Isochronicity of mechanical oscillators: teaching approaches and educational impact

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A. Filipponi, *"A non-isochronous rocking oscillator."*, Am. J. Phys. **82**, 1142-1148 (2014).

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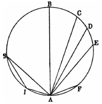
Introduction: definitions - history - mathematics

Definition: An oscillator is isochronous if the oscillation period is independent of the total mechanical energy E (and related oscillation amplitude A).

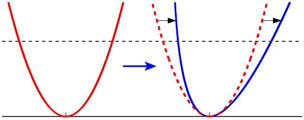
Harmonic oscillator: The point mass performs sinusoidal isochronous oscillations $x(t) = A\sin(\omega t + \phi)$, with constant $T = \frac{2\pi}{\omega}$ $\left(= 2\pi\sqrt{\frac{m}{k}}\right.$ for a harmonic spring-mass system).

Pendulum: G. Galilei believed it was isochronous (see the epistolary with Guidobaldo del Monte and the so called "Galileo's theorem") but he was wrong!



H.C. Huygens realized that if the point mass swings over a cycloid instead of a circle, the oscillator is actually isochronous. The potential energy is in this case quadratic in the curvilinear abscissa (Lagrangian coordinate).

Isochronicity theorem: Any one dimensional dynamical system with kinetic energy $E_K = \frac{1}{2}m\dot{x}^2$ and a potential energy $U(x)$ obtained by "shearing" a parabolic potential performs isochronous (in general not sinusoidal) oscillations.



Proof: By exploiting energy conservation, the oscillation period can be expressed as:

$$T(E) = 2\sqrt{\frac{m}{2}}\int_{x_a(E)}^{x_b(E)} \frac{dx}{\sqrt{E - U(x)}} = 2\sqrt{\frac{m}{2}}\int_0^1 \frac{d[x_b(U) - x_a(U)]}{dU}\sqrt{E}\frac{dy}{\sqrt{1-y}} \ .$$

The second alternative expression can be obtained by substitution of the integration variable with $y = U/E$ (the potential should display a single minimum at $U = 0$). The explicit E dependence is canceled if the distance between the motion inversion points grows like $x_b(E) - x_a(E) \propto \sqrt{E}$ (where E is measured from the minimum).

Educational remarks n.1

Exposing students to the harmonic oscillator (and to the pendulum in the $\sin(\theta) \simeq \theta$ limit) only, may induce the misconception that all oscillators are harmonic and isochronous!

$\Longrightarrow$ It is useful to propose experimental investigations of various types of oscillators: mass-spring, pendulum (also when $\sin(\theta) \neq \theta$), bouncing ping-pong ball, ... to realize that some of them are not isochronous.

$\Longrightarrow$ Using an analytical approach it is instructive to calculate the oscillator period for different potential wells (square, triangular, isotonic potential $U(x) = \frac{1}{2}ax^2 + \frac{b}{x^2}, \quad x > 0$, gravitational potential with centrifugal barrier $U(x) = -\frac{2}{x} + \frac{1}{x^2} + 1, \quad x > 0$, ...) by direct integration of:

$$T(E) = 2\sqrt{\frac{m}{2}} \int_{x_a(E)}^{x_b(E)} \frac{dx}{\sqrt{E - U(x)}}$$

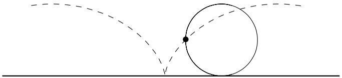
that is often elementary.

The unbalanced rocking cylinder

A very interesting mechanical oscillator is given by a cylindrical object with the center of mass displaced from the axis able to roll without slipping on an horizontal plane.

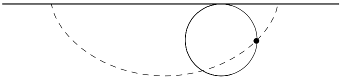
This system is the practical realization of the ideal system of a point mass fixed on a mass-less circle rolling on a horizontal line.

Rocking mass



$\neq$

Cycloidal pendulum



Two Italian groups worked independently on very similar systems.

A. Filipponi, *"A non-isochronous rocking oscillator."*, Am. J. Phys. **82**, 1142-1148 (2014).

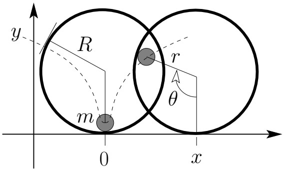
Pavia

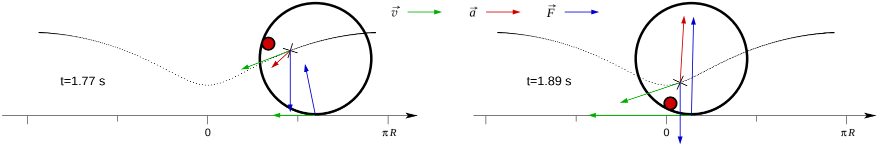
L'Aquila

The work from the Pavia group was published in:
P. Onorato, M. Malgieri, P. Mascheretti, and A. De Ambrosis, "The surprising rolling spool: experiments and theory from mechanics to phase transitions.", Eur. J. Phys. **35**, 055011 (2014).

