An asymptotic method for high-frequency waves in a tokamak plasma

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Main subject: application of suitable asymptotic methods to the equation which describes the propagation of a high-frequency electromagnetic wave in a tokamak plasma.

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Asymptotics for high-frequency waves in a plasma

Let's consider the following set of equations:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + c^2 \nabla \times (\nabla \times \mathbf{E}) + \frac{4\pi}{c^2} \sum_{i=1,2} q_i n_i \frac{\partial \mathbf{v}_i}{\partial t} = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}_i}{\partial t} = \frac{q_i}{m_i}(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B}), \qquad i = 1, 2 \tag{2}$$

Reduction of the equation

A Fourier analysis in time of the type $\mathbf{v}_i = e^{i\omega t}\tilde{\mathbf{v}}_i$, $\mathbf{E} = e^{i\omega t}\tilde{\mathbf{E}}$ allows (after calculation) to express the velocities in terms of the electric field:

$$\mathbf{v}_i(\mathbf{r}, t) = \frac{q_1}{m_1}\underline{\mathbf{C}}_i\mathbf{E}$$

where

$$\underline{\mathbf{C}}_i = \frac{1}{|\mathbf{F}_i|}\begin{pmatrix} -\omega^2 + \Omega_i^2 b_1^2 & \Omega_i^2 b_1 b_2 + i\omega\Omega_i b_3 & \Omega_i^2 b_1 b_3 - i\omega\Omega_i b_2 \\ \Omega_i^2 b_1 b_2 - i\omega\Omega_i b_3 & -\omega^2 + \Omega_i^2 b_2^2 & \Omega_i^2 b_2 b_3 + i\omega\Omega_i b_1 \\ \Omega_i^2 b_1 b_3 + i\omega\Omega_i b_2 & \Omega_i^2 b_2 b_3 - i\omega\Omega_i b_1 & -\omega^2 + \Omega_i^2 b_3^2 \end{pmatrix},$$

and $|\mathbf{F}_i| = \det(i\omega\underline{\mathbf{I}} - \underline{\mathbf{A}}_i) = -i\omega^3 + i\omega\Omega_i^2$.

- Defining $\omega_{pi}^2 = 4\pi \frac{q_i^2 n_i}{m_i}$, and then $\underline{\mathbf{D}} := \sum_{i=1,2} \omega_{pi}^2 \underline{\mathbf{C}}_i$, our starting equation becomes

$$-\omega^2 \mathbf{E} + c^2 \nabla \times (\nabla \times \mathbf{E}) + \mathrm{i}\omega \underline{\mathbf{D}}\mathbf{E} = 0. \quad (3)$$

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- We consider waves satisfying the condition

$$\omega \gg \Omega_i, \qquad i = 1, 2.$$

Under this hypothesis, we have

$$\underline{\mathbf{C}}_i \approx diag(-\mathrm{i}\omega^{-1}),$$

so that

$$\mathrm{i}\omega \underline{\mathbf{D}}(\omega) = \mathrm{i}\omega \sum_{i=1,2} \omega_{pi}^2 \underline{\mathbf{C}}_i(\omega) = \omega_p^2 \underline{\mathbf{I}},$$

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- The equation for the electric field is then simplified:

$$(\omega_p^2 - \omega^2)\mathbf{E} + c^2 \nabla \times (\nabla \times \mathbf{E}) = 0. \quad (4)$$

Asymptotic method

We seek for solutions of (4) in the form

$$\mathbf{E} = \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\omega\tau(M)} \sum_{j=0}^{\infty} \frac{\mathbf{U}_j(M)}{(-\mathrm{i}\omega)^j}.$$

Inserting this expression in (4), according to the vector identity $\nabla \times (\nabla \times \mathbf{E}) = -\Delta\mathbf{E} + \nabla \cdot (\nabla \cdot \mathbf{E})$, we separate the terms corresponding to different powers of the parameter ω; we get:

- for the phase:

$$(c^2|\nabla\tau|^2 - 1)\mathbf{U}_j - c^2\nabla\tau(\mathbf{U}_j \cdot \nabla\tau) = 0; \tag{5}$$

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- for the amplitudes $\mathbf{U}_j$:

$$\Big[2(\nabla\tau \cdot \nabla)\mathbf{U}_j + \mathbf{U}_j\Delta\tau - \nabla\tau(\nabla \cdot \mathbf{U}_j) - \nabla(\nabla\tau \cdot \mathbf{U}_j) - \Delta\mathbf{U}_{j-1} + \nabla(\nabla \cdot \mathbf{U}_{j-1}) + \frac{\omega_p^2}{c^2}\mathbf{U}_{j-1}\Big] = 0, \tag{6}$$

where $\mathbf{U}_{-1} = 0$.

