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Kinetic theory of the generalized self-consistent 3D plasma wake field excitation in overdense regime

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Introduction

- The propagation of a non-laminar, relativistic charged particle beam in a plasma

- The density and current perturbations of both plasma and beam excite the plasma wake field (PWF) that are travelling behind the beam itself

- For sufficiently long beam, the beam experiences the effect of the wake fields that itself created and it evolves according to a self-consistently which is described by Vlasov- Poisson-like pair of equations.

- We first consider the Lorentz-Maxwell system of equations governing the spatio-temporal evolution of the ‘*beam+plasma*’. Here, the beam acts as a source of both charge and current

Introduction

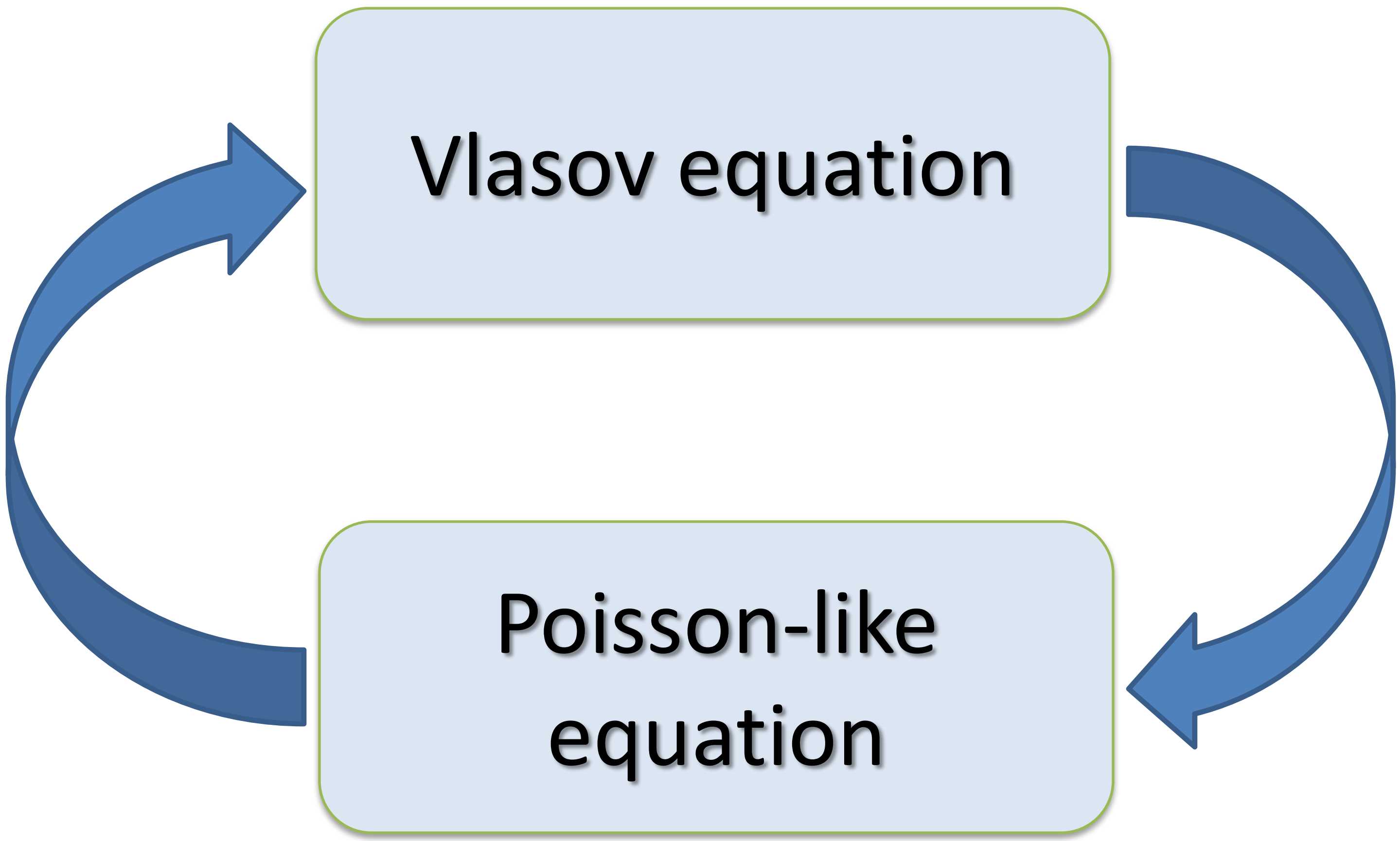
- In the co-moving frame a sort of electrostatic approximation can be provided, therefore the L-M system can be reduced to Poisson-like equation

- Poisson-like equation (PE) relates the beam density with the wake potential, providing this way an effective collective potential experienced by the beam itself

- Consequently, since here we assume that the collective and nonlinear beam dynamics is governed by the Vlasov equation, we provide an effective description of the beam+plasma system by adopting the pair of Vlasov and PE.

