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Mechanical analog of an over-damped Josephson junction

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Abstract

An over-damped pendulum can be adopted as a mechanical analogue of an over-damped Josephson junction (JJ). The basic equations leading to the driving torque versus the time average of the angular frequency are studied. The mechanical analogue can be used to provide additional insight into the current–voltage characteristics of over-damped JJs.

Keywords: Newtonian mechanics, simple pendulum, Josephson junction

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1973 Josephson received the Nobel Prize for having predicted the so called dc and ac Josephson effects [1] in a superconducting device that was named after him: the Josephson junction (JJ). A JJ consists of two weakly coupled superconductors. The dynamics of the superconducting phase difference ϕ across the junction is described by the following equations [2]:

$$I = I_J \sin\varphi, \tag{1a}$$

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar}V, \tag{1b}$$

where I is the current flowing through the junction (I_J being the maximum value that can flow in the zero-voltage state), $\hbar = h/2\pi$, h being Planck's constant, and V is the voltage across the two superconductors. In the dc Josephson effect a non-dissipative current can be seen to flow at zero voltage, as it can be shown by setting $V = 0$ in (1b), so that $\phi = \text{constant}$. In this way, I_J represents the maximum value of I flowing in the junction in the zero-voltage state. In the ac Josephson effect, the voltage across the JJ is kept at a fixed non-zero value V_0. Integrating both sides of equation (1*b*) we obtain $\phi(t) = (2e/\hbar)V_0\ t + \phi_0$, where ϕ_0 is the constant of integration. Therefore the current I is seen to oscillate at a frequency $\omega_J = (2e/\hbar)V_0$. Alternative derivations of equations (1*a*)–(1*b*) have been also proposed by Feynman [3] and by Ohta [4]. In the Feynman model a JJ is described as a weakly coupled two-level quantum system. Ohta noticed that Feynman model did not include an additional term due to energy contribution of the external classical circuit biasing the JJ. The latter author therefore introduced a semi-classical model based on a rigorous quantum derivation to attain full agreement between equations (1*a*)–(1*b*) and the corresponding final equations obtained by means of his valuable semi-classical analysis.

In order to describe the dynamics of the superconducting phase difference ϕ in an over-damped JJ, a resistively shunted junction (RSJ) model can be adopted [2]. In this model a purely superconducting element carrying a current I expressed in terms of ϕ as in equation (1*a*) is placed in parallel with a resistor of resistance R, as shown in figure 1. By injecting a current I_B in the system and by invoking charge conservation, we may write:

$$\frac{V}{R} + I_J \sin\varphi = I_B, \tag{2}$$

where V is the voltage across the JJ. By expressing V in terms of ϕ as in equation (1*b*) and by introducing the dimensionless quantities $i_B = I_B/I_J$ and $\tau = \frac{2\pi R I_J}{\Phi_0}t$ we may rewrite equation (2) as follows:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} + \sin\varphi = i_B. \tag{3}$$

The above equation also represents the dynamics of an over-damped simple pendulum. Therefore, starting from this analogy [2, 5], we consider the static and dynamic solutions of equation (3) referred to a simple pendulum with a constant forcing term, trying to grasp some physical insight from these expressions. Successively, we derive the curve of the driving torque versus the time average of the angular frequency. Finally, the analogy between the two systems is utilized to discuss the current-voltage characteristics of over-damped JJs.

2. An over-damped pendulum

Let us consider the pendulum hinged in O and consisting of a massless rod of length l and a spherical body of mass m, as shown in figure 2. Let us also assume that the sphere of mass m has radius R. This sphere is moving in a fluid of density ρ_F, so that it is subject to the buoyance force of intensity $F_B = \frac{4\pi R^3}{3} g\, \rho_F$, where g is the acceleration due to gravity. In addition, by assuming validity of Stoke's law, the sphere is taken to be subject to a viscous force, opposing its velocity and of intensity $F_S = 6\pi\, \eta\, R\, (l\ + R)\frac{\mathrm{d}\theta}{\mathrm{d}t}$, η being the coefficient of viscosity of the medium. The spherical body is also subject to the tension in the massless rod of length l and to its weight.

