The quantum effects on all Lagrangian points and prospects to measure them in the Earth-Moon system.

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Outline

- Restricted three-body problem.
- Quantum corrections on the coordintates of Lagrangian points.
- Full three-body problem.
- Displaced orbits.
- Laser Ranging.
- Conclusions and open problems.

Effective field theory (1)

- An Effective Field Theory (EFT) is a type of approximation to (or effective theory for) an underlying theory, such as a quantum field theory.
- An EFT includes the appropriate degrees of freedom to describe physical phenomena occurring at a chosen length scale or energy scale, while ignoring substructure and degrees of freedom at shorter distances (or equivalently at higher energies). Since EFTs are not valid at small length scales, they do not need to be renormalizable.
- EFTs are discussed in the context of the renormalization group where the process of integrating out short distance degrees of freedom is made systematic. This is done for example through the analysis of symmetries: if there is a single mass scale ***M*** in the microscopic theory, then the EFT can be seen as an expansion in ***1/M***. The construction of an EFT accurate to some power of ***1/M*** requires a new set of free parameters at each order of the expansion in ***1/M*.**

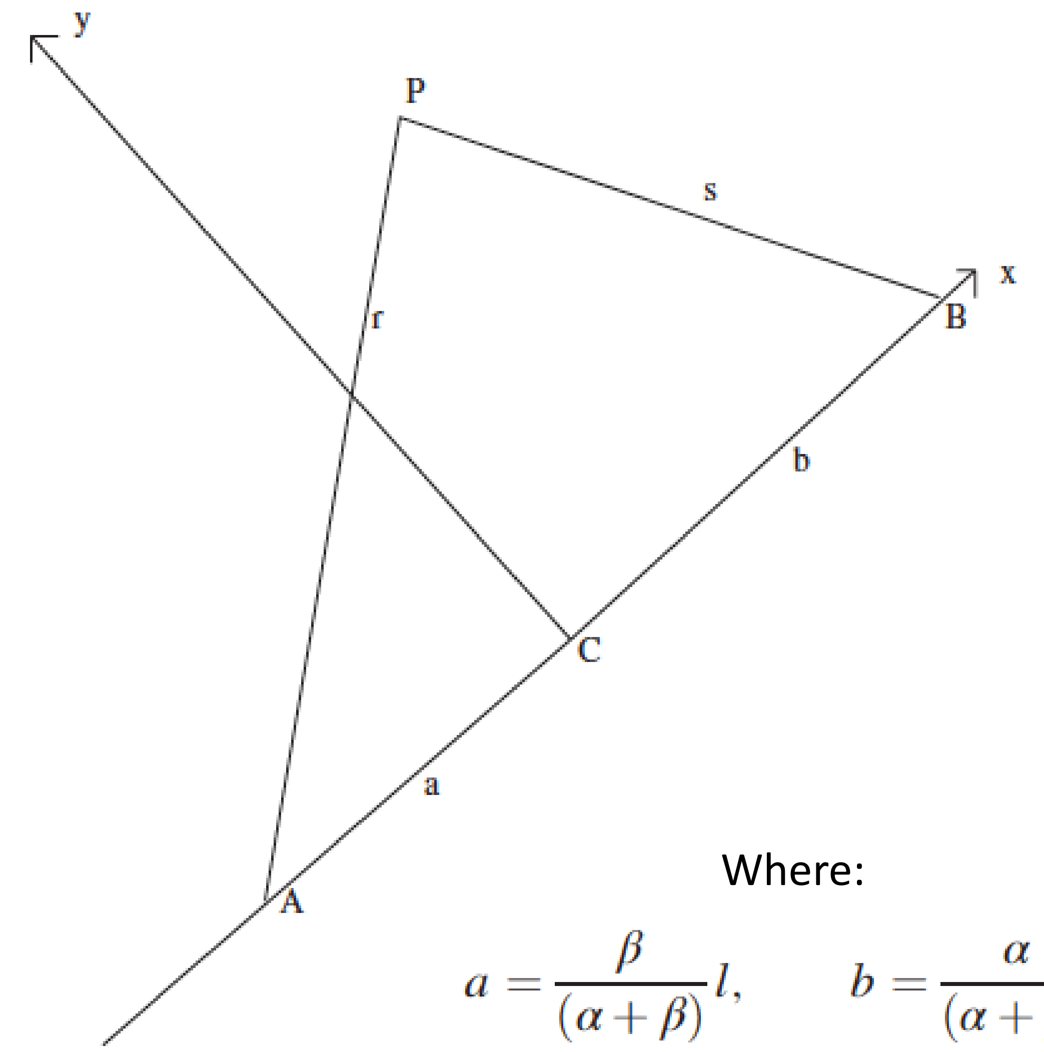
Effective field theory (2)

- General Relativity fits naturally into the context of EFT: gravitational interactions are proportional to energy and are easily organized into an energy expansion, where the expansion scale is the Planck length l_P.
- Treating gravity as a quantum EFT, allows a natural separation of the (known) low energy quantum effects from the (unknown) high energy effects. Within this framework, gravity is a well behaved quantum field theory at ordinary energy. The leading long-distance quantum corrections are independent of the eventually high energy theory of gravity and therefore represent necessary consequences of quantum gravity.

Introduction: Restricted three-body problem

The restricted three-body problem:

- Body A with mass α.
- Body B with mass β<α.
- A and B move under Newtonian potential considered without correction.
- Planetoid P with mass m such that m<<α, m<<β.
- Center of mass C.
- P is subjected to the quantum corrected Newtonian attraction of A and B.



Where:

$$a = \frac{\beta}{(\alpha+\beta)} l, \qquad b = \frac{\alpha}{(\alpha+\beta)} l$$

Quantum corrected Lagrangian (1)

The quantum corrected Lagrangian describing the motion of P assumes the form:

$$\begin{aligned}\frac{L}{m} &= \frac{1}{2}(\dot{x}^2+\dot{y}^2)+\omega(x\dot{y}-y\dot{x})+\frac{1}{2}\omega^2(x^2+y^2)\\ &\quad+\frac{G\alpha}{r}\left(1+\frac{k_1}{r}+\frac{k_2}{r^2}\right)+\frac{G\beta}{s}\left(1+\frac{k_3}{s}+\frac{k_2}{s^2}\right)\\ &= T-V=T_2+T_1+T_0-V,\end{aligned}$$

Where:

$$V=-\frac{Gm\alpha}{r}\left(1+\frac{k_1}{r}+\frac{k_2}{r^2}\right)-\frac{Gm\beta}{s}\left(1+\frac{k_3}{s}+\frac{k_4}{s^2}\right)$$

$$k_1=\kappa_1(l_m+l_\alpha),$$

$$k_2=k_4=\kappa_2\frac{G\hbar}{c^3}=\kappa_2 l_P^2,$$

$$k_3=\kappa_3(l_m+l_\beta).$$

$$\kappa_1=\kappa_3,$$

$$l_\alpha=\frac{G\alpha}{c^2},$$

$$l_\beta=\frac{G\beta}{c^2},$$

$$l_m=\frac{Gm}{c^2}$$

Quantum corrected Lagrangian (2)