Solution

- Equation (5) reduces, after some calculation, to the *eikonal* equation

$$|\nabla \tau|^2 - \frac{1}{c^2} = 0. \tag{7}$$

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- The recurrent set of equations described in (6) leads to

$$\mathbf{U}_0(\alpha,\beta,\tau) = \mathbf{U}_0(\alpha,\beta,\tau_0)$$
$$\mathbf{U}_j^{(k)} = \sqrt{c}\,\mathbf{U}_0^{(k)}\left[\bar{\psi}_j^{(k)}(\alpha,\beta) + \right.$$
$$\left. + \int_0^\tau \sqrt{\frac{1}{c}}\frac{c^2}{2}\left(\Delta\mathbf{U}_{j-1}^{(k)} - \partial_k(\nabla\cdot\mathbf{U}_{j-1}) - \frac{\omega_p^2}{c^2}\mathbf{U}_{j-1}^{(k)} + (\mathbf{U}_0^{\perp})^{(k)}\right)\right.$$

Dispersion relation for high-frequency waves

- If we compute the dispersion relation associated to Eq. (4) we get

$$-\omega^6 + \left(2c^2k^2 + 3\omega_p^2\right)\omega^4 - \left(c^4k^4 - 4c^2k^2\omega_p^2 - 3\omega_p^4\right)\omega^2 + \omega_p^6 + 2c^2k^2\omega_p^4 + c^4k^4\omega_p^2 = 0. \tag{8}$$

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- The solutions (for ω) are

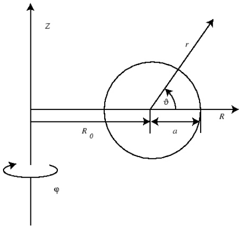
$$\omega_1^2 = \omega_3^2 = c^2k^2 + \omega_p^2, \qquad \omega_2^2 = \omega_p^2,$$

so that we have two positive real solutions:

$$\omega_1 = \sqrt{c^2k^2 + \omega_p^2}, \qquad \omega_2 = \omega_p. \tag{9}$$

We choose the coordinate system (r, φ, ϑ) on the torus of major radius R_0 and minor radius a, where

- r is the coordinate on the minor radius;
- φ is the toroidal angle;
- ϑ is the poloidal angle.



- Let $\mathbf{P} = (P_r, P_\varphi, P_\vartheta)$ be the wave vector expressed in the new coordinates. Our dispersion relation takes the form

$$\omega = \sqrt{c^2 P^2 + \omega_p^2}, \tag{10}$$

where

$$P^2 = P_\parallel^2 + P_\perp^2.$$

Hamiltonian analysis

The previous formulas give us a dependence

$$\omega = \omega(r, \vartheta, P_r, P_\varphi, P_\vartheta).$$

We consider this function as a Hamiltonian and we study the Hamilton-Jacobi equations associated to it:

$$\begin{cases} \dot{r}(t) = \frac{\partial \omega_1}{\partial P_r} \\ \dot{\vartheta}(t) = \frac{\partial \omega_1}{\partial P_\vartheta} \\ \dot{P}_r(t) = -\frac{\partial \omega_1}{\partial r} \\ \dot{P}_\varphi(t) = 0 \\ \dot{P}_\vartheta(t) = -\frac{\partial \omega_1}{\partial \vartheta} \\ r(0) = r_0, \quad \vartheta(0) = \vartheta_0, \\ P_r(0) = P_{r,0}, \quad P_\varphi(0) = P_{\varphi,0}, \quad P_\vartheta(0) = P_{\vartheta,0}. \end{cases} \tag{11}$$

The equations set can be solved using Mathematica, and it gives the time-dependency of each variable. We are interested on the wave fronts dynamics in 2-dimensional section of the torus, given by the parametrization

$$\begin{cases} x(t) = r(t)\cos(\vartheta(t)) \\ y(t) = r(t)\sin(\vartheta(t)), \end{cases} \tag{12}$$

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The single coordinates evolution looks like this:

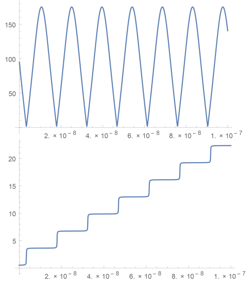


Figure: *Plots of $r(t)$ (above) and $\vartheta(t)$ (below).*

The propagation is along straight lines, and due to some cutoff regions (where the wave vector vanishes) we get reflections and the trajectory is deviated.

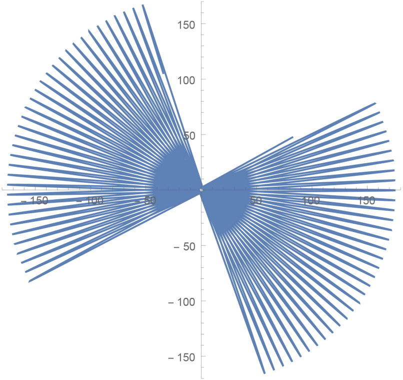
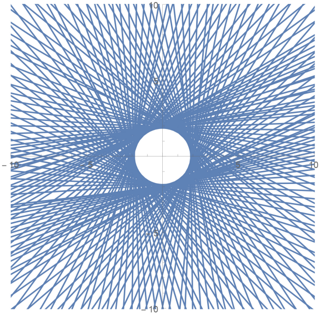


Figure: *Evolution of a wave front starting at* $r_0 = a$, $\vartheta_0 = \pi/6$.

A closer view of the region around the origin reveals the formation of a circular caustic:



Further studies

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- Explicit calculation of some terms of the expansion;
- Construction of the solution nearby the boarder of the domain and nearby the caustic (Lagrangian manifold);
- Generalization of the result to a more complete range of frequencies;
- Estimates of accuracy of the asymptotic solution.

Thanks for the attention!