By taking moments about point O, we may write:

$$I_O \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -F_S(l\ + R) - m^* g(l + R)\sin\theta + M_0(t), \tag{4}$$

where I_O is the moment of inertia, $m^* = m\left(1 - \frac{4\pi R^3}{3m}\rho_F\right)$ is the effective mass of the sphere, when buoyancy is taken into account, and M_0 is the applied torque. Considering the finite dimensions of the sphere, the moment of inertia I_O can be calculated by means of the parallel axes theorem [6], so that:

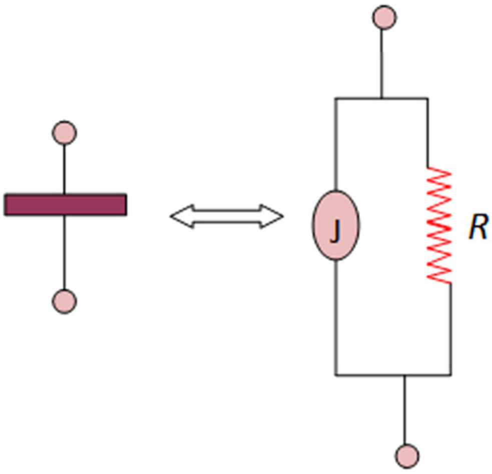

Figure 1. Resistively shunted model for a Josephson junction. The junction on the left is described by a parallel connection of a resistor with resistance R and an ideal Josephson element J. In the latter a current $I = I_J \sin\phi$ can flow.

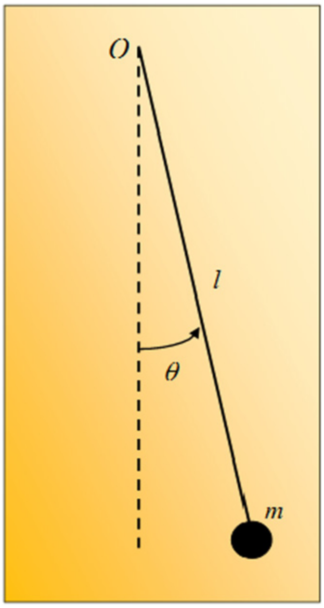

Figure 2. Schematic representation of a pendulum of mass m and length l displaced of an angle θ with respect to the vertical direction. Under certain conditions, the pendulum can be considered over-damped. This system realizes a mechanical analogue of an over-damped Josephson junction.

$$I_O = \frac{2}{5} m R^2 + m(l + R)^2. \tag{5}$$

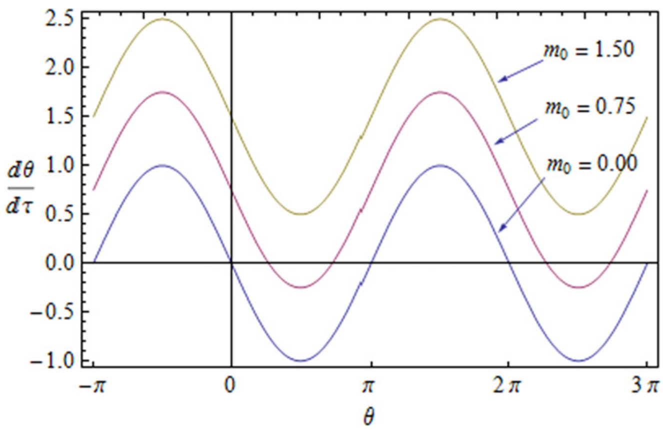

Figure 3. Phase-plane analysis for the over-damped pendulum. The constant forcing term is $m_0 = 0.0$ (bottom curve), $m_0 = 0.75$ (middle curve), and $m_0 = 1.50$ (top curve).