Choice of constants:

+ choice

$$\kappa_1 = 3 \qquad \text{or} \qquad -1,$$

$$\kappa_2 = \frac{41}{10\pi} \qquad \text{or} \qquad -\frac{127}{30\pi^2},$$

− choice

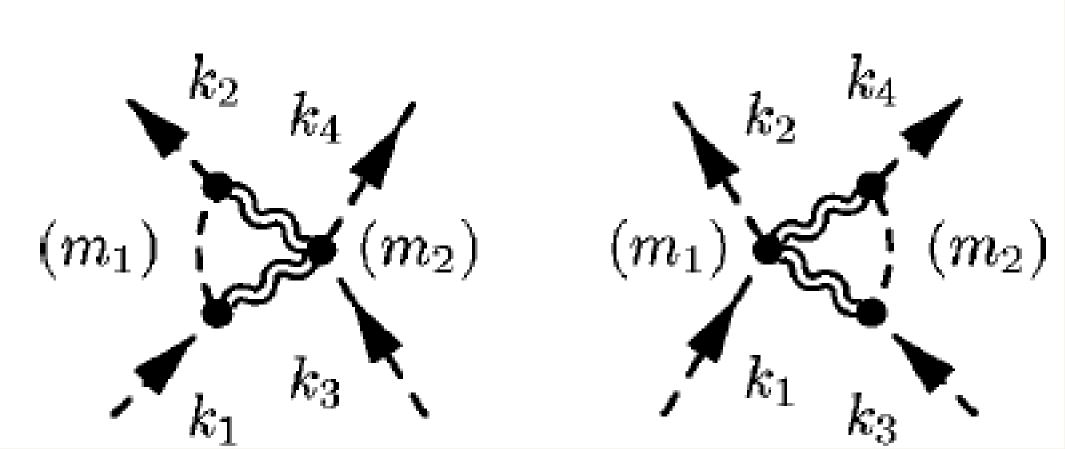
We define U as $T_0 - V = GU$, so that

$$U \equiv \frac{1}{2}\frac{(\alpha+\beta)}{l^3}(x^2+y^2) + \frac{\alpha}{r}\left(1+\frac{k_1}{r}+\frac{k_2}{r^2}\right) + \frac{\beta}{s}\left(1+\frac{k_3}{s}+\frac{k_2}{s^2}\right).$$

N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, Phys. Rev. D 67, 084033 (2003).
J. F. Donoghue, Phys. Rev. Lett. 72, 2996 (1994).

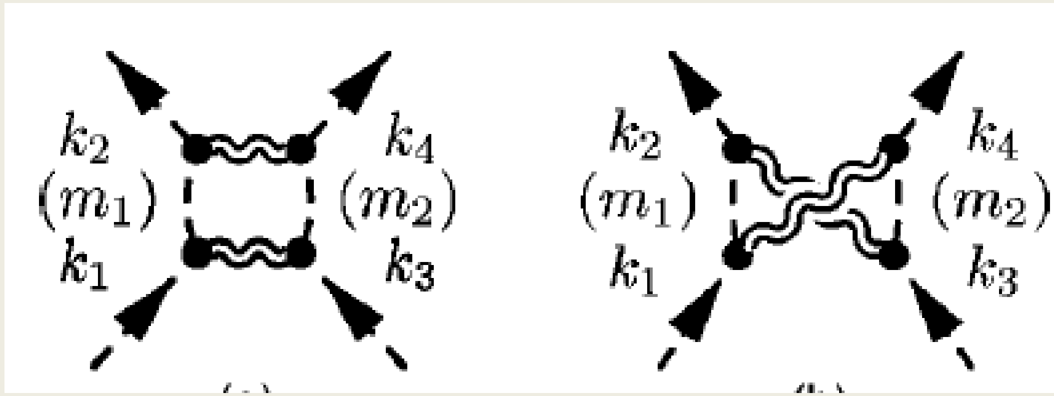
Feynman diagrams (1)

+ choice (scattering processes):

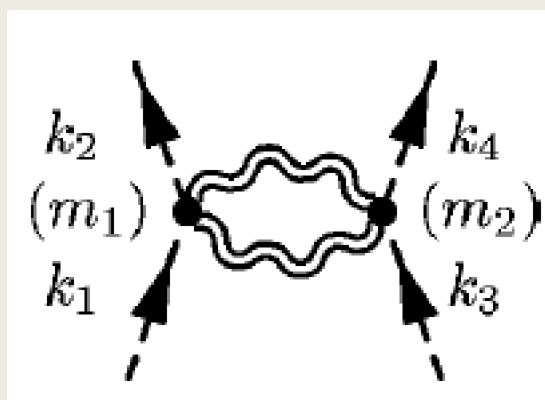


triangle diagrams

Feynman diagrams (2)



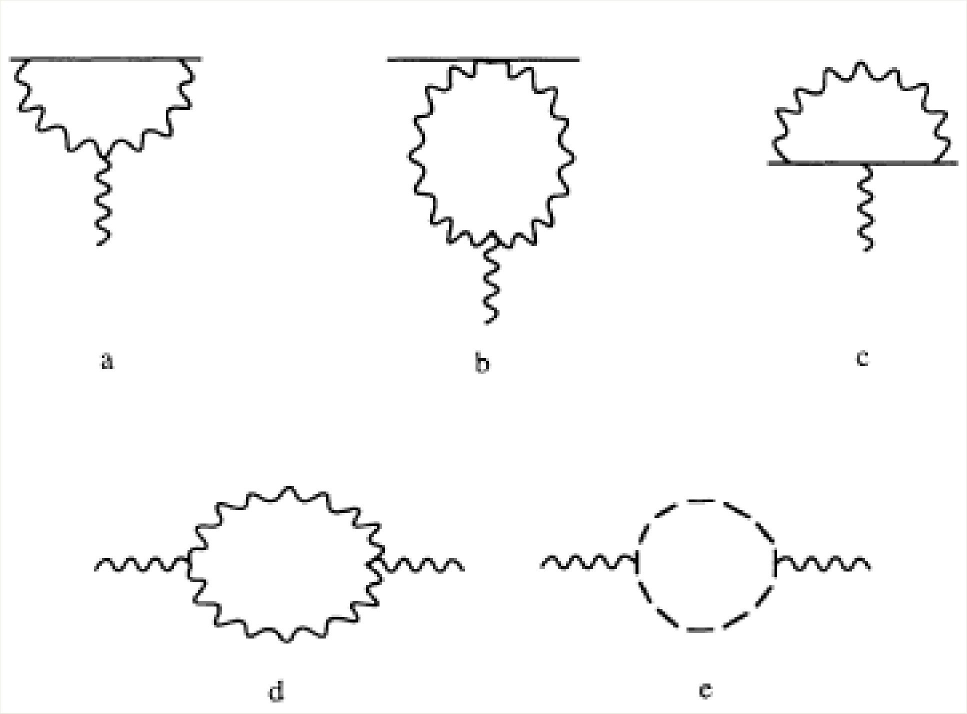
Box and crossed box diagrams



Double-seagull diagram

Feynman diagrams (3)

- choice (vertex and vacuum polarization diagrams)



One loop radiative corrections to the gravitational vertex (a)-(c) and vacuum polarization (d),(e).

