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Tomographic description of charged-particle coherent beam propagation

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Outline

- Thermal wave model
- Tomographic probability approach to quantum-like model of charged particle beam
- Tomographic representations of 2D coherent beams
- Tomography of accelerating wavepackets
- Conclusions

THERMAL WAVE MODEL ASSUMPTION

The transverse/longitudinal beam dynamics is governed by a (nonlinear) Schrödinger equation where the Planck's constant is replaced by the transverse/longitudinal emittance for a complex function, the beam wave function (BWF), whose squared modulus is proportional to the beam density

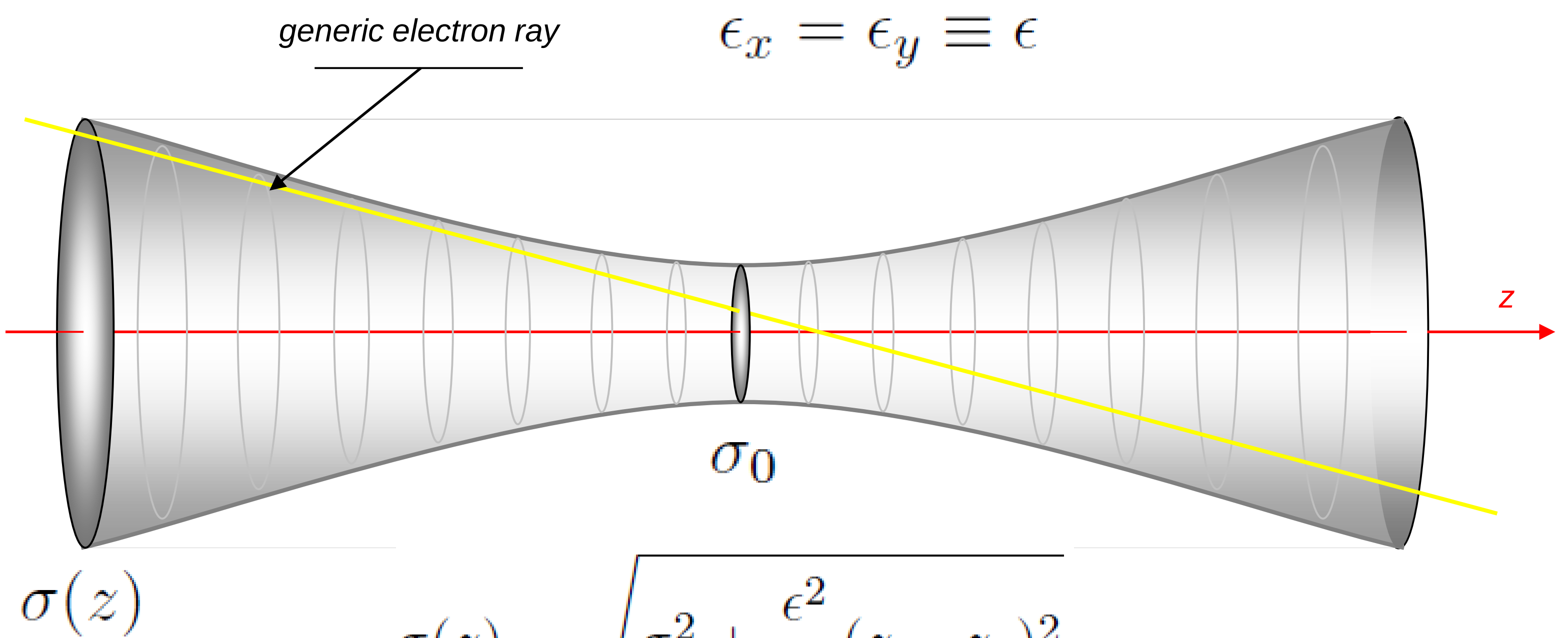
THE QUANTUM-LIKE UNCERTAINTY RELATION

The thermal spreading does not allow the focusing of the beam in a single point only (focal point).

$$\sigma_x \, \sigma_p \geq \frac{\epsilon}{2}$$

THERMAL SPREADING AMONG THE ELECTRON RAYS

Qualitative rappresentation of the free envelope motion (paraxial approximation) of a cilindrically-symmetric beam travelling in vacuo.