By dividing both members of equation (4) by $m^*g(l + R)$, by defining $\tau = \frac{m^* g}{6\pi\eta R(l+R)}t$ as a new dimensionless time variable, and by setting $m_0(\tau) = \frac{M_0(\tau)}{m^* g(l+R)}$, we may write

$$\frac{m^*m\, g\left(l^2 + 2Rl + \dfrac{7R^2}{5}\right)}{(6\pi\eta R)^2(l + R)^3}\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \sin\theta = m_0(\tau). \tag{6}$$

We immediately notice that equation (6) is equivalent to equation (3) for very small values of the pre-factor of the second derivative in equation (6); i.e., for

$$\frac{m^*m\, g\left(l^2 + 2Rl + \dfrac{7R^2}{5}\right)}{(6\pi\eta R)^2(l + R)^3} \ll 1.$$

In this way, the dynamical equation of an over-damped pendulum becomes formally equal to equation (3), reading:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \sin\theta = m_0(\tau). \tag{7}$$

Therefore, the analytic and experimental study of an over-damped pendulum allows us to derive important properties of an over-damped JJ. Naturally, in performing experimental studies, one needs to have negligible values of the pre-factor of the second derivative in equation (6). This can be obtained, for example, by using a fluid with high enough values of the coefficient of viscosity η. In the following sections we shall consider the forcing term as constant, obtaining a full analytic solution of the problem.

3. Constant driving moment

Let us take a constant forcing term of the over-damped pendulum in figure 2. In this case we can obtain analytic solutions for the differential equation (7). We start by noticing that, for $m_0 < 1$, we obtain two constant solutions, one stable, one unstable, as it can be argued by means of the phase-plane analysis shown in figure 3. The stable solution is given by

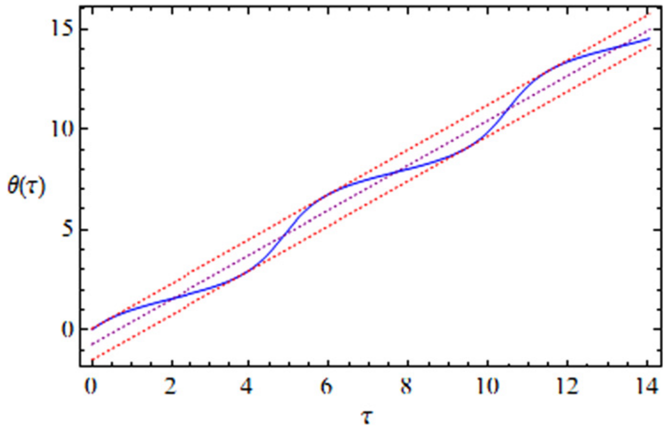

Figure 4. Normalized time dependence of the angular variable θ of an over-damped pendulum (full-line curve). The constant forcing term is $m_0 = 1.5$. The top and bottom dashed lines enclose. the undulatory behaviour of θ, whose oscillations take place about the middle dashed line.

$$\theta^* = \sin^{-1} m_0, \tag{8}$$

while the unstable solution is at $\theta = \pi - \theta^*$. The stability regime changes as the angle crosses the value $\theta = \frac{\pi}{2}$, as it can be noticed by analysing the sign of the derivative $\frac{\mathrm{d}\theta}{\mathrm{d}\tau}$ about these fixed points. For $m_0 = 1$ we have an half-stable solution: the pendulum may swirl around O whenever an arbitrary small positive perturbation arises. We may finally notice that, for $m_0 > 1$, the function $\theta = \theta(\tau)$ is monotonically increasing, given that the curves in figure 3 lie above the θ-axis and the derivative $\frac{\mathrm{d}\theta}{\mathrm{d}\tau}$ is always positive. In this 'running state' we solve the ordinary differential equation (7) by the method of separation of variables, as done in [2], by writing:

$$\int_{\theta_0}^{\theta(\tau)} \frac{\mathrm{d}\theta}{m_0 - \sin\theta} = \tau, \tag{9}$$

where $\theta_0 = \theta(0)$. By the substitution $x = \tan\frac{\theta}{2}$, we can write the integral in (9) as follows:

$$\frac{2}{m_0}\int_{\tan\frac{\theta_0}{2}}^{\tan\frac{\theta(\tau)}{2}} \frac{\mathrm{d}x}{x^2 + 1 - \dfrac{2}{m_0}x} = \frac{2}{m_0}\int_{\tan\frac{\theta_0}{2}}^{\tan\frac{\theta(\tau)}{2}} \frac{\mathrm{d}x}{\left(x - \dfrac{1}{m_0}\right)^2 + 1 - \dfrac{1}{{m_0}^2}}, \tag{10}$$

where we have completed the square in the denominator. The integral on the right-hand side of equation (9) can now be solved. By substituting this solution into (9) and by defining $\alpha_0 = \tan^{-1}\left[\frac{m_0}{\sqrt{{m_0}^2 - 1}}\left(\tan\frac{\theta_0}{2} - \frac{1}{m_0}\right)\right]$, we may write:

$$\frac{2}{\sqrt{{m_0}^2 - 1}}\left\{\tan^{-1}\left[\frac{m_0}{\sqrt{{m_0}^2 - 1}}\left(\tan\frac{\theta(\tau)}{2} - \frac{1}{m_0}\right)\right] - \alpha_0\right\} = \tau. \tag{11}$$

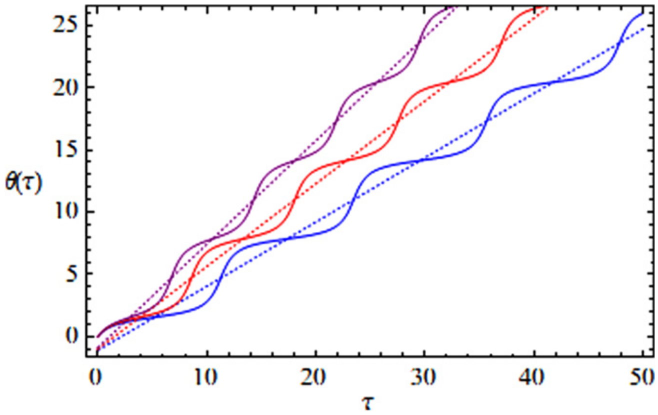

Figure 5. Normalized time dependence of the angular variable θ of an over-damped pendulum (full-line curves) represented together with the central dashed line about which oscillations take place. The constant forcing terms are as follows: $m_0 = 1.125$ (lower curve), $m_0 = 1.200$ (middle curve), $m_0 = 1.300$ (upper curve).

By finally extracting $\theta(\tau)$ from equation (11), we have:

$$\theta(\tau) = 2\tan^{-1}\left[\frac{1}{m_0} + \frac{\sqrt{{m_0}^2 - 1}}{m_0}\tan\left(\frac{\sqrt{{m_0}^2 - 1}}{2}\tau + \alpha_0\right)\right] + 2k\pi, \qquad (12)$$

where k is an integer. The above expression is represented in figure 4, as obtained from numerical solution of equation (7) with initial condition $\theta_0 = 0$, and for $m_0 = 1.5$. We notice that the numerical solution is easier to report on a graph, given the necessity of combining different pieces of the solution (12), one for each 2π shift of the angular variable $\theta(\tau)$. We also notice that this function oscillates within two lines of equation $\theta_{1,2}(\tau) = \sqrt{{m_0}^2 - 1}\ \tau + \gamma_{1,2}$, where $\gamma_{1,2} = \theta(\tau_{1,2}) - \sqrt{{m_0}^2 - 1}\ \tau_{1,2}$, $\tau_{1,2}$ being the times at which the lines are tangent to the oscillating curve $\theta(\tau)$ in the interval $[0, 2\pi]$. The quantities $\tau_{1,2}$ can be found by a straightforward, but rather cumbersome, calculation. Therefore, the curves of $\theta(\tau)$ are seen to oscillate about a central line $\theta_A(\tau) = \sqrt{{m_0}^2 - 1}\ \tau + \gamma_A$, whose intercept γ_A is the average value of γ_1 and γ_2. The solution of equation (7) is represented in figure 5, for various values of the constant forcing term m_0, along with the central lines $\theta_A(\tau)$ obtained by the procedure described above.