Equilibrium conditions

Derivative of U:

$$\frac{\partial U}{\partial y} = \lambda y$$

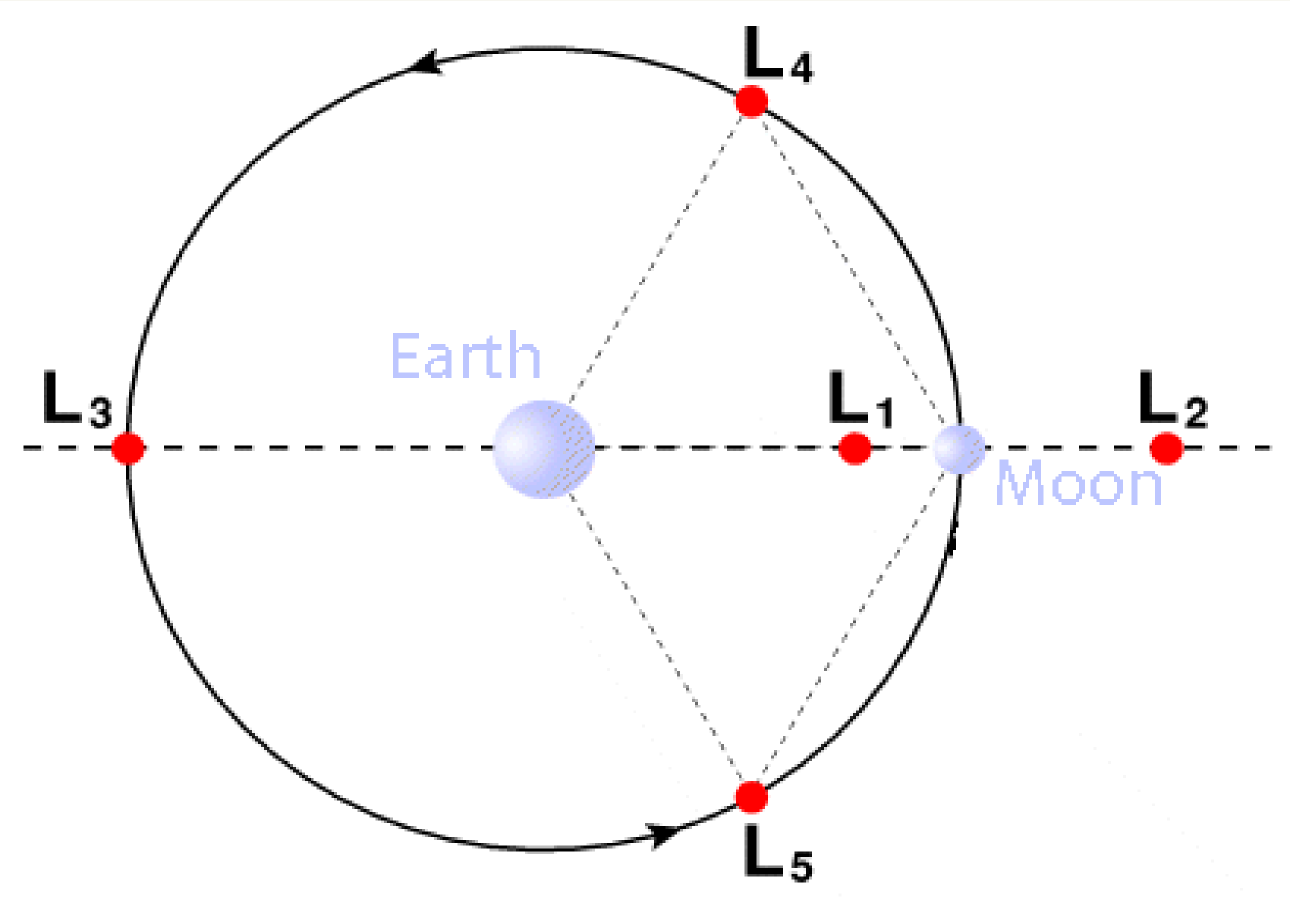
with

$$\lambda \equiv \frac{(\alpha+\beta)}{l^3} - \frac{\alpha}{r^3}\left(1 + 2\frac{k_1}{r} + 3\frac{k_2}{r^2}\right) - \frac{\beta}{s^3}\left(1 + 2\frac{k_3}{s} + 3\frac{k_2}{s^2}\right),$$

- *y=0* $\longrightarrow$ equilibrium points on the line joining A to B
- *λ =0* $\longrightarrow$ equilibrium points not lying on the line joining A to B

Classical Lagrangian points

Lagrangian points for the Moon-Earth system

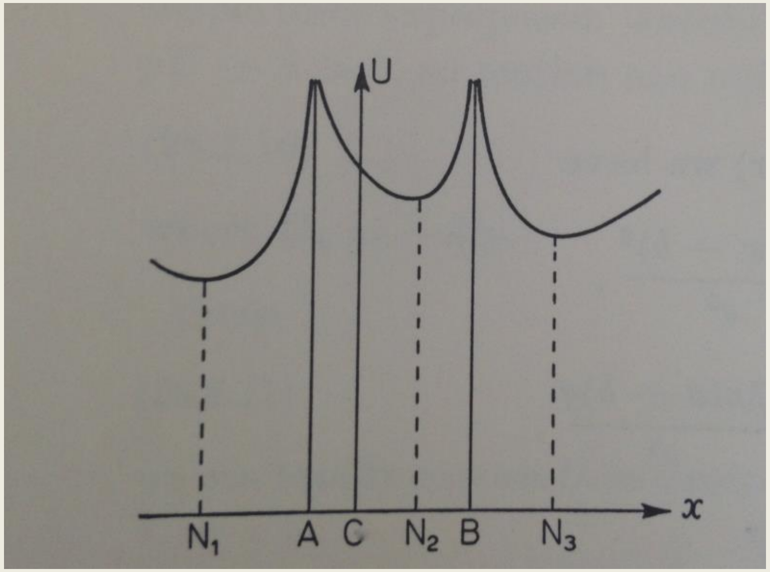


Equilibrium points on the line joining A to B (1)

