Spectral analysis of singular, inhomogeneous, collisionless plasma structures

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Outline

Origins and motivation

- Vlasov linear oscillations of an 1D-inhomogeneous collisionless plasma having singular particle distributions;
- oscillation eigenvalue problem with none of the cold, “fluid”, or “kinetic” limits;
- example: wave equation in a collisionless electrostatic double layer at finite temperature.

Cornerstones

- Unfolding singular particle distributions into ordinary functions by a fully spectral analysis, i.e. also in velocity;
- “spectroscopy” of the eigenfunction of the Liouville operator: their eigenvalues and degeneracy;
- Green function of the Vlasov operator;
- wave equation for the continuum electrostatic oscillations in a hot collisionless, inhomogeneous plasma.

Known facts

- General theory: the eigenfunctions of the Liouville operator are distributions;
- they have a purely real continuum as well as a discrete spectrum[1];
- warm "fluid" limit: BGK waves and electron holes are unstable against electron electrostatic perturbations[2, 3, 4];
- cold limit: purely real continuum spectrum, continuum-damped resonance-absorbing surface waves (quasi-modes) [5, 6].

Highlights of our work

- In the velocity Fourier transform space the eigenfunctions of the Liouville operator are ordinary functions;
- they have two continuous and up to three discrete degeneracies;
- they are algebraically singular (although integrable);
- they are **complete**;
- the Green function of the Vlasov operator has a spectral representation;
- a judicious closure gives novel hot plasma contributions in the electrostatic wave equation.

Notation

$$\alpha = \mathrm{e}, \mathrm{i}, \; Z_\alpha = Q_\alpha / |Q_\mathrm{e}|, \; \mu_\alpha = \mu_\alpha / \mu_\mathrm{e} : \textit{particle species}, \tag{1}$$

$$\Phi(x) : \textit{equilibrium potential}, \tag{2}$$

$$-V_\mathrm{e} = Z_\mathrm{e}\Phi, \; -V_\mathrm{i} = Z_\mathrm{i}(\Phi - 1) : \textit{potential energies}, \tag{3}$$

$$\omega, \; k_y : \textit{frequency and transverse wavenumber}, \tag{4}$$

$$\mathbf{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix} : \textit{Fourier} - \textit{conjugate of velocity vector}. \tag{5}$$

Fields

$$|F\rangle = \begin{bmatrix} e^{-\beta_{\mathrm{e}}(q_x^2+q_y^2)/2-V_{\mathrm{e}}} \\ e^{-\beta_{\mathrm{i}}(q_x^2+q_y^2)/2-V_{\mathrm{i}}} \end{bmatrix} : \textit{equilibrium distribution}, \qquad (6)$$

$$|f_{k_y\omega}\rangle = \begin{bmatrix} f_{\mathrm{e}k_y\omega}(x, q_x, q_y) \\ f_{\mathrm{i}k_y\omega}(x, q_x, q_y) \end{bmatrix} : \textit{perturbed distribution}, \qquad (7)$$

$$\mathbf{e}_{k_y\omega} = \begin{bmatrix} e_{xk_y\omega}(x) \\ e_{yk_y\omega}(x) \end{bmatrix} : \textit{electric field}. \qquad (8)$$

Vlasov equation in Fourier transformed velocity space [7]

$$\overbrace{[(S_{k_y} - \omega)|f_{k_y\omega}\rangle]_\alpha}^{\text{Vlasov operator}} = -\frac{Z_\alpha}{\mu_\alpha} F_\alpha \overbrace{[q_x e_{xk_y\omega} + q_y e_{yk_y\omega}]}^{\text{electric fields}}, \quad (9)$$

$$S_{\alpha\alpha' k_y} = \underbrace{\left[\frac{\partial^2}{\partial x \partial q_x} - ik_y \frac{\partial}{\partial q_y} + q_x \frac{V'_\alpha}{\mu_\alpha}\right] \delta_{\alpha\alpha'}}_{\text{Liouville operator}}. \quad (10)$$

Electrostatic Maxwell equations

$$\overbrace{\nabla^2[-\omega^2 + \omega_{\mathrm{p}}^2] e_{xk_y\omega} - k_y^2[\omega_{\mathrm{p}}^2]' \int \mathrm{d}x e_{xk_y\omega}}^{\text{cold plasma terms (e.g. [6])}} = \overbrace{-\sum_\alpha [S^2_{\alpha\alpha k_y} f_{\alpha k_y \omega}]'_{\mathbf{q}=0}}^{\text{hot plasma terms}},$$

$${}' = \partial/\partial x, \quad \omega_{\mathrm{p}} = \sqrt{\sum_\alpha Z_\alpha F_\alpha|_{\mathbf{q}=0}/\mu_\alpha} : \textit{plasma frequency}. \quad (11)$$