Scheme of the self consistency

Nonlinear and collective dynamics



Non-relativistic plasma + relativistic beam

Generalized Poisson-like equation

- ✓ plasma: *warm* (in adiabatic approximation), non-relativistic, ions are at rest (infinitely massive), *magnetized* $\mathbf{B}_0 = \hat{z} B_0$
- ✓ beam: *non-laminar*, *collisionless*, *relativistic* and *arbitrarily sharp*

Lorentz-Maxwell system of equations

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{e}{m_0}\mathbf{E} - \frac{e}{m_0\, c}\mathbf{u} \times \mathbf{B} - \frac{\nabla \cdot \hat{\mathcal{P}}}{m_0\, n},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{E} = 4\pi \left[e(n_0 - n) + q\rho_b \right],$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}(q\rho_b \mathbf{u}_b - en\mathbf{u}) + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

$$\hat{\mathcal{P}} = \begin{pmatrix} \mathcal{P}_\perp & 0 & 0 \\ 0 & \mathcal{P}_\perp & 0 \\ 0 & 0 & \mathcal{P}_z \end{pmatrix}$$

Non-relativistic plasma + relativistic beam

- linearize the set of equations around unperterbed state
- transform all the equations to the co-moving frame $\xi = z\text{-}\beta ct$
- split the variables into the longitudinal and transverse components.

Generalized Poisson-like equation for the wake potential

$$\left[\left(\frac{\partial^2}{\partial\xi^2} + k_{uh}^2 - \alpha_z\frac{\partial^2}{\partial\xi^2} - \alpha_\perp\nabla_\perp^2 - \alpha_z k_{ce}^2\right)\left(\frac{1}{\gamma_0^2}\frac{\partial^2}{\partial\xi^2} + \nabla_\perp^2 - k_p^2\right) + (1-\alpha_z)k_{pe}^2 k_{ce}^2\right]\Omega$$
$$= k_{pe}^2\frac{qm_0c^2}{e^2}\left[\frac{1}{\gamma_0^2}\left(\frac{\partial^2}{\partial\xi^2} + k_{ce}^2 - \alpha_z\frac{\partial^2}{\partial\xi^2} - \alpha_\perp\nabla_\perp^2 - \alpha_z k_{ce}^2\right) - (1-\alpha_z)k_{pe}^2\right]\frac{\rho_b}{n_0}$$

where $\Omega = (\beta A_{1z} - \phi_1)$, A_{1z} and ϕ_1 being the longitudinal components of the perturbation of vector and scalar potential respectively.

$$\alpha_z = (\Gamma_z v_z^2)/(\beta^2 c^2),\ \alpha_\perp = (\Gamma_\perp v_\perp^2)/(\beta^2 c^2)$$

$$\gamma_0 = (1-\beta^2)^{-1/2}$$

$$v_z^2 = (k_B T_z)/m_0,\ v_\perp^2 = (k_B T_\perp)/m_0$$

Γ_z, $\Gamma_\perp$ are adiabatic coefficients in longitudinal and transverse directions

Non-relativistic plasma + relativistic beam

Different limiting cases in PWF theory from generalized one

- If $\alpha_z = \alpha_\perp = 0$ (cold plasma),

$$\left[\left(\frac{\partial^2}{\partial\xi^2} + k_{uh}^2\right)\left(\frac{1}{\gamma_0^2}\frac{\partial^2}{\partial\xi^2} + \nabla_\perp^2 - k_{pe}^2\right) + k_{pe}^2 k_{ce}^2\right]\Omega = k_{pe}^2 \frac{q m_0 c^2}{e^2}\left[\frac{1}{\gamma_0^2}\left(\frac{\partial^2}{\partial\xi^2} + k_{ce}^2\right) - k_{pe}^2\right]\frac{\rho_b}{n_0}$$

- If $\frac{1}{\gamma_0}\frac{\partial}{\partial\xi} \to 0$ (limited beam sharpness) and $\alpha_z = \alpha_\perp = 0$

$$\left[\left(\frac{\partial^2}{\partial\xi^2} + k_{uh}^2\right)\left(\nabla_\perp^2 - k_{pe}^2\right) + k_{pe}^2 k_{ce}^2\right]\Omega = -k_{pe}^4 \frac{q m_0 c^2}{e^2}\frac{\rho_b}{n_0}$$

