Evaluating input parameter uncertainty affecting the Global Braginskii Solver (GBS) code

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Fusion Energy - What is it?

Nuclear fusion is the *fundamental* source of energy in the Universe.
All of our sources of energy (apart from nuclear *fission* and *geothermic*), involve some form of 'recycling' of the energy produced in the core of the stars by nuclear fusion.
Main nuclear fusion reactions occurring in the stars are

- Proton-Proton Reaction, $4\mathrm{p} \longrightarrow {}^4_2\mathrm{He} + 2\,\nu_\mathrm{e} + 2\,\gamma$ $(13.36\mathrm{MeV})$
- Triple-alpha process, $3\ {}^4_2\mathrm{He} \longrightarrow {}^8_4\mathrm{Be} + {}^{12}_6\mathrm{C} + 2\,\gamma$ $(\mathrm{total}\ 7.367\mathrm{MeV})$

On Earth, we would like to create artificial *controlled* thermonuclear fusion reactions, with the purpose of producing electrical energy.
The most likely candidate reactions are

- ${}^2_1\mathrm{D} + {}^3_1\mathrm{T} \longrightarrow {}^4_2\mathrm{He}\ (3.5\,\mathrm{MeV}) + \mathrm{n}^0(14.1\mathrm{MeV})$
- ${}^2_1\mathrm{D} + {}^2_1\mathrm{D} \longrightarrow {}^3_1\mathrm{T}\ (1.01\,\mathrm{MeV}) + \mathrm{p}^+(3.02\mathrm{MeV})$

Advantages of Fusion Energy - Why do we need it?

It's abundant

The **world energy consumption** is predicted to **double by 2050**. The amount of energy that can be produced with nuclear fusion is **almost limitless**. Current reserves of deuterium would last thousands of years.

It's clean

No CO_2 emission would be involved in the production of energy with thermonuclear fusion. There is **no risk of nuclear fall-out** and the radiation contamination is minimal.

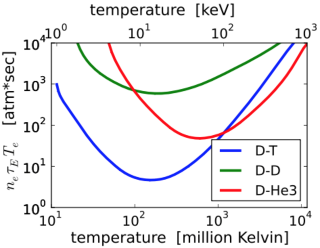
It's fair

The **basic fuel** will be deuterium, **readily available in sea water**. No more wars for control of energy resources. There are **no military application** of a nuclear fusion power plant.

Fusion Energy - How do we get it?

Achieving thermonuclear fusion is relatively easy by itself, it suffices to heat a plasma to **high enough temperatures** that ions can **overcome the Coulomb barrier**. Problems arise in:

1. Confinement
2. Control
3. Stability
4. Efficiency

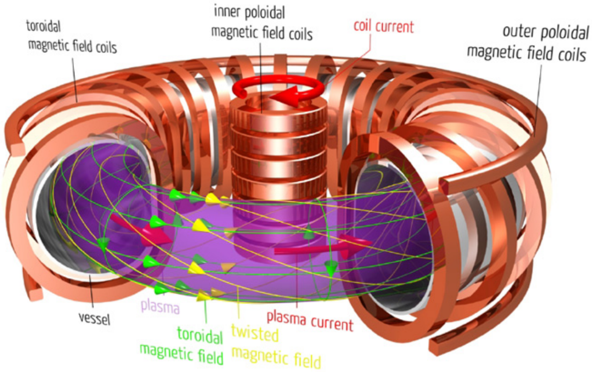


The Lawson criterion

According to John Lawson's pioneering analysis, the product of electron density n_e, confinement time τ_E and electron temperature T_e must exceed $3 \cdot 10^{21}$ keV s/m^3 in order to produce net energy from nuclear fusion.

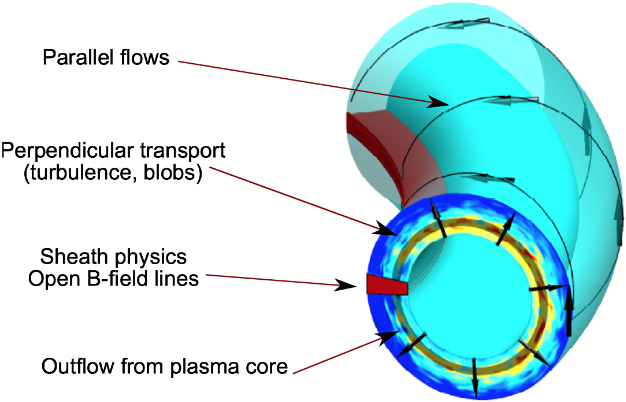
The tokamak concept

A tokamak is a doughnut-shaped device in which plasma is confined by **toroidal** and **poloidal magnetic fields**, generated by conducting **coils** and a **poloidal** current induced in the plasma.