$$\sigma(z) = \sqrt{\sigma_0^2 + \frac{\epsilon^2}{\sigma_0^2}(z - z_0)^2}$$

GENERAL TRANSVERSE DYNAMICS

$$i\varepsilon\frac{\partial\Psi}{\partial z}=-\frac{\varepsilon^2}{2}\nabla_\perp{}^2\Psi+U(x,y,z)\Psi$$

R. Fedele and G. Miele, *Nuovo Cim. D* **13,** 1527 (1991)

dimensionless potential energy (normalized with respect to $m\beta c^2$**)**

$$U_\perp(x,y,z)=U_\perp^{ext}(x,y,z)+U_\perp^{coll}(x,y,z)$$

external interaction

collective interaction
(*mean field approximation*)

TRANSVERSE BEAM DYNAMICS:
a generalized 2D nonlinear Schrödinger equation

$$i\epsilon\frac{\partial\Psi}{\partial z} = -\frac{\epsilon^2}{2}\nabla_\perp^2\Psi + U_{ext}(x,y,z)\Psi + U_{coll}(|\Psi(x,y,z)|^2)\Psi$$

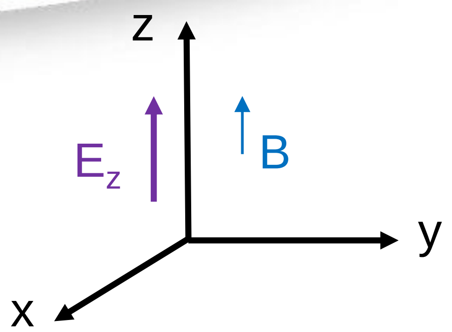
TWM has been applied to a number of linear and nonlinear problems, such as the luminosity estimates in final focusing stages of linear colliders in the presence of small aberrations

$$U_{ext} = \frac{1}{2}Kr^2\,, \quad U_{coll} = 0$$

$$U_{ext} = \frac{1}{2}Kr^2 + \lambda r^4$$

[R. Fedele and G. Miele, *Nuovo Cim. D* **13,** 1527 (1991)]
R. Fedele and G. Miele, *Phys.Rev.A* **46,** 6634 (1992)]
[D. Anderson et al., *Phys. Scr.* **58,** 608 (1998)]

3D DYNAMICS CHARGED PARTICLE BEAM DYNAMICS: HAMILTONIAN FORMULATION



$$-i\hbar\frac{\partial\Psi}{\partial t} = H_\perp\Psi + H_z\Psi$$

$$\Psi(x,y,z,t) = \psi_\perp(x,y,t)\psi_z(z,t)$$

$$K = \frac{1}{4}m\omega_c^2$$

$$-i\varepsilon\frac{\partial\Psi_\perp}{\partial t} = -\frac{\varepsilon^2}{2}\nabla_\perp^2\Psi_\perp + \frac{1}{2}Kr^2\Psi_\perp - \vec{a}_\perp\cdot\vec{r}\Psi_\perp + i\frac{1}{2}\omega_c\hat{z}\cdot(\vec{r}\times\nabla_\perp)\Psi_\perp$$

$$a_z = \frac{qE_z}{m\gamma c^c}$$

$$-i\hbar\frac{\partial\psi_z}{\partial t} = -\frac{\hbar^2}{2m}\nabla_\perp^2\psi_z + a_z z\psi_z$$

Transverse dynamics of 2-D coherent beams

$$-i\hbar\frac{\partial\Psi_\perp}{\partial t} = -\frac{\hbar^2}{2m}\nabla_\perp^2\Psi_\perp + \frac{1}{2}Kr^2\Psi_\perp$$

$$\Psi_{\perp,n,m}(x,y,t) = \psi_{\perp,n}(x,t)\,\psi_{\perp,m}(y,t)$$

$$\Psi_{\perp,n}(x,t) = \frac{1}{2^{(n/2)}\sqrt{n!}}\,e^{-\frac{1}{2}\left(\frac{\omega_c}{\varepsilon}\right)\left(x-\langle x\rangle\right)^2}\,e^{-i\left(n+\frac{1}{2}\right)\omega_c t + ix\langle p\rangle - \frac{i}{2}\langle x\rangle\langle p\rangle}\,H_n\left(\sqrt{\frac{\omega_c}{\varepsilon}}\left(x-\langle x\rangle\right)\right)$$

$$\langle x\rangle = x_m\cos\left(\omega_c t\right)$$

$$\langle p\rangle = -\omega_c x_m\sin\left(\omega_c t\right)$$

Tomographic probability representations of classical and quantum states: probability of $X = q\cos(\theta) + p\sin(\theta)$