Divide the line *y=0* into three regions

- $R_1 : x \in (-\infty,-a)$
- $R_2 : x \in (-a,b)$
- $R_3 : x \in (b,+\infty)$

In each region *U(x,0)* has one equilibrium point:



Equilibrium points on the line joining A to B (2)

Recall that

$$\frac{\partial U}{\partial y} = \lambda y$$

We also have that

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{(\alpha+\beta)}{l^3}x - \frac{\alpha x}{r^3}\left(1+2\frac{k_1}{r}+3\frac{k_2}{r^2}\right) - \frac{\beta x}{s^3}\left(1+2\frac{k_3}{s}+3\frac{k_2}{s^2}\right)\\ &+ \frac{\alpha\beta l}{(\alpha+\beta)}\left[\frac{1}{s^3}\left(1+2\frac{k_3}{s}+3\frac{k_2}{s^2}\right) - \frac{1}{r^3}\left(1+2\frac{k_1}{r}+3\frac{k_2}{r^2}\right)\right],\end{aligned}$$

the condition *y=0* guarantees the vanishing of $\frac{\partial U}{\partial y}$ and leads to the algebraic equation

$$x^2 + 2ax + a^2 - r^2 = 0.$$

Equilibrium points on the line joining A to B (3)

which is solved by the two roots

$$x_{\pm} = \epsilon r - a = \epsilon r - \frac{\beta l}{(\alpha + \beta)}, \quad \epsilon = \pm 1$$

Furthermore we have

$$s = r - \varepsilon l$$

Algebraic ninth degree equation

$$\sum_{n=0}^{9} B_n \gamma^n = 0$$

(where $r = \gamma l_P$)

Equilibrium points on the line joining A to B (4)

where

$$B_0 = -3(1+\rho_{\beta\alpha})^{-1}\rho^7\kappa_2,$$

$$B_1 = -(1+\rho_{\beta\alpha})^{-1}\Big[2(\rho_\alpha)\rho\kappa_1 - 12\varepsilon\kappa_2\Big]\rho^6,$$

$$B_2 = -(1+\rho_{\beta\alpha})^{-1}\Big[\rho^2 - 8\varepsilon(\rho_\alpha)\rho\kappa_1 + 18\kappa_2\Big]\rho^5,$$

$$B_3 = -(1+\rho_{\beta\alpha})^{-1}\Big[-4\varepsilon\rho^2 + 12(\rho_\alpha)\rho\kappa_1 - 12\varepsilon\kappa_2\Big]\rho^4,$$

$$B_4 = -(1+\rho_{\beta\alpha})^{-1}\Big[(6+(1+\varepsilon)\rho_{\beta\alpha})\rho^2 - \varepsilon(8\rho_\alpha\kappa_1 + 2(\rho_\beta)\rho_{\beta\alpha}\kappa_3)\rho + 3\kappa_2\Big]\rho^3,$$

$$B_5 = (1+\rho_{\beta\alpha})^{-1}\Big[((1+4\varepsilon)+(5+2\varepsilon)\rho_{\beta\alpha})\rho - 2\rho_\alpha\kappa_1 - 2(\rho_{\beta\alpha})\rho_\beta\kappa_3\Big]\rho^3,$$

$$B_6 = -(1+\rho_{\beta\alpha})^{-1}\Big[(1+4\varepsilon)+(1+10\varepsilon)\rho_{\beta\alpha}\Big]\rho^3,$$

$$B_7 = (1+\rho_{\beta\alpha})^{-1}\Big[2(3+5\rho_{\beta\alpha})\Big]\rho^2,$$

$$B_8 = -(1+\rho_{\beta\alpha})^{-1}\Big[\varepsilon(4+5\rho_{\beta\alpha})\Big]\rho,$$

$$B_9 = 1.$$

with

$$\rho_{\beta\alpha} \equiv \frac{\beta}{\alpha}$$

$$\rho_\alpha = \frac{(l_\alpha + l_m)}{l_P}, \quad \rho_\beta = \frac{(l_\beta + l_m)}{l_P}, \quad \rho = \frac{l}{l_P}.$$

Equilibrium points on the line joining A to B (5)

- In Newtonian theory $U_{,xx}|_{y=0} > 0$. In the quantum case we have

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{y=0} = \left[\frac{(\alpha+\beta)}{l^3} + 2\frac{\alpha}{r^3} + 2\frac{\beta}{s^3}\right] + 2\frac{\alpha}{r^4}\left(3k_1 + 6\frac{k_2}{r}\right) + 2\frac{\beta}{s^4}\left(3k_3 + 6\frac{k_2}{s}\right)$$

$$\left(3k_1 + 6\frac{k_2}{r}\right) + \frac{\beta}{\alpha}\left(\frac{r}{s}\right)^4\left(3k_3 + 6\frac{k_2}{s}\right) > 0$$

$\Rightarrow$ + choice

- In Newtonian theory $-a-l < l_1 < -a$. In our model we have

$$\left.\frac{\partial U}{\partial x}\right|_{x=-a-l} = -\frac{7}{4}\frac{\beta}{l^2} + \frac{1}{l^3}\left[\alpha\left(2k_1 + 3\frac{k_2}{l}\right) + \frac{\beta}{4}\left(k_3 + \frac{3}{4}\frac{k_2}{l}\right)\right]$$

$$2k_1 + 3\frac{k_2}{l} + \frac{\beta}{4\alpha}\left(k_3 + \frac{3}{4}\frac{k_2}{l}\right) < 0$$

$\Rightarrow$ – choice

Equilibrium points on the line joining A to B (6)

- Classically, L_2 lies between C and B. Thus

$$\left.\frac{\partial U}{\partial x}\right|_C = -(\alpha^3-\beta^3)\frac{(\alpha+\beta)^2}{\alpha^2\beta^2 l^2} - \left[\frac{\alpha}{a^3}\left(2k_1+3\frac{k_2}{a}\right) + \frac{\beta}{b^3}\left(2k_3+3\frac{k_2}{b}\right)\right].$$

$$k_1+\frac{3}{2}\frac{k_2}{a}+\frac{\beta}{\alpha}\left(\frac{a}{b}\right)^3\left(k_3+\frac{3}{2}\frac{k_2}{b}\right)>0$$

+ choice

- In Newtonian theory $\left.\frac{\partial^2 U}{\partial y^2}\right|_{L_1}$ <0. In quantum case

$$\left.\frac{\partial^2 U}{\partial y^2}\right|_{L_1} = \lambda = \frac{\alpha\beta l}{(\alpha+\beta)}\frac{1}{x}\left(\frac{1}{r^3}-\frac{1}{s^3}\right) + \frac{1}{x}\left[2\left(\frac{k_1}{r^4}-\frac{k_3}{s^4}\right)+3k_2\left(\frac{1}{r^5}-\frac{1}{s^5}\right)\right].$$

$$\left(\frac{k_1}{r^4}-\frac{k_3}{s^4}\right)+\frac{3}{2}k_2\left(\frac{1}{r^5}-\frac{1}{s^5}\right)>0$$

condition that can be violated with – choice

Equilibrium points on the line joining A to B (7)