4. Time average of the angular frequency

Let us study the time average $\left\langle \frac{d\theta}{d\tau} \right\rangle$ of the angular frequency $\frac{d\theta}{d\tau}$ as a function of the constant forcing term m_0. This analysis is important, given that the m_0 versus $\left\langle \frac{d\theta}{d\tau} \right\rangle$ curves correspond to the normalized current i_B versus average voltage $\left\langle \frac{d\varphi}{d\tau} \right\rangle$ characteristics of an over-damped JJ.

We may start by considering the function $\frac{d\theta}{d\tau}$, represented in figure 6 for $m_0 = 1.5$ along with the value of the slope $\sqrt{{m_0}^2 - 1}$ of the central line running through the solution as seen in details in figure 4. This slope corresponds to the average value of the curve in figure 6, as we shall see below. We notice that this function is periodic with period equal to

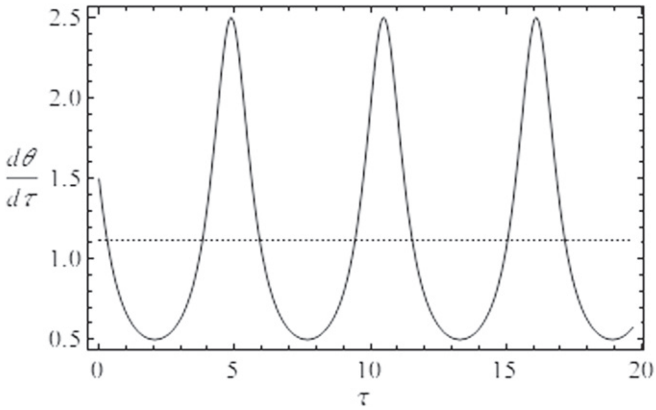

Figure 6. Normalized time dependence of the angular frequency (full line) of an over-damped pendulum subject to a constant forcing equal to $m_0 = 1.50$. Notice that the curve is periodic and the period is $T = 2\pi / \sqrt{{m_0}^2 - 1}$. The dashed line represents the slope $\sqrt{{m_0}^2 - 1}$ of the central dashed line of the primitive curve in figure 4 and, at the same time, the average value of the full-line curve.

$$T = \frac{2\pi}{\sqrt{{m_0}^2 - 1}} \tag{13}$$

as can be formally proven by calculating the derivative with respect to τ of $\theta(\tau)$ in equation (12). The slope of the central line can be written as $\frac{2\pi}{T}$. On the other hand, the time-averaged value of $\frac{\mathrm{d}\theta}{\mathrm{d}\tau}$ can be calculated as follows:

$$\left\langle \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\rangle = \frac{1}{T} \int_0^T \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \mathrm{d}\tau = \frac{\theta(T) - \theta(0)}{T} = \frac{2\pi}{T}, \tag{14}$$

so that it is proven that the average value of the angular frequency curves is $\sqrt{{m_0}^2 - 1}$. From equations (13) to (14) we can then argue that

$$m_0 = \sqrt{1 + \left\langle \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\rangle^2}. \tag{15}$$

The m_0 versus $\left\langle \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\rangle$ curve is represented in figure 7. We soon notice the role played by the static solution in equation (8). In fact, for $m_0 < 1$, the pendulum is in static equilibrium, so that $\left\langle \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\rangle = 0$. The same happens in a JJ: when the value of the normalized bias current i_B is less than one, the junction is said to be in the superconducting or zero-voltage state. Therefore, no current flows in the resistive branch of the RSJ model in figure 1, so that the curve climbs vertically from 0 to 1 just as shown in figure 7. However, when $i_B > 1$, the resistive branch is activated and a finite voltage appears across the junction, in the way described in figure 7. We also notice that the m_0 versus $\left\langle \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\rangle$ curve presents the oblique asymptote $m_0 = \left\langle \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\rangle$. In fact, for large enough values of m_0, this driving moment becomes predominant with respect to the nonlinear sine term in equation (7), thus justifying the observed asymptotic behaviour.