Radon transform of the probability density $f(q,p)$ in phase space

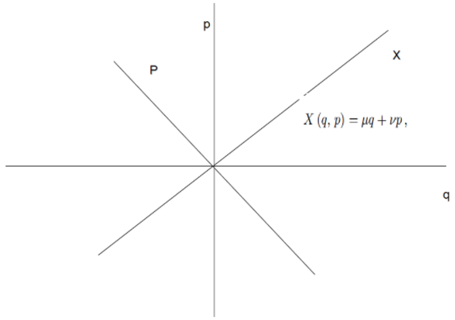
$$\int f(q,p)\,dq\,dp = 1.$$

$$w(X,\theta) = \langle \delta\left(X - q\cos\theta - p\sin\theta\right)\rangle = \int f(q,p)\delta\left(X - q\cos\theta - p\sin\theta\right)\,dq\,dp.$$

$$f(q,p) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{2\pi} r w(X,\theta) e^{ir(X - q\cos\theta - p\sin\theta)} dX\,dr\,d\theta \geq 0.$$

$w(X,\theta)$ probability density of quadrature $X = q\cos\theta + p\sin\theta$

$$\int w(X,\theta)\,dX = 1.$$



$\theta = 0,\ X = q$ and for $\theta = \pi/2,\ X = p$

Generalized tomographic representations:

symplectic tomography of $X = q\cos(\theta) + p\sin(\theta)$

$$\mu = \cos\theta, \qquad \nu = \sin\theta$$

Symplectic tomogram

$$w(X,\mu,\nu) = \mathrm{Tr}\,\hat{\rho}\,\delta(X - \mu\hat{q} - \nu\hat{p}) \qquad \int w(X,\mu,\nu)\,dX = 1$$

Inverse map

$$\hat{\rho} = \frac{1}{2\pi}\int w(X,\mu,\nu)\exp\left[i\left(X - \mu\hat{q} - \nu\hat{p}\right)\right] dX\,d\mu\,d\nu$$

$$\text{pure state } \hat{\rho}_\psi = |\psi\rangle\langle\psi|$$

$$w(X,\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right)dy\right|^2$$

G.M. D'Ariano, S. Mancini, V.I. Man'ko, P. Tombesi, Quantum Semiclass. Opt. 8, 1017 (1996)

S. De Nicola, R. Fedele, M.A. Man'ko, V.I. Man'ko, J. Opt. B: Quantum Semiclass. Opt. 5, 95 (2003

De Nicola S., Fedele R., Manko M,, Manko V., Meucci R., and Zito G. *Tomography and interferometric technique for characterizing Laguerre-Gaussian beams* Journal of Russian Laser Research, 31(2), pp. 139-151 (2010)

Fresnel tomography of classical and quantum states

Tomographic map in the phase space ν_x, ν_y

$$w_F\left(X,\ \mathrm{Y},\nu_x,\nu_y,\mathrm{t}\right)=\frac{1}{4\pi^2\left|\nu_x\nu_y\right|}\left|\int_{-\infty}^{\infty}\psi_{\perp}\left(x,\ y,\ \mathrm{t}\right)e^{i\frac{(X-x)^2}{2\nu_x}+\frac{(Y-y)^2}{2\nu_y}}dxdy\right|^2$$

De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2006 *Eur. Phys. J.* B **52** 191
De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2007 *Teor. Mat. Fiz.* **152** 225 [2007 *Theor. Math. Phys.***152** 1069 (Engl.Transl.)]
Man’ko M A 2006 *J. Russ. Laser Res.* **27** 405
De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2007 *J. Phys. Conf. Ser.* **70** 012007
Man’ko M A 2007 Tomographic entropy and new entropicuncertainty relations *Quantum Theory: Reconsideration ofFoundations–4 (Vaxjo, Sweden, June 2007)* ed G Adenier, A
Yu Khrennikov, P Lahti, V I Man’ko and T Nieuwenhuizen vol **962** (New York: AIP) p 132
Sergio De Nicola, Renato Fedele, Margarita A Man’ko Vladimir I Man’ko”Entropic uncertainty relations for electromagnetic beams, Phys Scripta **T135** 014053 (2009)