- We also note that at L_2 we have

$$\left.\frac{\partial^2 U}{\partial y^2}\right|_{L_2} = \frac{(\alpha+\beta)}{l^3} - \frac{\alpha}{r^3} - \frac{\beta}{s^3} - \left[2\left(\alpha\frac{k_1}{r^4} + \beta\frac{k_3}{s^4}\right) + 3k_2\left(\frac{\alpha}{r^5} + \frac{\beta}{s^5}\right)\right]$$

$$\alpha\frac{k_1}{r^4} + \beta\frac{k_3}{s^4} + \frac{3}{2}k_2\left(\frac{\alpha}{r^5} + \frac{\beta}{s^5}\right) > 0$$

+ choice

- Classically $U(l_2) > U(l_3) > U(l_1)$.

$$2f[k_3(f^2-a^2)^2 - k_1(f^2-b^2)^2] + \frac{k_2[(b^2+3f^2)(f^2-a^2)^3 - (a^2+3f^2)(f^2-b^2)^3]}{(f^2-a^2)(f^2-b^2)} > 0$$

$$k_1 + \frac{1}{2}k_2\frac{(j^2+3l^2)}{l(l^2-j^2)} > 0$$

+ choice

$j = L_2B = BQ_3$

$f = Q_1C = CL_3$

Equilibrium points not lying on the line joining A to B (1)

Two equations of fifth degree describing equilibrium points

$$\sum_{k=0}^{5} \zeta_k w^k = 0, \sum_{k=0}^{5} \tilde{\zeta}_k u^k = 0,$$

where

$$\zeta_5 \equiv 1, \;\; \zeta_4 \equiv \frac{2}{3}\frac{\kappa_1}{\kappa_2}\frac{(l_m + l_\alpha)}{l_P^2}, \; \zeta_3 \equiv \frac{1}{3\kappa_2}\frac{1}{l_P^2},$$

$$\zeta_2 = \zeta_1 \equiv 0, \;\; \zeta_0 \equiv -\frac{1}{3\kappa_2}\frac{1}{l_P^2 l^3},$$

$$\widetilde{\zeta}_k = \zeta_k \; \forall k = 0, 1, 2, 3, 5, \;\; \widetilde{\zeta}_4 \equiv \frac{2}{3}\frac{\kappa_1}{\kappa_2}\frac{(l_m + l_\beta)}{l_P^2}.$$

Equilibrium points not lying on the line joining A to B (2)

There are two equilibrium points not lying on the line joining A to B, which we write in the form $L_4 \equiv (x(l),y_+(l))$ and $L_5 \equiv (x(l),y_-(l))$, where

$$x(l) \equiv \frac{(r^2(l) - s^2(l) + b^2 - a^2)}{2(a+b)}$$

$$y_\pm(l) \equiv \pm\sqrt{r^2(l) - x^2(l) - 2ax(l) - a^2}.$$

In Newtonian theory at the points L_4 and L_5 the planetoid is at the same distance from A and B. Our quantum corrected model predicts tiny displacement from this case.

Equilibrium points not lying on the line joining A to B (3)

For the Sun-Earth system the quantum corrected planetoid coordinates are

fifth decimal digit

$$x_Q = 7,479978 \cdot 10^{10} m, \; y_Q = 1,29573 \cdot 10^{11} m$$

fourth decimal digit

$$x_C = 7,479955 \cdot 10^{10} m, \; y_C = 1,29557 \cdot 10^{11} m$$

For the Earth-Moon system we have

$$x_Q = 1.8732985834 \cdot 10^8 m \quad y_Q = 3.32553755084 \cdot 10^8 m$$

$$x_C = 1.8732985825 \cdot 10^8 m \quad y_C = 3.32553755032 \cdot 10^8 m$$

tenth decimal digit

eleventh decimal digit

Stability of equilibrium points

Using first-order stability criterion, for quantum case we have found that

- Points L_1, L_2 and L_3 remain points of unstable equilibrium provided we adopt the *+ choice*.

- Points L_4 and L_5 remain points of first-order stable equilibrium if we use the *+ choice*.

Analytical solution of fifth degree equations (1)

Fifth degree equation for *1/r*

$$w^5 + \zeta_4 w^4 + \zeta_3 w^3 + \zeta_0 = 0,$$

We pass to dimensionless units by defining

$$w = \frac{1}{r} \equiv \frac{\gamma}{l_P}$$

$$\gamma^5 + \rho_4 \gamma^4 + \rho_3 \gamma^3 + \rho_0 = 0,$$

where:

$$\rho_4 \equiv \zeta_4 l_P = \frac{2}{3}\frac{\kappa_1}{\kappa_2}\frac{G(m+\alpha)}{c^2 l_P},$$

$$\rho_3 \equiv \zeta_3 l_P^2 = \frac{1}{3\kappa_2},$$

$$\rho_0 \equiv \zeta_0 l_P^5 = -\frac{1}{3\kappa_2}\left(\frac{l_P}{l}\right)^3 = -\rho_3 \left(\frac{l_P}{l}\right)^3$$

Analytical solution of fifth degree equations (2)

By means of a cubic Tschirnhaus transformation

$$Y = \gamma^3 + c_2\gamma^2 + c_1\gamma + c_0$$

Our equation becomes

$$Y^5 + d_4Y^4 + d_3Y^3 + d_2Y^2 + d_1Y + d_0 = 0$$

We require that

$$\begin{cases} d_2 = 0 \\ d_3 = 0 \\ d_4 = 0 \end{cases} \Longrightarrow \quad Y^5 + d_1Y + d_0 = 0$$

Bring-Jerrard quintic equation

Analytical solution of fifth degree equations (3)

Write the Bring-Jerrard equation in the following way

$$\gamma^5 + d_1\gamma + d_0 = 0.$$

Define the number

$$\sigma \equiv \frac{3125}{256}\frac{(-d_0)^4}{(-d_1)^5}.$$

Rescaling γ according to $\gamma = \chi\tilde{\gamma}$ we obtain

$$\tilde{\gamma}^5 + \frac{d_1}{\chi^4}\tilde{\gamma} + \frac{d_0}{\chi^5} = 0$$

Analytical solution of fifth degree equations (4)

if χ is choosen in such a way that

$$-\frac{d_1}{\chi^4} = 1 \Longrightarrow \chi = \chi(d_1) = (-d_1)^{\frac{1}{4}}$$

we obtain the quintic

$$\tilde{\gamma}^5 - \tilde{\gamma} - \tilde{\beta} = 0$$

with

$$\tilde{\beta} \equiv -\frac{d_0}{(\chi(d_1))^5}$$

$$\tilde{\sigma} = \frac{3125}{256}\left(-\frac{d_0}{(\chi(d_1))^5}\right)^4 = \sigma.$$