- $|\partial/\partial\xi| \ll k_{pe}$, $B_0 = 0$, $\alpha_z = \alpha_\perp = 0$ and $\frac{1}{\gamma_0}\frac{\partial}{\partial\xi} \to 0$

$$\left(\nabla_\perp^2 - k_{pe}^2\right)\Omega = -k_{pe}^2 \frac{q m_0 c^2}{e^2}\frac{\rho_b}{n_0}$$

Non-relativistic plasma + relativistic beam

- **Equation for beam dynamics**

$$\frac{\partial f}{\partial t} + \mathbf{p}\cdot\nabla_r f + \nabla_r \Omega \cdot \nabla_p f = 0$$

$$\mathbf{p} = \text{single particle momentum conjugate to } \mathbf{r}$$

$$\mathbf{r} = \hat{z}\xi + \mathbf{r}_\perp,\ \nabla_r = \hat{z}\frac{\partial}{\partial \xi} + \nabla_\perp,\ \nabla_p = \hat{z}\frac{\partial}{\partial p_z} + \nabla_{p\perp}$$

- **Purely longitudinal self consistent system**
  - ✓ *Vlasov-Poisson-like pair of equation for PWF,*

$$\frac{\partial f}{\partial s} + p\frac{\partial f}{\partial \xi} + \frac{q}{m_0\gamma_0 c^2}\frac{\partial \Omega}{\partial \xi}\frac{\partial f}{\partial p} = 0$$

$$\left[\left(\frac{\partial^2}{\partial \xi^2} + k_{pe}^2 - \alpha_z \frac{\partial^2}{\partial \xi^2}\right)\left(\frac{1}{\gamma_0^2}\frac{\partial^2}{\partial \xi^2} - k_p^2\right)\right]\Omega = k_{pe}^2 \frac{q m_0 c^2}{e^2}\left[\frac{1}{\gamma_0^2}\left(\frac{\partial^2}{\partial \xi^2} - \alpha_z \frac{\partial^2}{\partial \xi^2}\right) - (1-\alpha_z)k_{pe}^2\right]\frac{\rho_b}{n_0}$$

Relativistic plasma and Relativistic beam

- <u>Assumptions</u>: *relativistic* charged particle beam entering a *relativistic*, collision-less, cold, unmagnetized plasma and producing the PWF excitation therein

- <u>Model</u>: *relativistic* L-M system of equations

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0\,,$$

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{p} = -e\mathbf{E} - \frac{e}{c}\mathbf{v} \times \mathbf{B}\,,$$

$$\mathbf{p} = m_0 \mathbf{v} / \sqrt{1 - \mathbf{v}^2/c^2} \equiv m_0 \gamma \mathbf{v}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\,,$$

γ = relativistic gamma factor

$$\nabla \times \mathbf{B} = -\frac{4\pi}{c} en\mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} q n_b \mathbf{v}_b\,,$$

v_b = beam current velocity

$$\nabla \cdot \mathbf{E} = 4\pi e(n_0 - n) + 4\pi q n_b\,,$$

$$\nabla \cdot \mathbf{B} = 0$$

Relativistic plasma and Relativistic beam

- Assume that all the quantities depend on the combined variable $\xi = z - \beta c t$
- reduce the L-M system to a set of ordinary differetial equations describing the system dynamics:
  - *the transverse motion*

$$\frac{d^2\rho_x}{d\xi^2} + \frac{k_p^2}{\beta^2 - 1}\frac{\beta u_x}{\beta - u_z} = -\frac{4\pi q e n_b}{m_0 c^2}\frac{\mathbf{v}_{by}/c}{\beta^2 - 1} - \frac{4\pi q e n_b}{m_0 c^2}\frac{u_x}{\beta^2 - 1}\frac{(\beta - v_{bz}/c)}{\beta - u_z}$$

$$\frac{d^2\rho_y}{d\xi^2} + \frac{k_p^2}{\beta^2 - 1}\frac{\beta u_y}{\beta - u_z} = \frac{4\pi q e n_b}{m_0 c^2}\frac{\mathbf{v}_{bx}/c}{\beta^2 - 1} - \frac{4\pi q e n_b}{m_0 c^2}\frac{u_y}{\beta^2 - 1}\frac{(\beta - v_{bz}/c)}{\beta - u_z}$$