Liouville eigenvalue problem

$$S_{k_y}|\chi_{k_y\sigma}\rangle = \sigma|\chi_{k_y\sigma}\rangle. \tag{12}$$

Eigenfunctions of the Liouville operator

$$\chi^{s_\alpha}_{\alpha\sigma\gamma_\alpha} = \frac{C_\alpha}{|u_{\alpha\gamma_\alpha}|} e^{-is_\alpha(\sigma-k_y c_\alpha)\xi_{\alpha\gamma_\alpha}+is_\alpha q_x|u_{\alpha\gamma_\alpha}|+iq_y c_\alpha}, \tag{13}$$

$$s_\alpha = \pm,\ u_{\alpha\gamma_\alpha}(x) = s_\alpha\sqrt{\{2[\gamma_\alpha + V_\alpha(x)]/\mu_\alpha\}},$$

$$\xi_{\alpha\gamma_\alpha}(x) = \int \mathrm{d}x/|u_{\alpha\gamma_\alpha}(x)|,$$

$\alpha = \mathrm{e}, \mathrm{i}, s_\alpha = \pm$: *discrete real degeneracy parameters*,

$$\left.\begin{array}{l} -\infty < c_\alpha < \infty \\ -\max(V_\alpha) < \gamma_\alpha < \infty \end{array}\right\} : \text{ continuous real degeneracy parameters}, \tag{14}$$

$0 < \gamma_\alpha < \infty : -\infty < \sigma < \infty$: *continuous spectrum*,

$-\max(V_\alpha) < \gamma_\alpha < 0 : \sigma = n\omega_\mathrm{b}$: *discrete spectrum*.

Liouville-Bloch eigenfunctions in a periodic potential

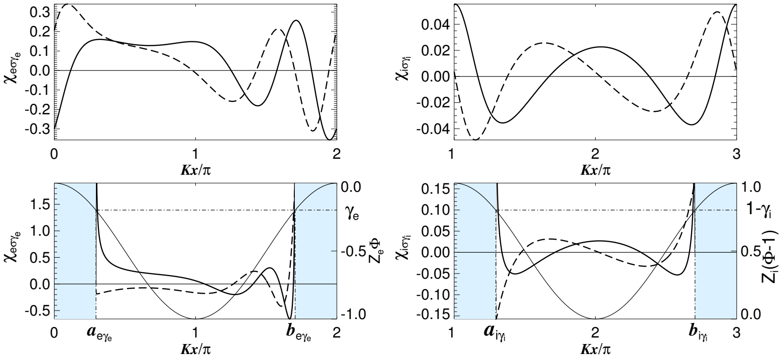

Figure 1 : Real (continuous line) and imaginary (dashed line) parts of the free electron (top left), free ion (top right), trapped electron (bottom left) and trapped ion (bottom right) eigenfunctions of the Liouville operator in a periodic potential. Particle motion is forbidden in the shaded areas. Note the *algebraic singularity* of the eigenfunctions at the reflection points.

Liouville eigenfunctions in a double layer

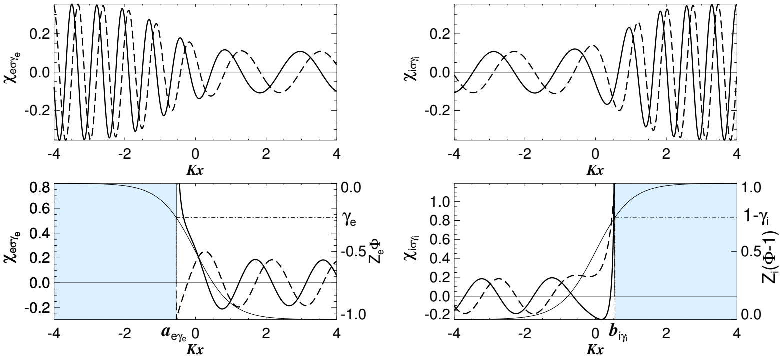

Figure 2 : same as in Fig. 1 for the eigenfunctions of the Liouville operator in a double layer potential.