Analytical solution of fifth degree equations (5)

if $|\tilde{\sigma}| < 1$ then the roots of the quintic for $\tilde{\gamma}$ are obtained from the higher hypergeometric functions of order 4

$$\begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \\ \tilde{\gamma}_3 \\ \tilde{\gamma}_4 \end{pmatrix} = \begin{pmatrix} \mathrm{i} & \frac{\tilde{\beta}}{4} & \frac{5}{32}\mathrm{i}\tilde{\beta}^2 & -\frac{5}{32}\tilde{\beta}^3 \\ -1 & \frac{\tilde{\beta}}{4} & \frac{5}{32}\tilde{\beta}^2 & \frac{5}{32}\tilde{\beta}^3 \\ -\mathrm{i} & \frac{\tilde{\beta}}{4} & -\frac{5}{32}\mathrm{i}\tilde{\beta}^2 & -\frac{5}{32}\tilde{\beta}^3 \\ 1 & \frac{\tilde{\beta}}{4} & -\frac{5}{32}\tilde{\beta}^2 & \frac{5}{32}\tilde{\beta}^3 \end{pmatrix} \begin{pmatrix} F_0(\tilde{\sigma}) \\ F_1(\tilde{\sigma}) \\ F_2(\tilde{\sigma}) \\ F_3(\tilde{\sigma}) \end{pmatrix}$$

$$\tilde{\gamma}_5 = -\tilde{\beta} F_1(\tilde{\sigma}),$$

Analytical solution of fifth degree equations (6)

where

$$F : \tilde{\sigma} \to F(\tilde{\sigma}) \equiv F \begin{pmatrix} a_1, & a_2, & ..., & a_{n-2}, & a_{n-1} \\ b_1, & b_2, & ..., & b_{n-2}, & \tilde{\sigma} \end{pmatrix} = \sum_{s=0}^{\infty} C_s \tilde{\sigma}^s$$

and the coefficients

$$C_0 \equiv 1,\ C_s \equiv \frac{(a_1, s)(a_2, s)...(a_{n-2}, s)(a_{n-1}, s)}{(1, s)(b_1, s)...(b_{n-3}, s)(b_{n-2}, s)},$$

$$(\lambda, \mu) \equiv \lambda(\lambda + 1)(\lambda + 2)...(\lambda + \mu - 1),$$

Analytical solution of fifth degree equations (7)

we have that

$$F_0(\tilde{\sigma}) \equiv F\begin{pmatrix} -\frac{1}{20}, & \frac{3}{20}, & \frac{7}{20}, & \frac{11}{20} \\ \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4}, & \tilde{\sigma} \end{pmatrix},$$

$$F_1(\tilde{\sigma}) \equiv F\begin{pmatrix} \frac{1}{5}, & \frac{2}{5}, & \frac{3}{5}, & \frac{4}{5} \\ \frac{1}{2}, & \frac{3}{4}, & \frac{5}{4}, & \tilde{\sigma} \end{pmatrix},$$

$$F_2(\tilde{\sigma}) \equiv F\begin{pmatrix} \frac{9}{20}, & \frac{13}{20}, & \frac{17}{20}, & \frac{21}{20} \\ \frac{3}{4}, & \frac{5}{4}, & \frac{3}{2}, & \tilde{\sigma} \end{pmatrix},$$

$$F_3(\tilde{\sigma}) \equiv F\begin{pmatrix} \frac{7}{10}, & \frac{9}{10}, & \frac{11}{10}, & \frac{13}{10} \\ \frac{5}{4}, & \frac{3}{2}, & \frac{7}{4}, & \tilde{\sigma} \end{pmatrix}.$$

Analytical solution of fifth degree equations (8)

With this refined analysis, we have obtained for the planetoid coordinates in the Earth-Moon system the corrections

$$x_Q - x_C \approx 8.7894 \text{ mm}, \ |y_Q| - |y_C| \approx -4 \text{ mm}$$

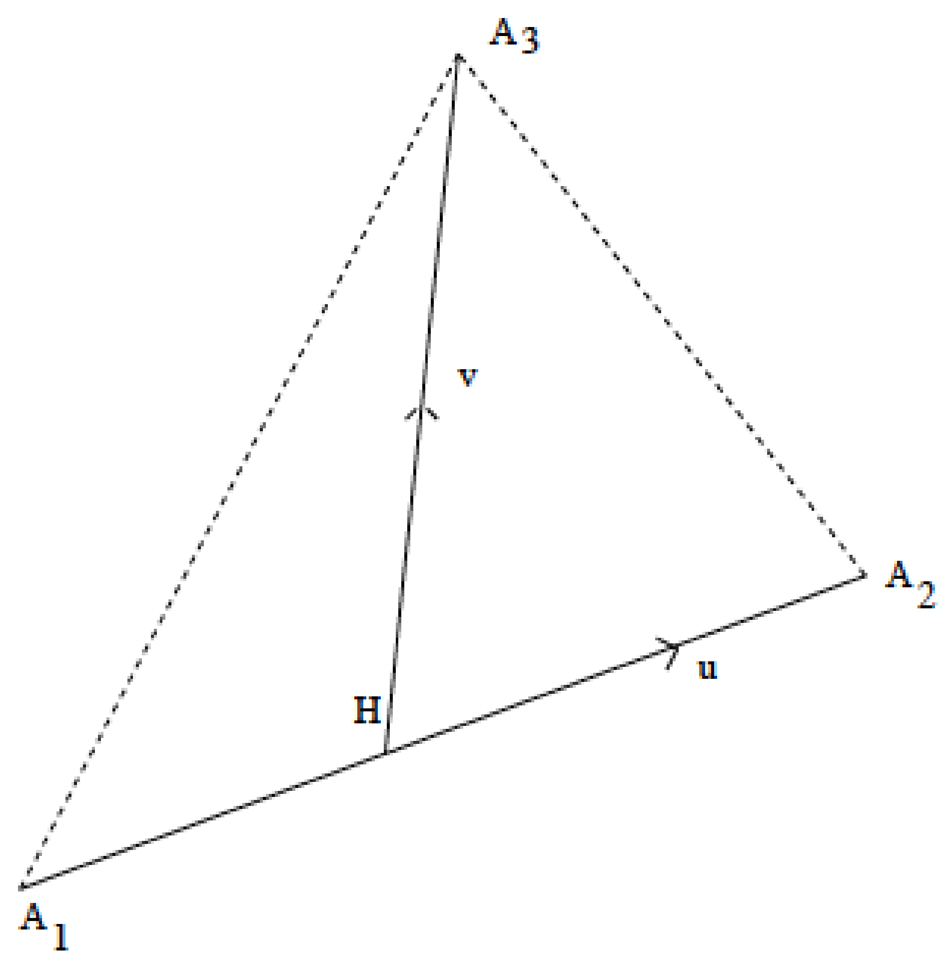
The numerical method gives

$$x_Q - x_C \approx 8.8 \text{ mm}, \ |y_Q| - |y_C| \approx 5.2 \text{ mm}$$

Introduction: Full 3-Body problem

- Bodies A_1, A_2 and A_3 with masses m_1, m_2 and m_3
- ***u***=*(x,y,z)* and ***v*** = *(ξ,η,ζ)*
- *H* center of mass of A_1 and A_2
- Equation of motion :