  - *the longitudinal motion*

$$\frac{d}{d\xi}\left[(u_z - \beta)\frac{d\rho_z}{d\xi} + u_x\frac{d\rho_x}{d\xi} + u_y\frac{d\rho_y}{d\xi}\right] = k_p^2\frac{u_z}{\beta - u_z} + \frac{4\pi q e n_b}{m_0 c^2}\frac{u_z - u_{bz}}{\beta - u_z}$$

$$u_x = v_x/c \qquad u_{bz} = v_{bz}/c \qquad \rho_x = p_x/m_0 c$$

Relativistic plasma and Relativistic beam

- **Purely longitudinal equation for electron motion** $(u_x = u_y = 0)$
  - *expressing momentum in terms of velocity and* $u_z = u$

$$\frac{d^2}{d\xi^2}\left[\frac{1-\beta u}{\sqrt{1-u^2}}\right] - k_p^2 \frac{u}{\beta - u} = k_p^2 \frac{q}{e}\frac{n_b}{n_0}\frac{u-u_b}{\beta - u}$$

  - *from moment equation:* $\frac{1-\beta u}{\sqrt{1-u^2}} = -\frac{e}{m_0 c^2}\Omega + K_0$

at boundary: $K_0 = 1 + \frac{e}{m_0 c^2}\bar{\Omega}$ $\quad \frac{1-\beta u}{\sqrt{1-u^2}} = 1 - \alpha(\Omega - \bar{\Omega})$ $\quad \alpha = e/m_0 c^2$

Fully relativistic equation for PWF in beam-plasma interaction

$$\frac{d^2\Omega}{d\xi^2} + \frac{k_p^2}{\alpha}\left(\frac{\beta^2 + \alpha\Omega\left(\alpha\Omega - 2\right) \mp \beta\sqrt{(\alpha\Omega - 1)^2\left[\beta^2 + \alpha\Omega\left(\alpha\Omega - 2\right)\right]}}{\left(\beta^2 - 1\right)\left[\beta^2 + \alpha\Omega\left(\alpha\Omega - 2\right)\right]}\right) = -\frac{k_p^2}{\alpha}\frac{q}{e}\frac{n_b}{n_0}\frac{u-u_b}{\beta - u}$$

Relativistic plasma and Relativistic beam

- expanding wake field around relativistic unperturbed state,

$$u = u_0 \qquad \Omega = \Omega_0(\xi)$$

- Zeroth order relativistic PWF equation

$$\frac{d^2\Omega_0}{d\xi^2} + \frac{k_p^2}{\alpha}\frac{\beta^2 + \alpha\Omega_0\left(\alpha\Omega_0 - 2\right) \mp \beta\sqrt{\left(\alpha\Omega_0 - 1\right)^2\left[\beta^2 + \alpha\Omega_0\left(\alpha\Omega_0 - 2\right)\right]}}{\left(\beta^2 - 1\right)\left[\beta^2 + \alpha\Omega_0\left(\alpha\Omega_0 - 2\right)\right]} = -\frac{k_p^2}{\alpha}\frac{q}{e}\frac{n_{b0}}{n_0}\frac{u_0 - u_{b0}}{\beta - u_0}$$

$$\frac{d^2\Omega_0}{d\xi^2} + \frac{k_p^2}{\alpha}\frac{2\beta - \alpha\Omega_0(\beta + 1)}{(\beta^2 - 1)(\beta - \alpha\Omega_0)} = \frac{k_p^2}{\alpha}\frac{q}{e}\frac{n_{b0}}{n_0} \qquad \textit{(for ' – ' sign)}$$

$$\frac{d^2\Omega_0}{d\xi^2} + \frac{k_p^2\Omega_0}{(\beta + 1)(\beta - \alpha\Omega_0)} = \frac{k_p^2}{\alpha}\frac{q}{e}\frac{n_{b0}}{n_0} \qquad \textit{(for ' + ' sign)}$$

- First order relativistic PWF equation (provided that rigidity condition, $u_{b0} = \beta$, is satisfied)

$$\frac{d^2\Omega_1}{d\xi^2} \mp \frac{k_p^2\beta}{(\beta - \alpha\Omega_0)^3}\Omega_1 = \frac{k_p^2}{\alpha}\frac{q}{e}\frac{n_{b1}}{n_0}$$

Relativistic plasma and Relativistic beam

- **Longitudinal relativistic kinetic equation for beam dynamics**
  - *Relativistic Hamiltonian in z-direction*

$$H = c\left[(p - \frac{q}{c}A)^2 + m_0^2 c^2\right]^{1/2} + q\phi$$

$$H_0 - q\phi_0 = c(p_0^2 + m_0^2 c^2)^{1/2} = m_0 \gamma_0 c^2$$ *(unperturbed Hamiltonian)*