Tomographic representation of fundamental 2D coherent beam $\psi_{\perp 0}$ (x,y) and associated tomogram

$$\Psi_{\perp,0}(x,t) = e^{-\frac{1}{2}\left(\frac{\omega_c}{\varepsilon}\right)(x-\langle x\rangle)^2} e^{-i\left(\frac{1}{2}\right)\omega_c t + ix\langle p\rangle - \frac{i}{2}\langle x\rangle\langle p\rangle}$$

probability density $\left|\Psi_{\perp 0}(x,t)\Psi_{\perp 0}(y,t)\right|^2$
:

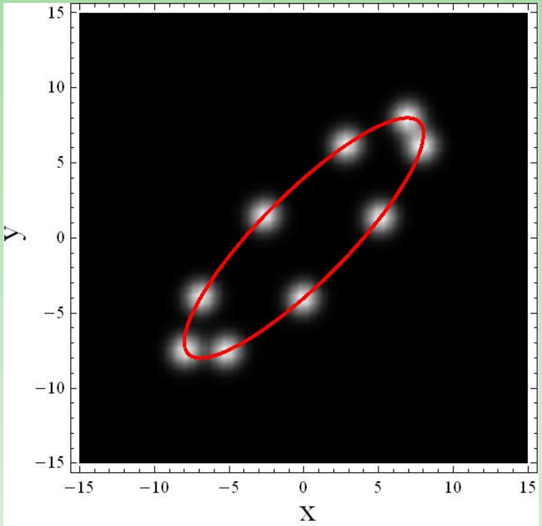
$$W\left(X,Y,\mu_x,\mu_y,\nu_x,\nu_y\right) = \frac{e^{-\frac{\left(X-\mu_x\langle x\rangle-\nu_x\langle p_x\rangle\right)^2}{\left(\mu_x^2+\nu_x^2\right)}-\frac{\left(Y-\mu_y\langle y\rangle-\nu_y\langle p_y\rangle\right)^2}{\left(\mu_y^2+\nu_y^2\right)}}}{\pi\sqrt{\left(\mu_x^2+\nu_x^2\right)\left(\mu_y^2+\nu_y^2\right)}}$$

Tomographic evolution in phase space (μ,ν)

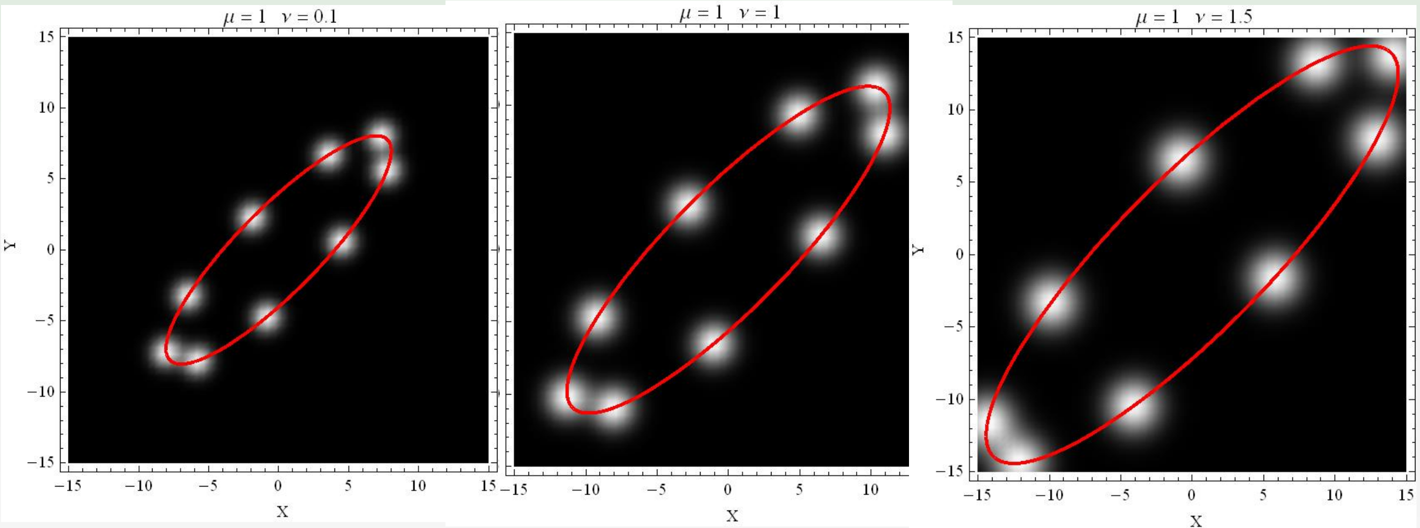
$$\boxed{\frac{\partial W}{\partial \tau} - \vec{\mu}\cdot\frac{\partial W}{\partial \vec{\nu}} + \vec{\nu}\cdot\frac{\partial W}{\partial \vec{\mu}} = 0}$$