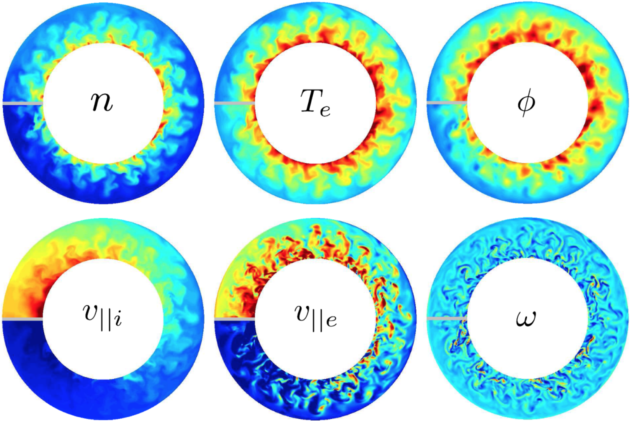
The Scrape-Off Layer (SOL)

The scrape off layer is defined as the region of the tokamak characterized by **open magnetic field lines**.
The magnetic field lines that cross the tokamak's walls are responsible for a significant **increase** in particle - and thus heat! - **flux**.

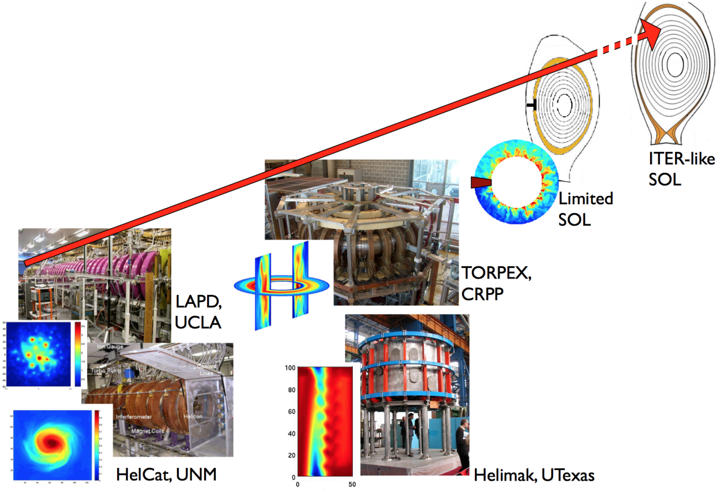


The GBS simulations

The GBS code is specifically tailored to model the plasma dynamics in the scrape-off layer.



The GBS code has been benchmarked against several experimental devices.



The Braginskii Equations

- Can be derived by taking moments of a Maxwellian distribution describing an ensemble of particles.
- Model the plasma dynamics in edge conditions well.

Conservation of mass

$$\frac{dn}{dt} + n\nabla \cdot \mathbf{v}_e = 0 \tag{1}$$

Momentum Equation

$$m_e n \frac{d\mathbf{v}_e}{dt} + \nabla \cdot \pi_e + en(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) = \mathbf{F} \tag{2}$$

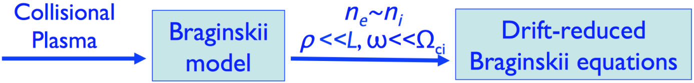
Energy Flux Equation

$$\frac{3}{2}\frac{dp_e}{dt} + \frac{5}{2}p_e\nabla \cdot \mathbf{v}_e + \pi_e : \nabla\mathbf{v}_e + \nabla\mathbf{q}_e = W_e \tag{3}$$

Reduction to the drift motion

Approximation regime

$$\frac{\partial}{\partial t} \approx \frac{\rho_i}{L_\perp} \ll \omega_{ci}, \quad \omega_{ci}\tau \ll 1 \tag{4}$$



Convection — Magnetic curvature — Parallel dynamics — Source

$$\frac{\partial n}{\partial t} + [\phi, n] = \hat{C}(nT_e) - n\hat{C}(\phi) - \nabla_\parallel (nV_{\parallel e}) + S$$

Error from Input Parameter Uncertainty - What is it?

- Uncertainty on the output observables stemming from the error on the physical input parameters.

Example

Say our pressure equilibrium numerical values are 20% off the experimental data. Can this be justified by the uncertainty in determining the exact value of the plasma density (and other parameters) as an input of the code?

Inapplicability of Monte Carlo simulations

When dealing with simpler codes, the answer could be found out by using Monte Carlo simulations. However in the GBS case this is not possible due to very long computing times.

How does the spectral method work?