  - *small displacements of quantities around the relativistic zero-th order state* and get normalized Hamiltonian as

$$\mathcal{H} = \frac{H}{m_0 \gamma_0 c^2} \equiv \mathcal{H}_0 + \mathcal{H}_1 \Rightarrow \begin{aligned} \mathcal{H}_0 &= 1 + \varphi_0 \\ \mathcal{H}_1 &= \frac{1}{2}\mathcal{P}^2 + \beta_0 \mathcal{P} - \frac{q\Omega}{m_0 \gamma_0 c^2} \end{aligned}$$

$$\frac{1}{c}\frac{\partial f_0}{\partial \tau} + (\beta_0 - \beta)\frac{\partial f_0}{\partial \xi} + \frac{q}{c}\frac{\partial \Omega_0}{\partial \xi}\frac{\partial f_0}{\partial p_0} = 0$$ *Zero-th order motion*

$$\frac{\partial f_1}{\partial s} + \mathcal{P}\frac{\partial f_1}{\partial \xi} + \frac{q}{m_0 \gamma_0 c^2}\frac{\partial \Omega_1}{\partial \xi}\frac{\partial f_0}{\partial \mathcal{P}} = 0$$ *First-order motion*

EXAMPLES

SELF MODULATION OF A LONG BEAM

$$|\partial/\partial\xi| \ll k_{pe} \qquad \alpha_z = \alpha_\perp = 0 \qquad \frac{1}{\gamma_0}\frac{\partial}{\partial\xi} \to 0 \qquad B_0 \neq 0$$

- Poisson-like equation:

$$\nabla_\perp^2 U_w - k_s^2 U_w = \frac{k_s^2}{n_0\gamma_0}\rho_b$$

- Vlasov equation:

$$\frac{\partial f}{\partial \xi} + \left[\mathbf{p}_\perp + \frac{1}{2}k_c(\hat{z}\times\mathbf{r}_\perp)\right]\cdot\frac{\partial f}{\partial r_\perp} - \left[K\mathbf{r}_\perp + \frac{\partial U_w}{\partial \mathbf{r}_\perp} - \frac{1}{2}k_c(\hat{z}\times\mathbf{p}_\perp)\right]\cdot\frac{\partial f}{\partial \mathbf{p}_\perp} = 0$$

$$U_w(\mathbf{r}_\perp,\xi) = -\frac{q\Omega}{m_0\gamma_0 c^2}$$

$$\rho_b(\mathbf{r}_\perp,\xi) = \frac{N}{\sigma_z}\int f(\mathbf{r}_\perp,\mathbf{p}_\perp,\xi)\, d^2\mathbf{p}_\perp$$

$$\sigma_z = \text{bunch length}$$

$$k_s = k_p^2/k_{uh}, \quad k_{uh} = \omega_{uh}/c$$

$$K = (k_c/2)^2 = \left(-\frac{qB_0}{2m_0\gamma_0 c^2}\right)^2$$

$$N = \text{number of beam particle}$$

SELF MODULATION OF A LONG BEAM

- Virial description:

$$\sigma_\perp(\xi) = \langle r_\perp^2 \rangle^{1/2} = \left[\int r_\perp^2 \, f \, d^2 r_\perp \, d^2 p_\perp \right]^{1/2}$$

$$\sigma_{p_\perp}(\xi) = \langle p_\perp^2 \rangle^{1/2} = \left[\int p_\perp^2 \, f \, d^2 r_\perp \, d^2 p_\perp \right]^{1/2}$$

- Envelope equation:

$$\frac{d^2 \sigma_\perp^2}{d\xi^2} + 4K\sigma_\perp^2 = \begin{cases} 4\mathcal{C} + 2\langle U_w \rangle & \text{(NLC)}: \begin{cases} |\nabla_\perp| \approx k_s & \text{(moderately NLC)} \\ |\nabla_\perp| \gg k_s & \text{(strongly NLC)} \end{cases} \\ 4\mathcal{C} & \text{(LC)}: \quad |\nabla_\perp| \ll k_s \end{cases}$$

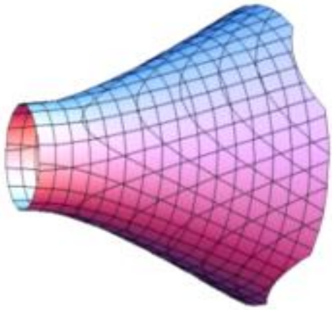
$$\mathcal{C} = \frac{1}{2}\sigma_{p_\perp}^2(\xi) + \frac{1}{2}K\sigma_\perp^2(\xi) + \frac{1}{2}\langle U_w \rangle = constant \qquad \lambda_0 = \frac{N}{n_0 \gamma_0 \sigma_z}$$