$$\left(\vec{\nu} = \left(\nu_x, \nu_y\right) \quad \vec{\mu} = \left(\mu_x, \theta_y\right) \quad \tau = \omega t\right)$$

Tomographic representation of 2-D coherent beam



Probability density



Tomographics representations

Gaussian wavepackets in a constant force field

$$-i\varepsilon\frac{\partial\psi_z}{\partial t} = -\frac{\varepsilon^2}{2}\nabla_\perp^2\psi_z - az\psi_z \qquad a = qE_z/m\gamma c^2$$

$$\psi_z(z,t) = \frac{\sigma_0}{\sqrt{\sigma_0^2 + i\varepsilon t}}\, e^{i\left[\frac{at}{\varepsilon^2}\left(z+\frac{1}{6}at^2\right)\right]} e^{i\left[\frac{v}{\varepsilon}\left(z-z_0-\frac{1}{2}vt+\frac{1}{6}at^2\right)\right]}\, e^{-\frac{\left(z-z_0-vt+\frac{1}{2}at^2\right)}{2\left(\sigma_0^2+i\varepsilon t\right)}}$$

$$\langle z\rangle = z_0 + vt - \frac{1}{2}at^2 \qquad \langle p\rangle = (v - at) \qquad \langle E\rangle = \frac{\varepsilon^2}{4\sigma_0^2} + \frac{1}{2}v^2 + ax_0$$

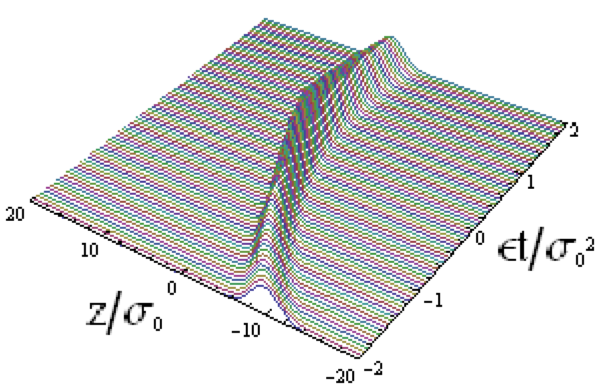
Variance of position and momentum

$$(\Delta z)^2 = \frac{1}{2}\left[\sigma_0^2 + \left(\frac{\varepsilon t}{\sigma_0}\right)^2\right] \qquad (\Delta p)^2 = \frac{\varepsilon^2}{2\sigma_0^2}$$

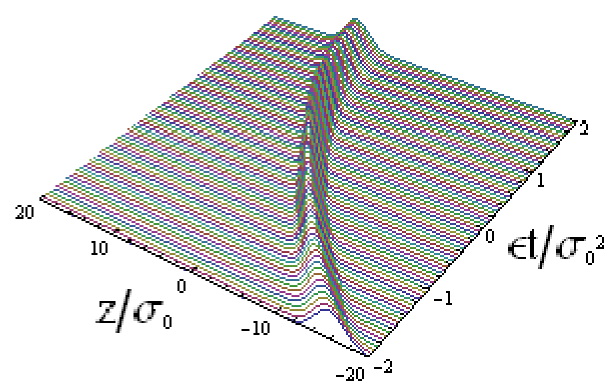
Accelerating Gaussian wavepackets in a constant force field