Liouville eigenfunctions in an electron hole

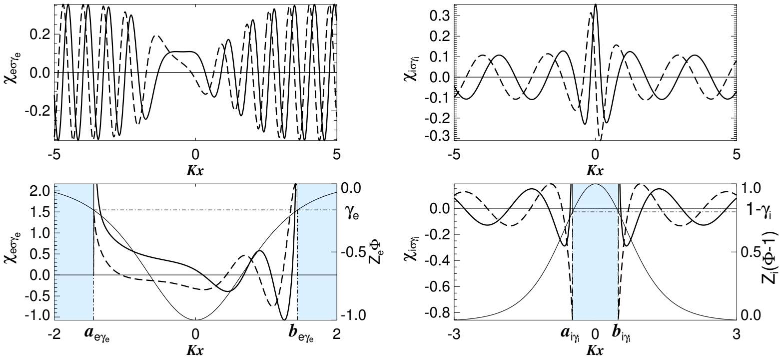

Figure 3 : same as in Fig. 1 for the eigenfunctions of the Liouville operator in a electron hole potential.

Liouville eigenfunctions in a non nonotonic ion hole

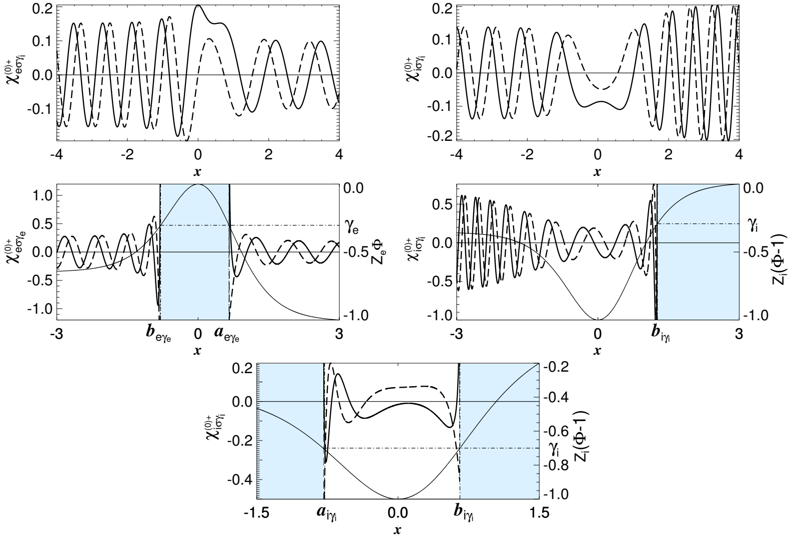

Figure 4 : same as in Fig. 1 for the eigenfunctions of the Liouville operator in an asymmetric ion hole.

Green function of the Vlasov operator

$$G_{\alpha\alpha' k_y \omega}(x, q_x, q_y; s, p_x, p_y) =$$

$$\left. \begin{array}{l} \delta_{\alpha\alpha'} \sum_{s_\alpha=\pm} \int_{-\infty}^{\infty} \mathrm{d}c_\alpha \int_{-V_\alpha(s)}^{\infty} \mathrm{d}\gamma_\alpha \int_{-\infty}^{\infty} \frac{A_\alpha^{s_\alpha}\,\mathrm{d}\sigma}{\sigma-\omega+is_\alpha 0^+} \times \\ \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y) \bar{\chi}^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(s, p_x, p_y) + \end{array} \right\} x > s$$

$$\left. \begin{array}{l} \delta_{\alpha\alpha'} \sum_{s_\alpha=\pm} \int_{-\infty}^{\infty} \mathrm{d}c_\alpha \int_{-V_\alpha(x)}^{\infty} \mathrm{d}\gamma_\alpha \int_{-\infty}^{\infty} \frac{B_\alpha^{s_\alpha}\,\mathrm{d}\sigma}{\sigma-\omega+is_\alpha 0^+} \times \\ \bar{\chi}^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y) \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(s, p_x, p_y), \sum_{s_\alpha=\pm} \end{array} \right\} x < s \qquad (15)$$