$$\begin{pmatrix} m\frac{\mathrm{d}^2}{\mathrm{d}t^2}+A & -B \\ -B & \mu\frac{\mathrm{d}^2}{\mathrm{d}t^2}+C \end{pmatrix}\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix}=0$$



where ($\alpha_1=m_1/(m_1+m_2)$, $\alpha_2=1-m_1$):

$$m \equiv \frac{m_1 m_2}{(m_1+m_2)},\ \mu \equiv \frac{(m_1+m_2)m_3}{(m_1+m_2+m_3)}$$

$$A \equiv -\frac{\alpha_1^2}{r_1}U_{,r_1} - \frac{\alpha_2^2}{r_2}U_{,r_2} - \frac{1}{r_3}U_{,r_3},$$

$$B \equiv \frac{\alpha_2}{r_2}U_{,r_2} - \frac{\alpha_1}{r_1}U_{,r_1},$$

$$C \equiv -\frac{1}{r_1}U_{,r_1} - \frac{1}{r_2}U_{,r_2}.$$

$$(r_1)^2 \equiv (-\alpha_1\vec{u}+\vec{v})\cdot(-\alpha_1\vec{u}+\vec{v}) = (-\alpha_1 x+\xi)^2+(-\alpha_1 y+\eta)^2+(-\alpha_1 z+\zeta)^2,$$

$$(r_2)^2 \equiv (\alpha_2\vec{u}+\vec{v})\cdot(\alpha_2\vec{u}+\vec{v}) = (\alpha_2 x+\xi)^2+(\alpha_2 y+\eta)^2+(\alpha_2 z+\zeta)^2,$$

$$(r_3)^2 \equiv \vec{u}\cdot\vec{u} = x^2+y^2+z^2.$$

A choice of quantum corrected potential

the potential $U(r_1, r_2, r_3)$ assumes the form

$$\begin{aligned} U(r_1, r_2, r_3) = &\frac{Gm_2m_3}{r_1}\left(1 + \kappa_{23}\frac{G}{c^2}\frac{(m_2+m_3)}{r_1} + \kappa\frac{l_P^2}{(r_1)^2}\right) \\ &+ \frac{Gm_1m_3}{r_2}\left(1 + \kappa_{13}\frac{G}{c^2}\frac{(m_1+m_3)}{r_2} + \kappa\frac{l_P^2}{(r_2)^2}\right) \\ &+ \frac{Gm_1m_2}{r_3}\left(1 + \kappa_{12}\frac{G}{c^2}\frac{(m_1+m_2)}{r_3} + \kappa\frac{l_P^2}{(r_3)^2}\right) \end{aligned}$$

Therefore we have

$$U_{,r_1} = -\frac{Gm_2m_3}{(r_1)^2}\left(1 + 2\kappa_{23}\frac{G}{c^2}\frac{(m_2+m_3)}{r_1} + 3\kappa\frac{l_P^2}{(r_1)^2}\right)$$

$$U_{,r_2} = -\frac{Gm_1m_3}{(r_2)^2}\left(1 + 2\kappa_{13}\frac{G}{c^2}\frac{(m_1+m_3)}{r_2} + 3\kappa\frac{l_P^2}{(r_2)^2}\right)$$

$$U_{,r_3} = -\frac{Gm_1m_2}{(r_3)^2}\left(1 + 2\kappa_{12}\frac{G}{c^2}\frac{(m_1+m_2)}{r_3} + 3\kappa\frac{l_P^2}{(r_3)^2}\right)$$

Hamiltonian equations of motion (1)

Hamiltonian equations of motion read as

$$\frac{\mathrm{d}}{\mathrm{d}t}x = p_x \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_x = -\frac{1}{m}(Ax - B\xi)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}y = p_y \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_y = -\frac{1}{m}(Ay - B\eta)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z = p_z \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_z = -\frac{1}{m}(Az - B\zeta)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi = p_\xi \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_\xi = -\frac{1}{\mu}(C\xi - Bx)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta = p_\eta \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_\eta = -\frac{1}{\mu}(C\eta - By)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta = p_\zeta \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_\zeta = -\frac{1}{\mu}(C\zeta - Bz)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i = \frac{\partial F}{\partial y_i}, \quad \frac{\mathrm{d}}{\mathrm{d}t}y_i = -\frac{\partial F}{\partial x_i}$$

where:

$$x_i \equiv (x, y, z, \xi, \eta, \zeta) \equiv (x_1, \ldots, x_6)$$

$$y_i \equiv (p_x, p_y, p_z, p_\xi, p_\eta, p_\zeta) \equiv (p_1, \ldots, p_6)$$

$$F(x_1, \ldots, x_6, y_1, \ldots, y_6) = \sum_{i=1}^{6} \frac{y_i^2}{2} + f(x_1, \ldots, x_6)$$

Hamiltonian equations of motion (2)

- At this stage we can exploit the Poincaré theorem:

if our Hamiltonian equations, which depend on a parameter $\rho = l_P$, possess for $\rho = 0$ a periodic solution whose characteristic exponents are all nonvanishing, they have again a periodic solution for small values of ρ.

- In our case for $\rho = 0$ we revert to three-body problem in post-Newtonian mechanics.

A. Chenciner and R. Montgomery, Ann. Math. 152, 881 (2000).

G. Huang and X. Wu, Phys. Rev. D 89, 124034 (2014).

Variational equations (1)

Assume a periodic solution of our Hamiltonian equations has been found

$$x_i = \varphi_i(t),\ y_i = \psi_i(t)$$

Consider small disturbances of these periodic solutions

$$\tilde{x}_i = \varphi_i(t) + \xi_i,\ \tilde{y}_i = \psi_i(t) + \eta_i$$

Variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_i = \sum_{k=1}^{6}\left[F_{,y_i x_k}\xi_k + F_{,y_i y_k}\eta_k\right],$$

$$\frac{\mathrm{d}\eta_i}{\mathrm{d}t} = -\sum_{k=1}^{6}\left[F_{,x_i x_k}\xi_k + F_{,x_i y_k}\eta_k\right]$$

Variational equations (2)

We try to integrate variational equations by setting

$$\xi_i = \mathrm{e}^{\alpha t} S_i, \;\; \eta_i = \mathrm{e}^{\alpha t} T_i$$

the constant *α* is called characteristic exponent.