- Choose a set of basis functions. In our case, Chebyshev polynomials of first kind, defined as $T_n(x) = cos(n \cdot cos^{-1}(x))$.
- Expand the solution of the *differential equation* as $f(t) = \sum_{k=0}^{K} a_k T_k(\tau)$, where the a_k's are the coefficients of the Chebyshev polynomials, and τ represents time in Chebyshev space.
- Derive *analytically* a system of *algebraic equations* for these coefficients.
- Find *numerically* the solution of this system of algebraic equations.

Advantages of the spectral method

Extremely high accuracy

The theoretically proved *minMax* property of Chebyshev-polynomials based spectral methods ensures that the solution found has the **smallest possible infinite norm of the error** given the chosen degree of the polynomial.

Speed

Very likely **faster than finite difference codes** when we don't want to resolve the oscillation on the equilibrium configuration. (There is good evidence, but a comprehensive quantitative comparison is still to be carried out)

Semi-analytic

The **solution** is found as a **set of coefficients** that multiply Chebyshev polynomials, thus it is readily available for further manipulation.

Spectral method example

Differential form of equation

$$\frac{\partial \nabla^2 \phi}{\partial t} = \frac{c}{B_0}\frac{\partial \phi}{\partial r}\frac{\partial \nabla^2 \phi}{\partial z} - \frac{c}{B_0}\frac{\partial \phi}{\partial z}\frac{\partial \nabla^2 \phi}{\partial r} \tag{5}$$

Algebraic form of equation

$$\frac{\partial g}{\partial t} = A{\sum_{k=0}^{K}}'{\sum_{x=0}^{X-1}}'{\sum_{y=0}^{Y}}'\sum_{\substack{\xi=x+1\\ \xi-x=odd}}^{X} 2\xi a_{k\xi y} T_k(\tau)T_x(\chi)T_y(\phi)\cdot{\sum_{k'=0}^{K}}'{\sum_{x'=0}^{X}}'{\sum_{y'=0}^{Y-1}}'\sum_{\substack{\sigma=y'+1\\ \sigma-y'=odd}}^{Y} 2\sigma b_{k'x'\sigma} T_{k'}(\tau)T_{x'}(\chi)T_{y'}(\phi)+$$

$$-A{\sum_{k'=0}^{K}}'{\sum_{x'=0}^{X-1}}'{\sum_{y'=0}^{Y}}'\sum_{\substack{\xi=x'+1\\ \xi-x'=odd}}^{X} 2\xi b_{k'\xi y'} T_{k'}(\tau)T_{x'}(\chi)T_{y'}(\phi)\cdot{\sum_{k=0}^{K}}'{\sum_{x=0}^{X}}'{\sum_{y=0}^{Y-1}}'\sum_{\substack{\sigma=y+1\\ \sigma-y=odd}}^{Y} 2\sigma a_{kx\sigma} T_k(\tau)T_x(\chi)T_y(\phi)$$

Chebyshev polynomials linearisation & Time Integration

$$T_{k_1}(\tau)\cdot T_{k_2}(\tau) = \frac{T_{k_1+k_2}(\tau)+T_{|k_1-k_2|}(\tau)}{2}$$

$$\frac{\partial g(t)}{\partial t} = \sum_{k=0}^{K} c_k T_k(\tau) \implies g(t) = \sum_{k=0}^{K}(c_{k-1}T_{k-1}(\tau) - c_{k+1}T_{k+1}(\tau))$$

Spectral Method Examples

Non-linearity

We can solve non-linear differential equations with the spectral method.

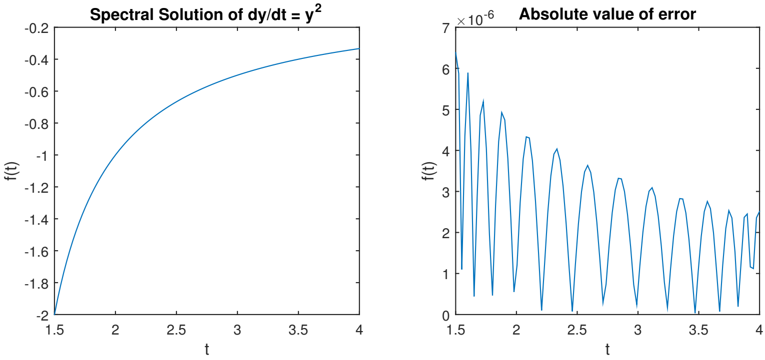


Figure : Solution to the equation $dy/dt = y^2$

Extremely high accuracy

It's much easier to get very accurate results with the spectral method than with any finite difference or finite elements methods.