The physical description at 1^{st} year level

The system has one degree of freedom described by a linear or angular coordinate $x = R\theta$. The external forces are the weight and the support force (i.e. the one required to satisfy the rolling constraint). A couple of instantaneous configurations during a wide amplitude oscillation with the relevant velocities $\vec{v}$, acceleration $\vec{a}$, and forces $\vec{F}$, are illustrated below:





Educational remarks n.2

In order to eliminate the effect of the "unknown" support force it is convenient to evaluate moments about the instantaneous contact point C, similarly to the rolling cylinder case, and use the equation:

$$\frac{d\vec{L}_C}{dt} = \boxed{-\vec{v}_C \times \vec{P}} + \vec{M}_{ext} \tag{1}$$

Here, however, the red term cannot be omitted since $\vec{v}_C$ and $\vec{v}_{cm}$ are not parallel! An additional complication is that the moment of inertia about the contact axis depends on the coordinate θ.

The physical description at 1^{st} year level (and beyond)

The moment of inertia about the contact axis is given by

$$I(\theta) = I_0 + MR^2 + I_m + m(R^2 + r^2 - 2rR\cos\theta) = I\left[1 + \frac{\alpha}{2}(1 - \cos\theta)\right]$$

where $I = I_0 + MR^2 + I_m + m(R-r)^2$ and $\alpha = 4mrR/I$ is a dimensionless parameter weighting the magnitude of its variable part.
The non trivial component of Eq. (1) provides the differential equation of motion:

$$\frac{d}{dt}\left[I(\theta)\dot{\theta}\right] = -mgr\sin\theta + mrR\dot{\theta}^2\sin\theta \quad \Leftrightarrow \quad I(\theta)\ddot{\theta} = -mgr\sin\theta - mrR\dot{\theta}^2\sin\theta \qquad (2)$$

The kinetic and potential energies are given by:

$$K(\theta,\dot{\theta}) = \frac{1}{2}I\dot{\theta}^2\left[1 + \frac{\alpha}{2}(1 - \cos\theta)\right] \quad \textit{and} \quad U(\theta) = mgr(1 - \cos\theta)$$

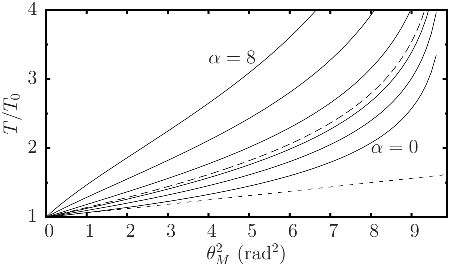
$U(\theta)$ is identical to the pendulum case which is retrieved in the $\alpha \to 0$ limit.
In the Lagrangian formalism Eq. (2) can be readily obtained by applying the Euler-Lagrange equation to $K - U$.
In the ideal conservative limit the total mechanical energy E is constant and equals the potential energy at the angle of maximum swing $U(\theta_M)$.
The resulting energy conservation equation is:

$$U(\theta) + K(\theta,\dot{\theta}) = mgr(1 - \cos\theta_M) = E$$

The period calculation

An integral expression for the oscillation period, suitable for numerical evaluation, can be obtained from energy conservation as:

$$T(\theta_M) = T_0 \, \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sqrt{1+\alpha\epsilon \sin^2(x)}}{\sqrt{1-\epsilon \sin^2(x)}} \, dx,$$

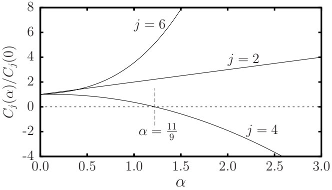


where $T_0 = T(0) = 2\pi\sqrt{\frac{I}{mgr}}$, $\epsilon = \sin^2\left(\frac{\theta_M}{2}\right) = \frac{1-\cos\theta_M}{2} = \frac{E}{2mgr}$ and $\alpha = \frac{4mrR}{I}$

Similarly to the pendulum ($\alpha = 0$) case, an analytical expansion for T in even powers of θ_M holds:

$$\frac{T}{T_0} = 1 + C_2\,\theta_M^2 + C_4\,\theta_M^4 + C_6\,\theta_M^6 + O(\theta_M^8)$$

The dependence of the coefficients on the parameter α is shown below:

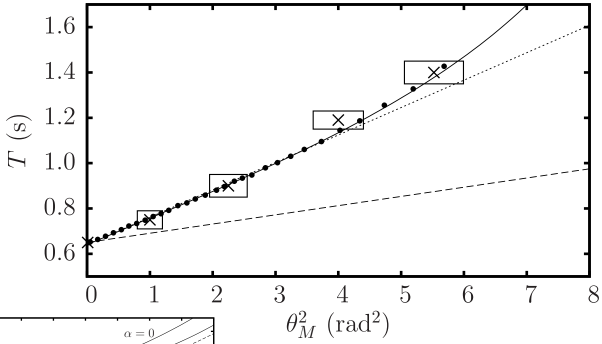


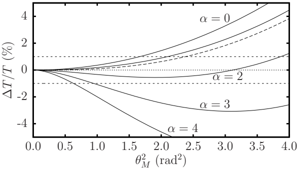
$$C_2 = \frac{1+\alpha}{16}, \quad C_4 = \frac{11+2\alpha-9\alpha^2}{3072},$$

$$C_6 = \frac{173+83\alpha+135\alpha^2+225\alpha^3}{737280}$$

The period measurements

The system investigated in L'Aquila has $T_0 \simeq 0.65$ s and $\alpha \simeq 2$. The coefficient C_2 is about 3 times that for the pendulum, therefore the $T(\theta_M)$ dependence can be easily detected also with manual measurements $\boxed{\times}$. Higher precision $T(\theta_M)$ measurements • performed with a computer acquisition method are in perfect agreement with the numerical integration of the analytical expression.





Notice also that for $\alpha \approx 2$ the deviation from a linear approximation in θ_M^2 is below 1% up to $\theta_M \simeq 2$ rad.

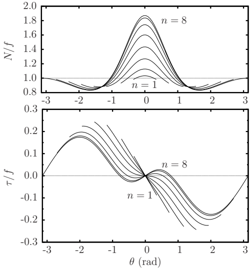
The support forces: theory and measurements

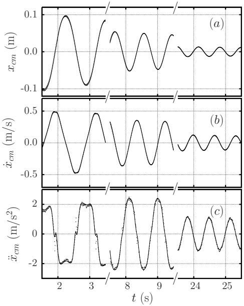
The reaction force $\vec{R}$ of the support plane and its normal N and horizontal τ components are obtained from Eq. (3):

$$\frac{d\vec{P}}{dt} = (M+m)\vec{g} + \vec{R} \qquad (3)$$

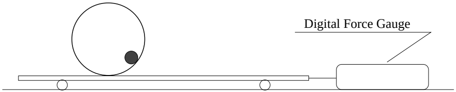
$$N = (M+m)(g+\ddot{y}_{cm}) = (M+m)g + mr(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

$$\tau = (M+m)\ddot{x}_{cm} = [(M+m)R - mr\cos\theta]\,\ddot{\theta} + mr\dot{\theta}^2\sin\theta$$





τ can be measured with a simple set-up using a digital force gauge. It is proportional to the center of mass horizontal acceleration. All other kinematic quantities are easily obtained by integration.

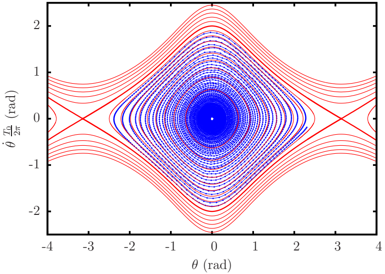


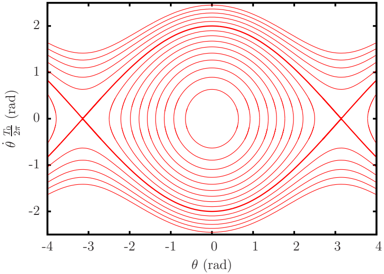
The phase space

A typical phase space plot corresponding to the constant energy curves

$$\frac{1}{4}\left(\frac{T_0}{2\pi}\right)^2 \dot{\theta}^2 \left[1 + \alpha\frac{1-\cos\theta}{2}\right] + \frac{1-\cos\theta}{2} = \epsilon$$

in dimensionless units (at $\epsilon = 0.1, 0.2, 0.3, \ldots, 1.0, \ldots 1.5$)
is reported below, together with experimental data (●).





The phase space for the pendulum ($\alpha = 0$)
is reported on the right hand side, for comparison.

Conclusions

- The unbalanced rocking cylinder is an interesting mechanical system with one degree of freedom able to perform non-isochronous oscillation.
- The $T(\theta_M^2)$ dependence can be tuned from the pendulum limit to a slope at least three times greater with an approximate (1%) linear dependence up to $\theta_M \approx 2$ rad. In contrast to the pendulum case no limitations in the swing angle are present and no additional support is needed.
- The system is well suited to perform both manual timing and high precision computer acquisition experiments.
- The system was proposed as a first year laboratory assessment: the students where challenged to investigate the system and tell if it was isochronous or not and (in this case) tell if it was similar to the investigated pendulum case.
- The physical description provides an instructive case in which the term $-\vec{v}_0 \times \vec{P} \neq 0$ cannot be omitted. The system was introduced as an example in General Physics courses and good students where able to discern cases in which $-\vec{v}_0 \times \vec{P} = 0$ or not.

Thank you for your attention!

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