SELF MODULATION OF A LONG BEAM

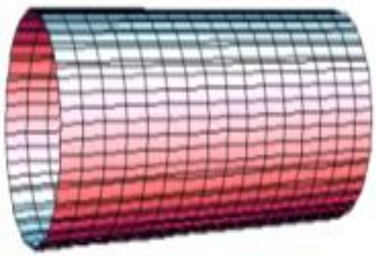
- Stability analysis in **purely local regime,** for unmagnetized plasma ($B_0 = 0$)

$$\sigma_{\perp}^2(\xi) = \sigma_{0\perp}^2 + 2\mathcal{C}(\xi - \xi_0)^2 \Rightarrow \begin{cases} \bullet\ \mathcal{C} > 0: \quad \sigma_{\perp}(\xi) > \sigma_{0\perp}, \quad \text{for any } \xi > \xi_0 \ (\textit{self-defocusing}) \\ \bullet\ \mathcal{C} = 0: \quad \sigma_{\perp}(\xi) = \sigma_{0\perp}, \quad \text{for any } \xi > \xi_0 \ (\textit{stationary state}) \\ \bullet\ \mathcal{C} < 0: \quad \sigma_{\perp}(\xi) < \sigma_{0\perp}, \quad \text{for any } \xi : \xi_0 < \xi < \bar{\xi} \ (\textit{self-focusing}) \end{cases}$$

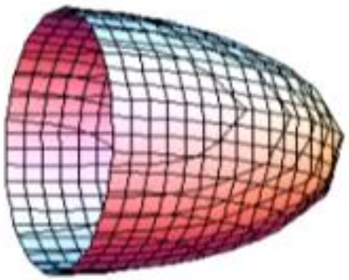
$$\textit{beam collapse:}\ \sigma_{\perp}(\bar{\xi}) = 0 \quad \bar{\xi} = \xi_0 + \sigma_{0\perp}/\sqrt{|\mathcal{C}|}$$



self-defocusing



stationary



self-focusing

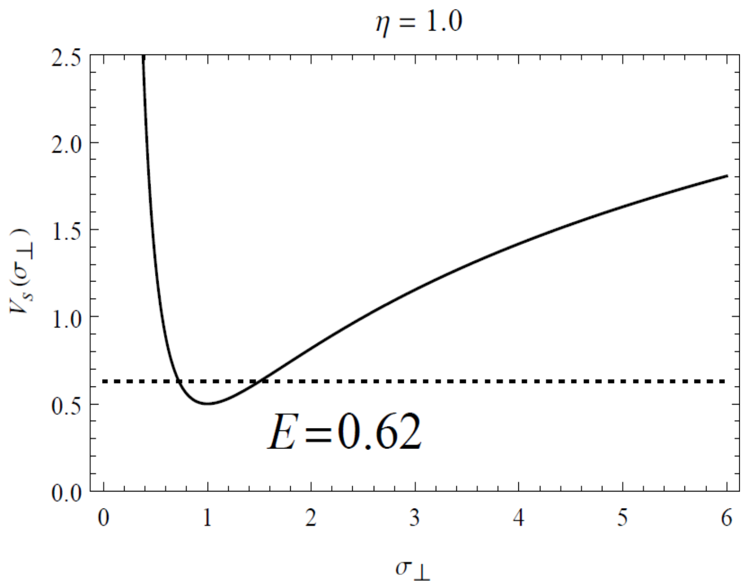
SELF MODULATION OF A LONG BEAM

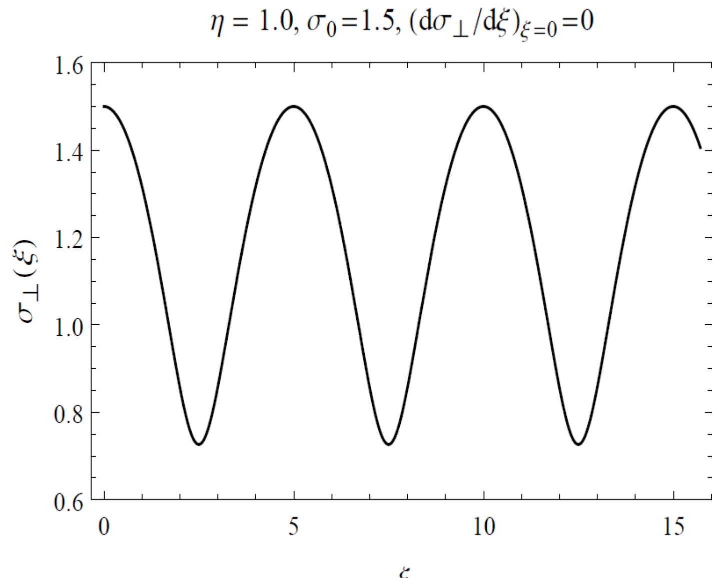
- Stability analysis in **strongly nonlocal regime** (in cylindrical symmetry)