Scattering amplitudes (related to boundary conditions)

$$\overbrace{A_\alpha^+ + A_\alpha^- + B_\alpha^+ + B_\alpha^- = 1}^{\text{normalization}}, \; \overbrace{A_\alpha^+ - A_\alpha^- - (B_\alpha^+ - B_\alpha^-) = 0}^{\text{conservation}}, \qquad (16)$$

$$[S_{\alpha\alpha k_y} - \omega] G_{\alpha\alpha' k_y} = \delta_{\alpha\alpha'} \delta(x-s) \delta(q_x - p_x) \delta(q_y - p_y). \qquad (17)$$

Initial value theorem for Laplace transforms

$$(i\partial/\partial t + S_{k_y})|f\rangle = 0,\quad f_\alpha(x,q_x,q_y,0) = -ih_\alpha(x,q_x,q_y), \tag{18}$$

$$f_\alpha(x,q_x,q_y,0^+) = \lim_{\omega\to -i\infty} i\omega \underbrace{g_{\alpha\omega}(x,q_x,q_y)}_{\text{Laplace transform of } f_\alpha}, \tag{19}$$

$$(S_{k_y} - \omega)|g_\omega\rangle = |h\rangle. \tag{20}$$

⬇

Expansion of an arbitrary function ⇒ COMPLETENESS

$$\overbrace{h_\alpha(s,q_x,q_y)}^{\text{arbitrary function}} = \overbrace{\sum_{s_\alpha=\pm}\int_{-\infty}^{\infty}\mathrm{d}c_\alpha \int_{-V_\alpha(x)}^{\infty}\mathrm{d}\gamma_\alpha}^{\text{superposition}}$$

$$\underbrace{(A_\alpha^{s_\alpha} + B_\alpha^{-s_\alpha})}_{\text{arbitrary coefficients}} \underbrace{\langle h|\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}\rangle_{x,q_x,q_y}}_{\text{scalar product}} \underbrace{\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(x,q_x,q_y)}_{\text{eigenfunction}}. \tag{21}$$

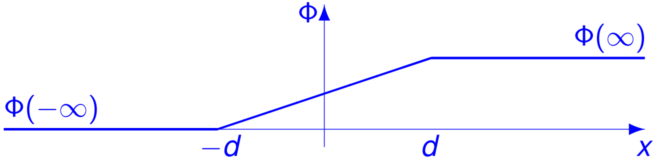

Figure 5 : Piecewise continuous model equilibrium potential

Summation of the Green function (not $\omega \gg S_{k_y}$ or $\omega \ll S_{k_y}$)

$$G_{\alpha\alpha' k_y \omega}(x, q_x, q_y; s, p_x, p_y) =$$

$$\left.\frac{1}{2\pi}\delta_{\alpha\alpha'} \sum_{s_\alpha=\pm} s_\alpha \frac{|k_y|}{|p_y|} A_\alpha^{s_\alpha} e^{-i\omega p_y/k_y - i s_\alpha p_x |k_y|(x-s)/|p_y|} \right\} \begin{array}{l} x > s \\ s_\alpha p_y > 0 \end{array} \tag{22}$$

$$\left.\frac{1}{2\pi}\delta_{\alpha\alpha'} \sum_{s_\alpha=\pm} s_\alpha \frac{|k_y|}{|p_y|} B_\alpha^{s_\alpha} e^{i\omega p_y/k_y + i s_\alpha p_x |k_y|(x-s)/|p_y|} \right\} \begin{array}{l} x < s \\ s_\alpha p_y < 0 \end{array}$$

Asymptotic analysis: hot plasma ($p_y \simeq 1/\sqrt{\beta_{\mathrm{e,i}}} \ll 1$)

$$F_\alpha = e^{-\beta_\alpha (q_x^2+q_y^2)/2} e^{-V_\alpha(x)}, \tag{23}$$

$$\theta = \sum_\alpha \frac{1}{\beta_\alpha}, \tag{24}$$

$$\omega_{\mathrm{p}} = \sqrt{\sum_\alpha Z_\alpha F_\alpha|_{\mathbf{q}=0}/\mu_\alpha} : \textit{plasma frequency}. \tag{25}$$

Electrostatic Maxwell equations

$$\nabla^2[-\omega^2 + \omega_{\mathrm{p}}^2(1+\theta)] e_{xk_y\omega} = k_y^2[\omega_{\mathrm{p}}^2]' \int \mathrm{d}x e_{xk_y\omega} + \theta\left[k_y^2[\omega_{\mathrm{p}}^2]' - \omega^2\omega_{\mathrm{p}}^2 \frac{\partial}{\partial x}\right] e_{xk_y\omega}. \tag{26}$$

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