 0$ marginal stability.

Resistive layer

The plasma dynamics can be obtained by dividing the plasma in two regions: a region in which the resistivity is negligible, and a microscopic resistive layer around the surface

$$q(r) := r\,B_\phi/(R_0\,B_\theta) = 1, \qquad \text{where } \vec{B} = B_\phi\,\hat{e}_\phi + B_\theta\,\hat{e}_\theta,$$

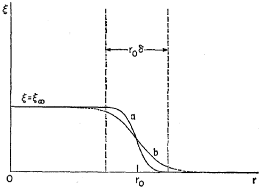
whose dynamics equations are

$$\epsilon\,\lambda\,\frac{1}{x}\,\xi''''(x) - 2\,\epsilon\,\lambda\,\frac{1}{x^2}\,\xi'''(x) + + \left(2\,\epsilon\,\lambda\,\frac{1}{x^3} - x - \lambda^2\,\frac{1}{x}\right)\xi''(x) - 2\,\xi'(x) = 0, \quad (1)$$

- $x = \frac{r - r_0}{r_0}$ where r_0 is such that $q(r_0) = 1$,
- $\epsilon = \tau_H/\tau_R < 10^{-5}$ is a small parameter ($\tau_H < 10^{-6}\,s$ is an Alfvén time and $\tau_R > 10^{-1}\,s$ is the resistive time),
- λ is the growth rate divided by τ_H.

Boundary condition

The boundary condition are the matching condition of the solution ξ, on the edges of the resistive layer, with the solution outside the resistive layer:



$$\lim_{x \to -\infty} \xi(x) = \xi_\infty$$

$$\lim_{x \to +\infty} \xi(x) = 0,$$

$$\lim_{x \to -\infty} -\frac{\pi x^2}{2} \frac{d \ln(\xi)}{dx} = \lambda_H,$$

where λ_H is a real parameter which depends on the tokamak profiles of current and pressure.

Eigenvalue equation

A solution of the resitive dynamics equations in integral form was found by Ara, Basu, Coppi et al, Annals of Physics 1978, [ABC] in the following.
From this solution one obtains

$$\hat{\lambda} = \hat{\lambda}_H \left\{ \frac{\hat{\lambda}^{9/4}}{8} \frac{\Gamma\left[(\hat{\lambda}^{3/2} - 1)/4\right]}{\Gamma\left[(\hat{\lambda}^{3/2} + 5)/4\right]} \right\}, \quad \begin{array}{l} \hat{\lambda} := \lambda/\epsilon^{1/3} \\ \hat{\lambda}_H := \lambda_H/\epsilon^{1/3} \end{array}$$

valid for $\Re\left(\hat{\lambda}^{3/2}\right) > 1$ and $\lambda_H > 0$, which allows to find the growth rate λ, when λ_H is assigned.

WKB method

To get a solution in explicit form, taking advantage of the small parameter ϵ, we tried with a WKB expansion:

$$\xi = \exp\left[\frac{i}{\epsilon}\, S(x)\right] \sum_{i=0}^{+\infty} \epsilon^i\, \xi_i(x),$$

Putting the expansion in the dynamycs equations we get $S(x) = 0$ and a recursive system of differential equations:

$$\left(x + \frac{\lambda^2}{x}\right) \xi_0''(x) + 2\,\xi_0'(x) = 0, \tag{2a}$$

$$\left(x + \frac{\lambda^2}{x}\right) \xi_i''(x) + 2\,\xi_i'(x) =$$

$$= \frac{\lambda}{x}\,\xi_{i-1}''''(x) - \frac{2\,\lambda}{x^2}\,\xi_{i-1}'''(x) + \frac{2\,\lambda}{x^3}\,\xi_{i-1}''(x). \tag{2b}$$

First order solution

The perturbation equations can be easily solved for the lower orders; at first order one gets

$$\xi_0(x) = \frac{\xi_\infty}{2}\left[1 - \frac{2}{\pi}\arctan(x/\lambda)\right]$$

and

$$\xi_1(x) = \frac{\xi_\infty}{\pi}\left[\frac{4\,\lambda^2\,x}{3\,(x^2+\lambda^2)^3} - \frac{5\,x}{6\,(x^2+\lambda^2)^2} - \frac{5\,x}{4\,\lambda^2\,(x^2+\lambda^2)}\right].$$

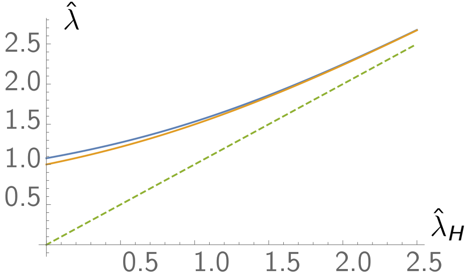
Using the boundary condition for λ_H we get the perturbative eigenvalue equation:

$$\lambda_H = \lambda - \frac{5}{4}\,\frac{\epsilon}{\lambda^2},$$

that can be solved by radicals (while the [ABC] eigenvalue equation can be solved only numerically) and gives rise to three solutions.

Comparison between analytic and perturbative solutions

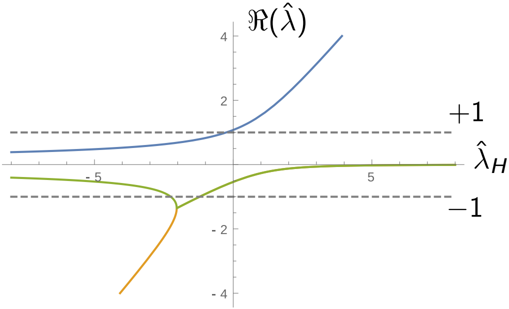
Solving the [ABC] eigenvalue equation (in its range of validity) for real values of $\hat{\lambda}$ and comparing it to the perturbative eigenvalue equation, a good agreement is obtained:



The maximum difference is for $\lambda = \epsilon^{1/3}$ (it is easy to check that when λ tends to $\epsilon^{1/3}$ the terms of the perturbative expansion have the same orders of magnitude, so the perturbative method is no longer available).

Growth rate

The perturbative method shows a stable branch of the solution that wasn't taken into account by the [ABC] solution:



(in the plot the dotted lines $\lambda = \pm\epsilon^{1/3}$ point out that for $|\lambda|$ approaching $\epsilon^{1/3}$ the perturbative solution is no longer available).

Conclusions

- The stable branch could be experimentally investigated in the future, to test the validity limits of the MHD;
- the knowledge of the stable branch in the linear theory is the basis for a nonlinear theory study.