$$\frac{d^2\sigma_\perp}{d\xi^2} + \frac{\eta}{\sigma_\perp} - \frac{\epsilon^2}{\sigma_\perp^3} = 0 \quad \Rightarrow \quad \frac{d^2\sigma_\perp}{d\xi^2} = -\frac{\partial V_s(\sigma_\perp)}{\partial \sigma_\perp}$$

Sagdeev Potential: $$V_s = \frac{1}{2}\eta \ln \sigma_\perp^2 + \frac{1}{2\sigma_\perp^2} \qquad \eta = k_s^2 \lambda_0 / 2\pi$$

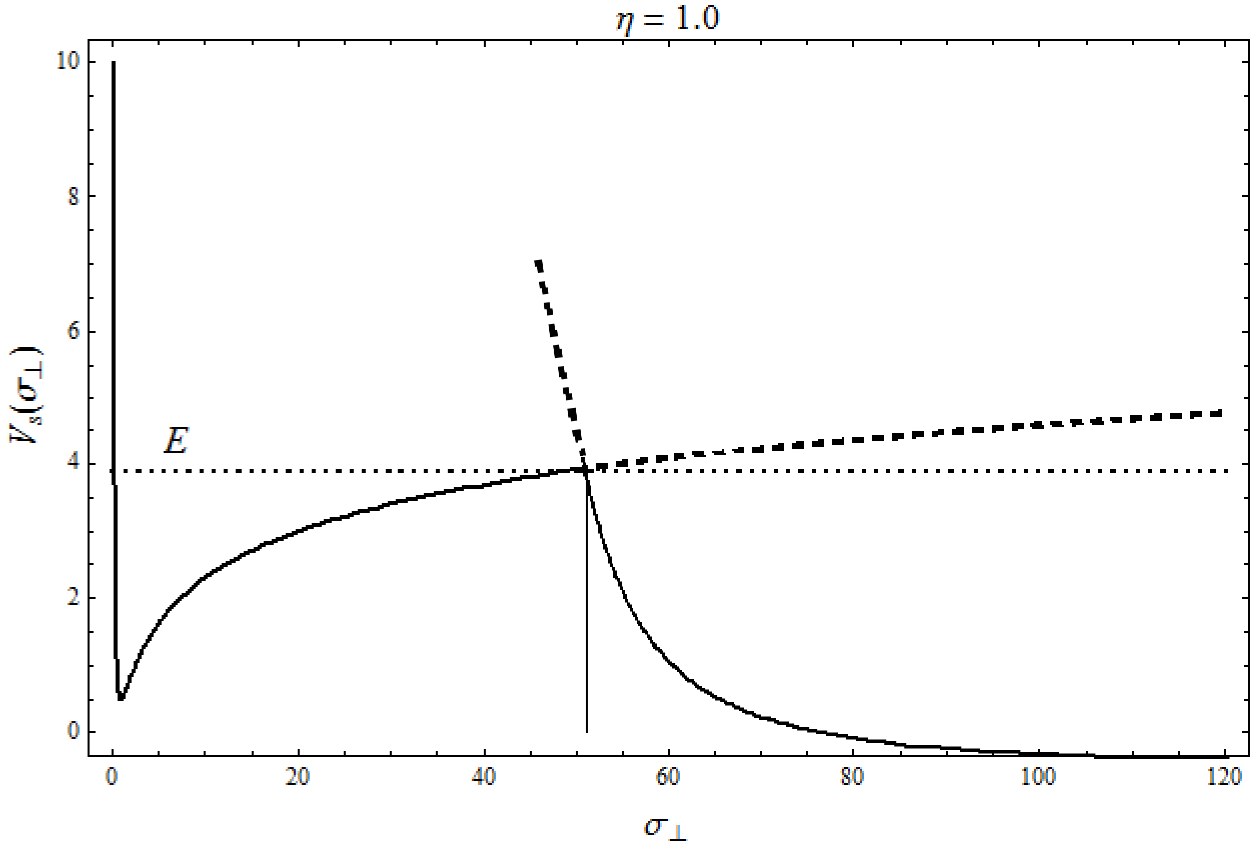
(arbitrary units)