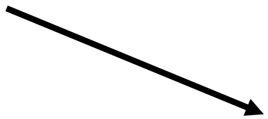
$$\psi_z(z,t)=\frac{\sigma_0}{\sqrt{\sigma_0^2+i\varepsilon t}}\,e^{i\left[\frac{at}{\varepsilon}\left(z+\frac{1}{6}at^2\right)\right]}e^{i\left[\frac{v}{\varepsilon}\left(z-z_0-\frac{1}{2}vt+\frac{1}{2}at^2\right)\right]}\,e^{-\frac{\left(z-z_0-vt+\frac{1}{2}at^2\right)}{2\left(\sigma_0^2+i\varepsilon t\right)}}$$

$v=\frac{3\epsilon}{\sigma_0}$



$v=\frac{6\epsilon}{\sigma_0}$

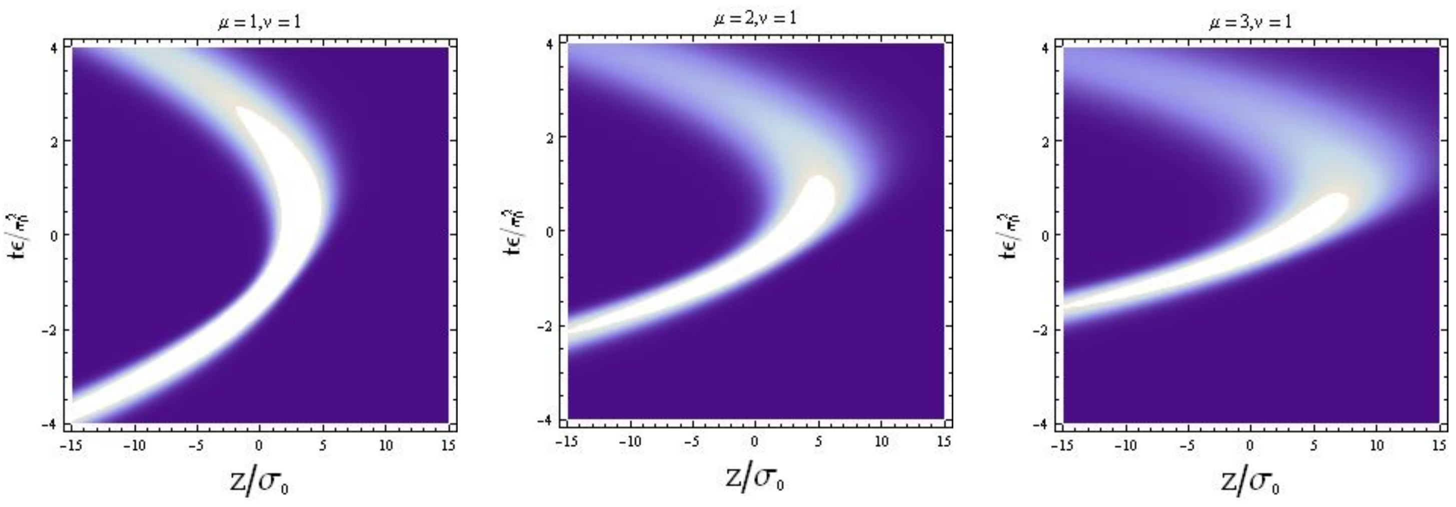




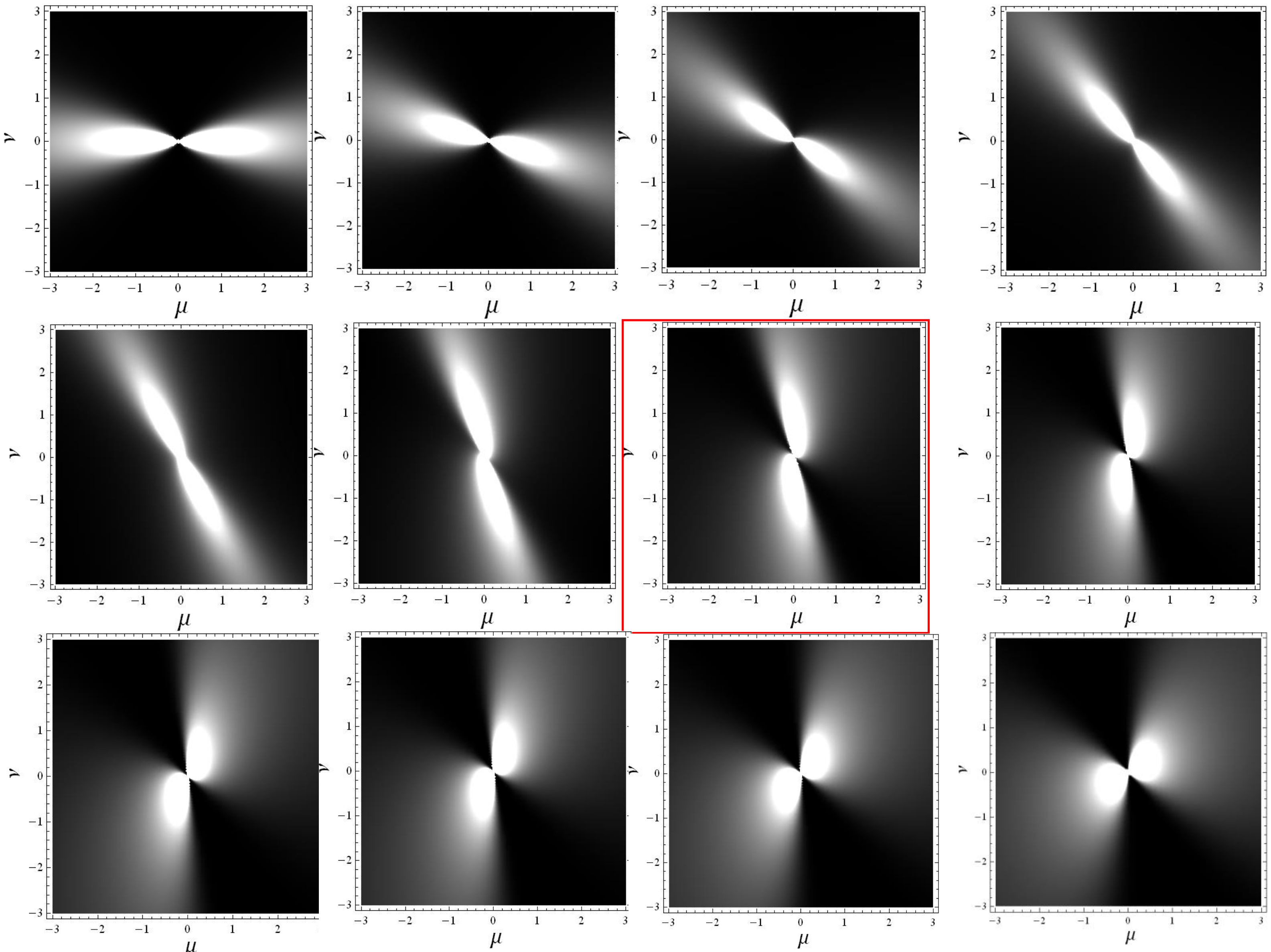
E_z

Tomographic *W(Z,t,μ,ν)* map of accelerating Gaussian wavepackets *ψ(Z,t)*

$$\psi(Z,t) \quad \leftrightarrow \quad W(Z,t;\mu,\nu) = \frac{1}{\sqrt{\pi}} \frac{e^{-\frac{\left(Z-\mu\langle z\rangle-\nu\langle p\rangle\right)^2}{\mu^2+(t\mu+\nu)^2}}}{\sqrt{\mu^2+(t\mu+\nu)^2}}$$



Time evolution of the tomographic representation of an accelerating wavepacket



Conclusions and perspectives

- **Symplectic tomography for characterizing classical and quantum fields allows to determine the probability density distribution of the values of the "composite" (classical and quantum) variable** $X = \mu q + \nu p$

- **Tomography of 3D beams is expressed in terms of six phase parameters. The tomogram of an accelerating wavepacket can be explicitly calculated for an accelerating Gaussian wavepacket**

- **The tomographic approach can be further extended by using integral descriptor such as the tomographic entropy associated to the tomogram.**

  De Nicola S, Fedele R, Manko M A and Manko Fresnel entropic characterization of optical Laguerre–Gaussian beams,Phys Letters A 375 (2011) De Nicola S, Fedele R, Manko M A and Manko V I 2009 Entropic uncertainty relations for electromagnetic beams *Phys. Scr.* T135 014053

- **The tomographic entropy approach can be applied for studying the accelerating wavepackets**

Thanks for your attention!