An asymptotic method for high-frequency waves in a tokamak plasma

L. Guidi, B. Tirozzi, A. Cardinali

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An asymptotic method for high-frequency waves in a tokamak plasma

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Main subject: application of suitable asymptotic methods to the equation which describes the propagation of a high-frequency electromagnetic wave in a tokamak plasma.
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Asymptotics for high-frequency waves in a plasma

Let's consider the following set of equations:

\[ \frac{\partial^2 E}{\partial t^2} + c^2 \nabla \times (\nabla \times E) + \frac{4\pi}{c^2} \sum_{i=1,2} q_i n_i \frac{\partial v_i}{\partial t} = 0, \]  
\[ \frac{\partial v_i}{\partial t} = \frac{q_i}{m_i} (E + \frac{1}{c} v_i \times B), \quad i = 1, 2 \]
Reduction of the equation

A Fourier analysis in time of the type \( \mathbf{v}_i = e^{i\omega t}\tilde{\mathbf{v}}_i, \mathbf{E} = e^{i\omega t}\tilde{\mathbf{E}} \) allows (after calculation) to express the velocities in terms of the electric field:

\[
\mathbf{v}_i(\mathbf{r}, t) = \frac{q_1}{m_1} \mathbf{C}_i \mathbf{E}
\]

where

\[
\mathbf{C}_i = \frac{1}{|\mathbf{F}_i|} \begin{pmatrix}
-\omega^2 + \Omega_i^2 b_1^2 & \Omega_i^2 b_1 b_2 + i\omega\Omega_i b_3 & \Omega_i^2 b_1 b_3 - i\omega\Omega_i b_2 \\
\Omega_i^2 b_1 b_2 - i\omega\Omega_i b_3 & -\omega^2 + \Omega_i^2 b_2^2 & \Omega_i^2 b_2 b_3 + i\omega\Omega_i b_1 \\
\Omega_i^2 b_1 b_3 + i\omega\Omega_i b_2 & \Omega_i^2 b_2 b_3 - i\omega\Omega_i b_1 & -\omega^2 + \Omega_i^2 b_3^2
\end{pmatrix},
\]

and \( |\mathbf{F}_i| = \det(i\omega \mathbf{I} - \mathbf{A}_i) = -i\omega^3 + i\omega\Omega_i^2 \).
Defining $\omega_{pi}^2 = 4\pi \frac{q_i^2 n_i}{m_i}$, and then $D := \sum_{i=1,2} \omega_{pi}^2 C_i$, our starting equation becomes

$$-\omega^2 E + c^2 \nabla \times (\nabla \times E) + i\omega D E = 0.$$  

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$$-\omega^2 E + c^2 \nabla \times (\nabla \times E) + i\omega DE = 0. \quad (3)$$

We consider waves satisfying the condition

$$\omega \gg \Omega_i, \quad i = 1, 2.$$ 

Under this hypothesis, we have

$$C_i \approx diag(-i\omega^{-1}),$$

so that

$$i\omega D(\omega) = i\omega \sum_{i=1,2} \omega_{pi}^2 C_i(\omega) = \omega_{pi}^2 I,$$

where $I$ is the identity matrix.
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$$i\omega D(\omega) = i\omega \sum_{i=1,2} \omega_{pi}^2 C_i(\omega) = \omega_p^2 I,$$

where $I$ is the identity matrix.

The equation for the electric field is then simplified:

$$(\omega_p^2 - \omega^2) E + c^2 \nabla \times (\nabla \times E) = 0. \quad (4)$$
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Asymptotic method

We seek for solutions of (4) in the form

$$E = e^{i\omega t + i\tau(M)} \sum_{j=0}^{\infty} \frac{U_j(M)}{(-i\omega)^j}.$$ 

Inserting this expression in (4), according to the vector identity

$$\nabla \times (\nabla \times E) = -\Delta E + \nabla \cdot (\nabla \cdot E),$$

we separate the terms corresponding to different powers of the parameter $\omega$; we get:

- for the phase:

$$\left(c^2|\nabla \tau|^2 - 1\right)U_j - c^2 \nabla \tau (U_j \cdot \nabla \tau) = 0; \quad (5)$$
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Inserting this expression in (4), according to the vector identity \( \nabla \times (\nabla \times E) = -\Delta E + \nabla \cdot (\nabla \cdot E) \), we separate the terms corresponding to different powers of the parameter \( \omega \); we get:

- for the phase:
  \[ (c^2|\nabla \tau|^2 - 1) U_j - c^2 \nabla \tau (U_j \cdot \nabla \tau) = 0; \tag{5} \]

- for the amplitudes \( U_j \):
  \[ 2(\nabla \tau \cdot \nabla) U_j + U_j \Delta \tau - \nabla \tau (\nabla \cdot U_j) - \nabla (\nabla \tau \cdot U_j) - \Delta U_{j-1} + \nabla (\nabla \cdot U_{j-1}) + \frac{\omega_p^2}{c^2} U_{j-1} \right] = 0, \tag{6} \]

where \( U_{-1} = 0 \).
Solution

Equation (5) reduces, after some calculation, to the *eikonal* equation

$$|\nabla \tau|^2 - \frac{1}{c^2} = 0. \quad (7)$$

which can be interpreted as a Hamilton-Jacobi equation. We obtain (solving the associated Hamilton equations system) that the trajectories (rays) are straight lines.
Solution