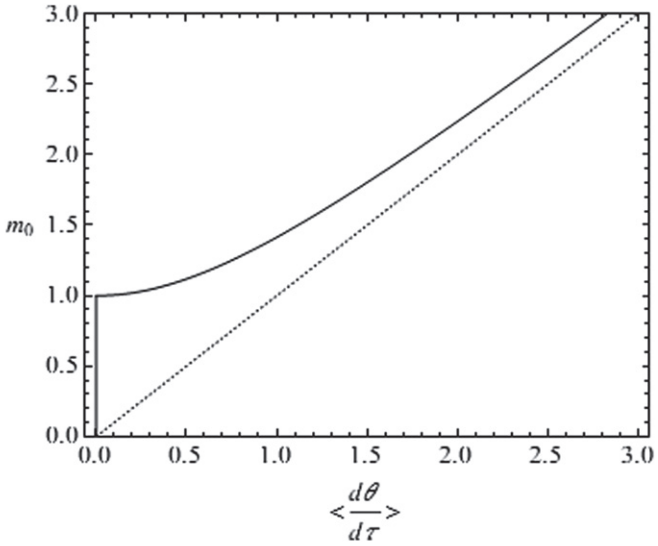

Figure 7. Normalized forcing term versus the time average of the angular frequency (full line) of an over-damped pendulum. Notice that the curve is anchored at null angular frequency if $m_0 \leqslant 1$. On the other hand, for $m_0 > 1$, the values of the curve tend towards the asymptote, given by the dashed line, as the abscissa increases.

5. The washboard potential

Very useful physics can be finally recovered by writing down the energy balance equation for the system. We start by noticing that energy is furnished from the externally applied moment at a rate $P_{ext} = M_0 \frac{d\theta}{dt}$. This energy is in part dissipated because of the presence of the viscous force $F_S = 6\pi\ \eta\ R\ (l + R)\ \frac{d\theta}{dt}$ at a rate $P_d = -6\pi\ \eta\ R(l + R)^2 \left(\frac{d\theta}{dt}\right)^2$, the minus sign meaning that energy is flowing out from the system. Therefore, the mechanical energy E_M, being the sum of the kinetic energy $\frac{1}{2} I_O \left(\frac{d\theta}{dt}\right)^2$ and of the potential energy $m^* g(l + R)(1 - \cos\theta)$, varies in time according to the following energy balance equation:

$$\frac{dE_M}{dt} = P_d + P_{ext}. \tag{16}$$

By explicitly writing down all terms, we have:

$$\frac{d}{dt}\left[\frac{1}{2} I_O \left(\frac{d\theta}{dt}\right)^2 + m^* g(l + R)(1 - \cos\theta) - M_0\theta\right] = -6\pi\ \eta\ R(l + R)^2 \left(\frac{d\theta}{dt}\right)^2, \tag{17}$$

where we have taken M_0 constant and have included the external forcing term under the derivative operator on the left-hand side. Of course, we can obtain the dynamical equation (4) from equation (17) by factoring out the angular frequency. However, we are here interested in highlighting the role of the forcing term in the system. Therefore, we consider a normalized effective potential $u_{\rm eff}$ defined as follows:

$$u_{\rm eff} = \frac{U_{\rm eff}}{m^* g(l + R)} = 1 - \cos\theta - m_0\theta. \tag{18}$$