- *the self-interaction of the charged particle beam leads always to its self modulation which prevents the beam collapse*

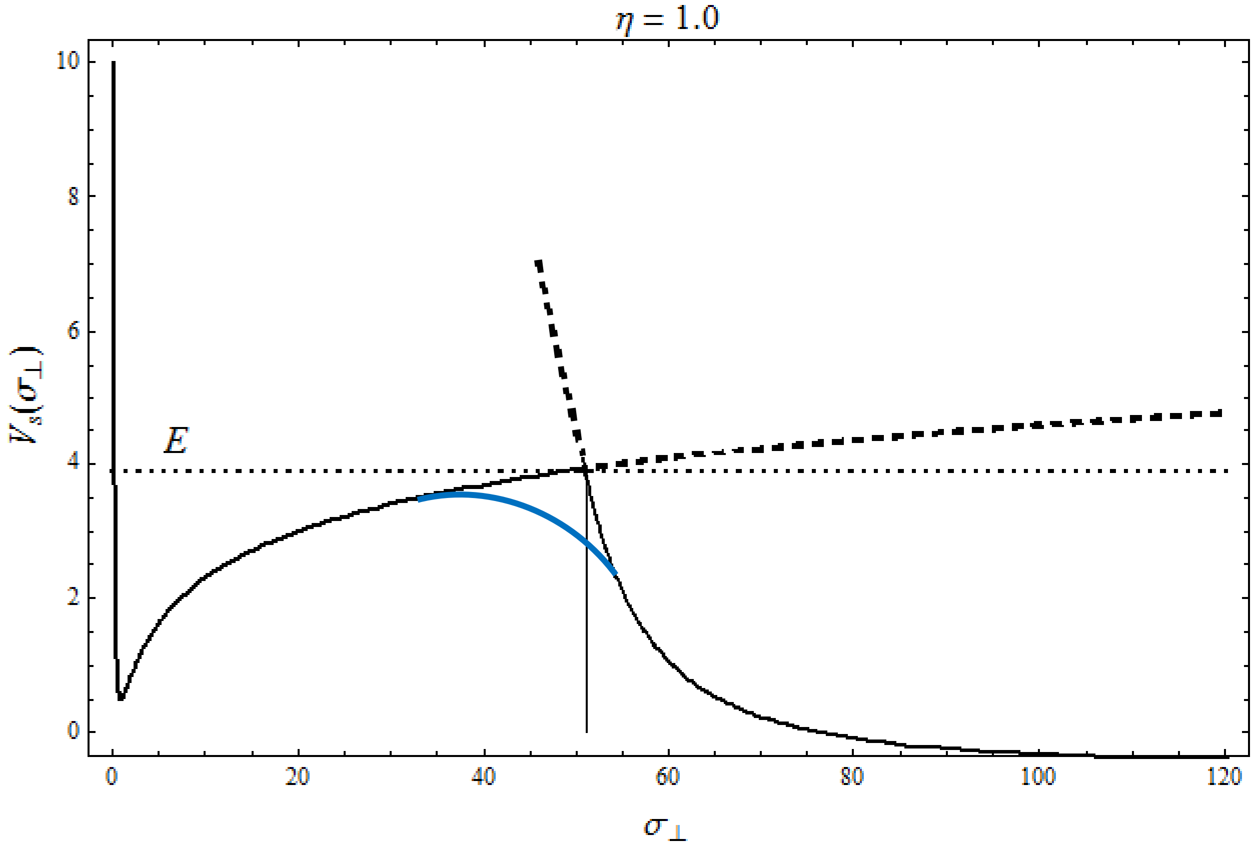
SELF MODULATION OF A LONG BEAM



✓ *Nonlocal regime is valid up to some small region, after that local regime starts.*
✓ *In overlapping region fixes the moderately nonlocal regime*

✓ *Qualitatively, we can understand that after the critical region, for a fixed value of energy, above E, it no longer oscillates and start evolve*

SELF MODULATION OF A LONG BEAM



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SUMMARY

- ☑ we generalized of PWF theory for warm plasma and arbitrarily sharp beam
- ☑ some special cases of the generalized PWF were discussed
- ☑ we provided the equations for fully *relativistic* self-consistent beam-pasma system in both *transverse and longitudinal* directions
- ☑ we discussed the self modulation for a long beam for local and strongly nonlocal regimes