- Equation (5) reduces, after some calculation, to the *eikonal* equation

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- The recurrent set of equations described in (6) leads to

\[
U_0(\alpha, \beta, \tau) = U_0(\alpha, \beta, \tau_0)
\]

\[
U_j^{(k)} = \sqrt{c}U_0^{(k)} \left[ \psi_j^{(k)}(\alpha, \beta) + \right. \\
+ \left. \int_0^\tau \sqrt{\frac{1}{c} \frac{c^2}{2}} \left( \Delta U_j^{(k)} - \partial_k (\nabla \cdot U_{j-1}) - \frac{\omega_p^2}{c^2} U_j^{(k)} + (U_0^\perp)^{(k)} \right) d\tau \right]
\]
Dispersion relation for high-frequency waves

If we compute the dispersion relation associated to Eq. (4) we get

$$-\omega^6 + (2c^2 k^2 + 3\omega_p^2) \omega^4 - (c^4 k^4 - 4c^2 k^2 \omega_p^2 - 3\omega_p^4) \omega^2 + \omega_p^6 + 2c^2 k^2 \omega_p^4 + c^4 k^4 \omega_p^2 = 0.$$
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(8)

The solutions (for \(\omega\)) are

\[\omega_1^2 = \omega_3^2 = c^2 k^2 + \omega_p^2, \quad \omega_2^2 = \omega_p^2,\]

so that we have two positive real solutions:

\[\omega_1 = \sqrt{c^2 k^2 + \omega_p^2}, \quad \omega_2 = \omega_p.\]  

(9)
We choose the coordinate system \((r, \varphi, \vartheta)\) on the torus of major radius \(R_0\) and minor radius \(a\), where

- \(r\) is the coordinate on the minor radius;
- \(\varphi\) is the toroidal angle;
- \(\vartheta\) is the poloidal angle.
Let $\mathbf{P} = (P_r, P_\varphi, P_\theta)$ be the wave vector expressed in the new coordinates. Our dispersion relation takes the form

$$\omega = \sqrt{c^2 P^2 + \omega_p^2},$$  \hspace{1cm} (10)$$

where

$$P^2 = P_\parallel^2 + P_\perp^2.$$
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Hamiltonian analysis

The previous formulas give us a dependence

$$\omega = \omega(r, \vartheta, P_r, P_\varphi, P_\vartheta).$$

We consider this function as a Hamiltonian and we study the Hamilton-Jacobi equations associated to it:

$$\begin{align*}
\dot{r}(t) &= \frac{\partial \omega_1}{\partial P_r}, \\
\dot{\vartheta}(t) &= \frac{\partial \omega_1}{\partial P_\vartheta}, \\
\dot{P}_r(t) &= -\frac{\partial \omega_1}{\partial r}, \\
\dot{P}_\varphi(t) &= 0, \\
\dot{P}_\vartheta(t) &= -\frac{\partial \omega_1}{\partial \vartheta}, \\
r(0) &= r_0, \quad \vartheta(0) = \vartheta_0, \\
P_r(0) &= P_{r,0}, \quad P_\varphi(0) = P_{\varphi,0}, \quad P_\vartheta(0) &= P_{\vartheta,0}.
\end{align*}$$

(11)
The equations set can be solved using Mathematica, and it gives the time-dependency of each variable. We are interested on the wave fronts dynamics in 2-dimensional section of the torus, given by the parametrization

\[
\begin{align*}
    x(t) &= r(t) \cos(\vartheta(t)) \\
    y(t) &= r(t) \sin(\vartheta(t)),
\end{align*}
\]  

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  x(t) &= r(t) \cos(\vartheta(t)) \\
  y(t) &= r(t) \sin(\vartheta(t)),
\end{align*}
\]

(12)

The single coordinates evolution looks like this:

Figure: Plots of \(r(t)\) (above) and \(\vartheta(t)\) (below).
The propagation is along straight lines, and due to some cutoff regions (where the wave vector vanishes) we get reflections and the trajectory is deviated.

**Figure:** *Evolution of a wave front starting at $r_0 = a$, $\theta_0 = \pi/6$.***
A closer view of the region around the origin reveals the formation of a circular caustic:
Further studies

The results shown can hopefully be extended in different directions:
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- Construction of the solution nearby the border of the domain and nearby the caustic (Lagrangian manifold);
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- Explicit calculation of some terms of the expansion;
- Construction of the solution nearby the boarder of the domain and nearby the caustic (Lagrangian manifold);
- Generalization of the result to a more complete range of frequencies;
Further studies

The results shown can hopefully be extended in different directions:

- Explicit calculation of some terms of the expansion;
- Construction of the solution nearby the boarder of the domain and nearby the caustic (Lagrangian manifold);
- Generalization of the result to a more complete range of frequencies;
- Estimates of accuracy of the asymptotic solution.
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Thanks for the attention!