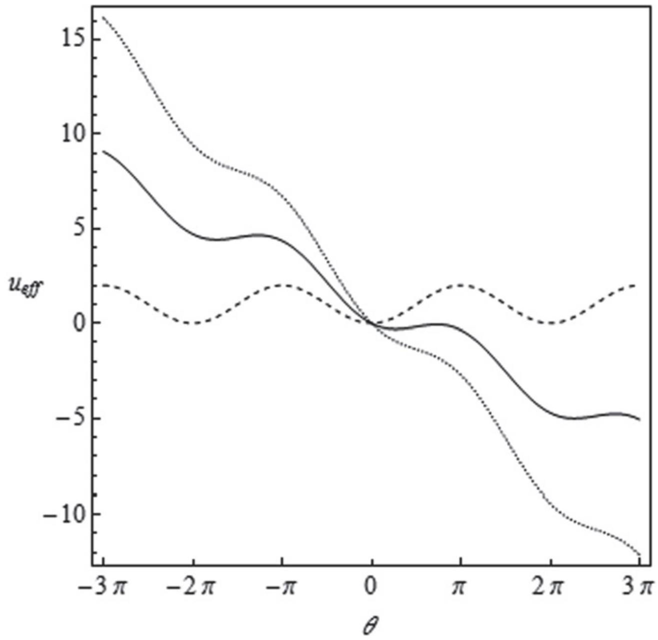

Figure 8. Normalized effective potential as a function of the angle θ for the following three values of the parameter m_0: 0.0 (dashed line); 0.75 (full line); 1.5 (dotted line). Notice that the parameter m_0 determines the degree of tilting and stretching of the undulating curves.

This normalized potential, called washboard potential because of its shape, is represented in figure 8 as a function of the variable θ and for various values of the parameter m_0. One word of comment on the origin of the term 'washboard potential' is in order. By looking at the tilted full-line curve in figure 8, we have the impression to see the board used by our great-grandmother to wash clothes, before the washing machine came into use. The above representation is useful, since it clarifies, once more, the crossover from static to dynamic solutions of the system. In fact, by looking at figure 8, we first notice that the parameter m_0 affects the degree of tilting and stretching of the washboard potential. This can be seen by starting from the dashed curve obtained for $m_0 = 0.0$ and by considering the remaining curves obtained for increasing values of this parameter. In the horizontal washboard all minima fall exactly at $2k\pi$, with k integer. The number of minima fitting in the graph shown in figure 8 are three. The same number of minima, though their abscissa are slightly displaced with respect to the above specified positions, are still present in the stretched and tilted curve for $m_0 = 0.75$ (full line in figure 8). Therefore, a point-like body could still be in static equilibrium in the angular positions corresponding to the minima and given by equation (8) in the interval $[0, 2\pi]$. Static equilibrium is not anymore possible for point-like particles on the washboard potential for $m_0 = 1.5$ (dotted curve), because of excessive tilting and stretching. This feature can be also derived analytically from equation (18), by just taking the derivative with respect to θ and by setting it to zero. Of course, this corresponds to finding the fixed point of the dynamical equation (7) and brings us to the same results as in section 3.

6. Conclusions

The properties of an over-damped JJ have been analysed by means of a mechanical analogue: an over-damped pendulum. The strict analogy between the dynamical equations of the two systems [2, 5] has been first reviewed. Being the physical properties of a simple pendulum more familiar to students, the JJ dynamics in the over-damped limit may be derived by analogy. Therefore, we have analysed some interesting features of an over-damped JJ by means of the corresponding physical properties of the over-damped pendulum. As an example, we have noticed that the current-voltage characteristics of the superconducting device can be obtained by means of an analytical expression derived for the normalized driving moment as a function of the time average of the angular frequency. Finally, by considering the energy balance equation for the system, we have seen that it is possible to describe the effect of the driving moment on the pendulum through the tilting and stretching of the washboard potential.

Apart from the analogy between the over-damped JJ and the over-damped pendulum, this work can be adopted as a lecture for first-year college physics students, in order to integrate the usual description of the pendulum made by means of the small oscillations approximation. In addition, starting from a mechanical system devised in such a way that the pre-factor of the second derivative in equation (6) is negligible, teachers may experiment on the effect of a constant applied torque on the pendulum, adding to direct observation the simple comment that similar response is expected in an over-damped JJ. In the future, experimental work based on the present analysis will be performed, after a careful fabrication of the mechanical system. Extension of the present analysis to non-constant applied torque will also be sought.

Acknowledgements

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