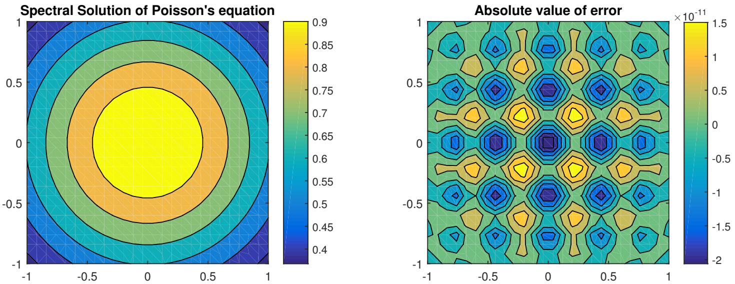


Figure : Solution to the Poisson equation $\nabla^2\phi(x,y) = (x^2+y^2-2)exp\left(-\frac{x^2+y^2}{2}\right)$

Spectral Method Examples

Benchmarking (stationary solution)

The code developed to solve the Braginskii equations spectrally has been benchmarked against available analytic solution.

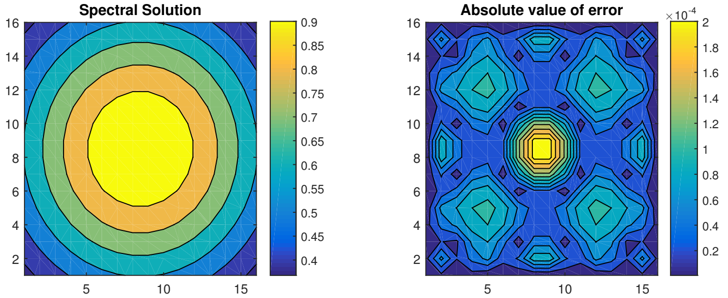


Figure : Stationary solution to the equation $\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\frac{\partial(\nabla_\perp^2)^{-1}g}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial(\nabla_\perp^2)^{-1}g}{\partial y}$

Benchmarking (dynamic solution)

The method of *manufactured solutions* has been used to test the code for description of plasma dynamics.

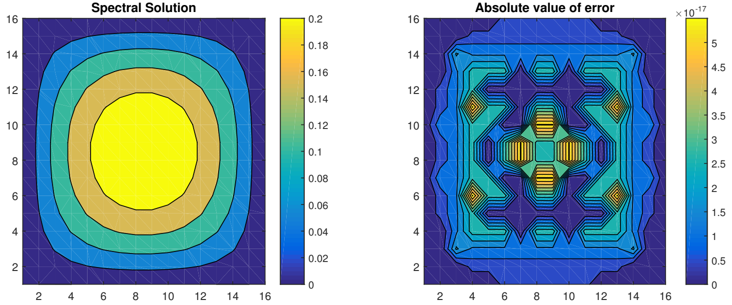
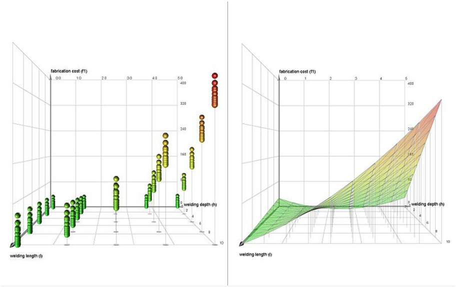


Figure : Time-dependent solution of $\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\frac{\partial (\nabla_\perp^2)^{-1} g}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial (\nabla_\perp^2)^{-1} g}{\partial y}$

Overall code performance evaluation

Surface Response Methodology

The main idea is to create a map which relates the main input parameters - with appropriate weightings - to the observable whose uncertainty we are interested in.



A set of dedicated numerical experiments is necessary to create this map. These investigations can be carried out in a **fast** and **accurate** way with the new spectral code developed.

Overall GBS code performance evaluation

Quantitative performance evaluation

One can quantitatively evaluate the overall performance by putting together the single "marks" for the different output parameters in a single metric $\chi = \frac{\sum_j R_j H_j S_j}{\sum_j H_j S_j}$, where $R_j = \frac{tanh[(d_j - d_0)/\lambda]+1}{2}$.

Simulation uncertainty estimation

In the formula to evaluate the distance between numerical results and experimental data $d_j = \sqrt{\frac{1}{N_j} \cdot \sum_{i=1}^{N_j} \frac{(e_{i,j} - s_{j,i})^2}{\Delta e_{j,i}^2 + \Delta s_{j,i}^2}}$

This project has focused on finding accurate $\Delta^2 s_{i,j}$.

Metric use example

Overall evaluation

Performance on the quality of simulations of different observables are combined together.

Configuration	Electric Potential	Density	Temperature	Metric χ
Symmetric Gaussian	0.958	0.986	0.945	**0.974**
Non-Symmetric Gaussian	0.825	0.910	0.854	**0.879**
Sinusoidal Perturbation	0.756	0.578	0.658	**0.687**

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Thanks for the attention!