If, when *ρ=0*, the characteristic exponents are vanishing, then for small but nonvanishing values of *ρ* we have the asymptotic expansions

$$\alpha \sim \sum_{j=1}^{N} \alpha_j \rho^{\frac{j}{2}}$$

$$S_i \sim \sum_{l=0}^{N} S_i^l \rho^{\frac{l}{2}}$$

$$T_i \sim \sum_{l=0}^{N} T_i^l \rho^{\frac{l}{2}}$$

Variational equations (3)

If the Hamiltonian has the asymptotic expansion

$$F \sim F_0 + \rho F_1 + \rho^2 F_2 + \mathrm{O}(\rho^3) \quad ,$$

by virtue of the previous expansions, variational equations give

$$\frac{\mathrm{d}S_i^0}{\mathrm{d}t} = \sum_{k=1}^{6} \Big(F_{0,y_i x_k} S_k^0 + F_{0,y_i y_k} T_k^0 \Big),$$

$$\alpha_1 S_i^0 + \frac{\mathrm{d}S_i^1}{\mathrm{d}t} = \sum_{k=1}^{6} \Big(F_{0,y_i x_k} S_k^1 + F_{0,y_i y_k} T_k^1 \Big),$$

$$\alpha_1 S_i^1 + \alpha_2 S_i^0 + \frac{\mathrm{d}S_i^2}{\mathrm{d}t} = \sum_{k=1}^{6} \Big(F_{0,y_i x_k} S_k^2 + F_{1,y_i x_k} S_k^0 + F_{0,y_i y_k} T_k^2 + F_{1,y_i y_k} T_k^0 \Big),$$

$$\frac{\mathrm{d}T_i^0}{\mathrm{d}t} = -\sum_{k=1}^{6} \Big(F_{0,x_i x_k} S_k^0 + F_{0,x_i y_k} T_k^0 \Big),$$

$$\alpha_1 T_i^0 + \frac{\mathrm{d}T_i^1}{\mathrm{d}t} = -\sum_{k=1}^{6} \Big(F_{0,x_i x_k} S_k^1 + F_{0,x_i y_k} T_k^1 \Big),$$

$$\alpha_1 T_i^1 + \alpha_2 T_i^0 + \frac{\mathrm{d}T_i^2}{\mathrm{d}t} = -\sum_{k=1}^{6} \Big(F_{0,x_i x_k} S_k^2 + F_{1,x_i x_k} S_k^0 + F_{0,x_i y_k} T_k^2 + F_{1,x_i y_k} T_k^0 \Big)$$

Variational equations (4)

In our case we have

$$F_{0,x_i y_k} = f_{0,x_i y_k} = 0,\ F_{0,y_i x_k} = (y_i)_{,x_k} = 0,\ F_{0,y_i y_k} = \delta_{ik},\ F_{0,x_i x_k} = f_{0,x_i x_k}$$

Therefore, on writing

$$f(x_1, \ldots, x_6) = f_0(x_1, \ldots, x_6) + f_2(x_1, \ldots, x_6)\rho^2$$

$$A = A_0 + \rho^2 A_2,\ B = B_0 + \rho^2 B_2, C = C_0 + \rho^2 C_2$$

we have found, for all *i=1,…,6* the general form of variational equations

$$\sum_{k=1}^{6} \begin{pmatrix} \delta_{ik}\frac{\mathrm{d}}{\mathrm{d}t} & -\delta_{ik} \\ M^0_{ik} & \delta_{ik}\frac{\mathrm{d}}{\mathrm{d}t} \end{pmatrix} \begin{pmatrix} S^0_k \\ T^0_k \end{pmatrix} = 0$$

Variational equations (5)

where $M_{ik}^{0} \equiv f_{0,x_i x_k}$

while for higher-order terms we find the inhomogeneous equations

$$\sum_{k=1}^{6} \begin{pmatrix} \delta_{ik} \frac{\mathrm{d}}{\mathrm{d}t} & -\delta_{ik} \\ M_{ik}^{0} & \delta_{ik} \frac{\mathrm{d}}{\mathrm{d}t} \end{pmatrix} \begin{pmatrix} S_k^n \\ T_k^n \end{pmatrix} = -\sum_{l=0}^{n-1} \alpha_{n-l} \begin{pmatrix} S_i^l \\ T_i^l \end{pmatrix}$$

Variational equations (6)

What if the characteristic exponent does not vanish at $\rho=0$?

In that case the asymptotic expansion should be generalized by adding α_0

$$\alpha \sim \sum_{l=0}^{N} \alpha_l \rho^{\frac{l}{2}}$$

$$\sum_{k=1}^{6} \begin{pmatrix} \delta_{ik}\left(\frac{\mathrm{d}}{\mathrm{d}t} + \alpha_0\right) & -\delta_{ik} \\ M_{ik}^{0} & \delta_{ik}\left(\frac{\mathrm{d}}{\mathrm{d}t} + \alpha_0\right) \end{pmatrix} \begin{pmatrix} S_k^0 \\ T_k^0 \end{pmatrix} = 0,$$

$$\sum_{k=1}^{6} \begin{pmatrix} \delta_{ik}\left(\frac{\mathrm{d}}{\mathrm{d}t} + \alpha_0\right) & -\delta_{ik} \\ M_{ik}^{0} & \delta_{ik}\left(\frac{\mathrm{d}}{\mathrm{d}t} + \alpha_0\right) \end{pmatrix} \begin{pmatrix} S_k^n \\ T_k^n \end{pmatrix} = -\sum_{l=0}^{n-1} \alpha_{n-l} \begin{pmatrix} S_i^l \\ T_i^l \end{pmatrix}$$

Variational equations (7)

The periodic solutions

$$x_i = \varphi_i(t),\ y_i = \psi_i(t)$$

can be taken to be solutions of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i = \frac{\partial F}{\partial y_i},\ \frac{\mathrm{d}}{\mathrm{d}t}y_i = -\frac{\partial F}{\partial x_i}$$

when $\rho = 0$. Therefore the matrix M^0_{ik} should be evaluated along the solutions of the coupled equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = y_i\ \forall i = 1, ..., 6,$$

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{1}{m}(A_0 x_i - B_0 x_{i+3})\ \forall i = 1, 2, 3,$$

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{1}{\mu}(C_0 x_i - B_0 x_{i-3})\ \forall i = 4, 5, 6.$$

Variational equations (8)

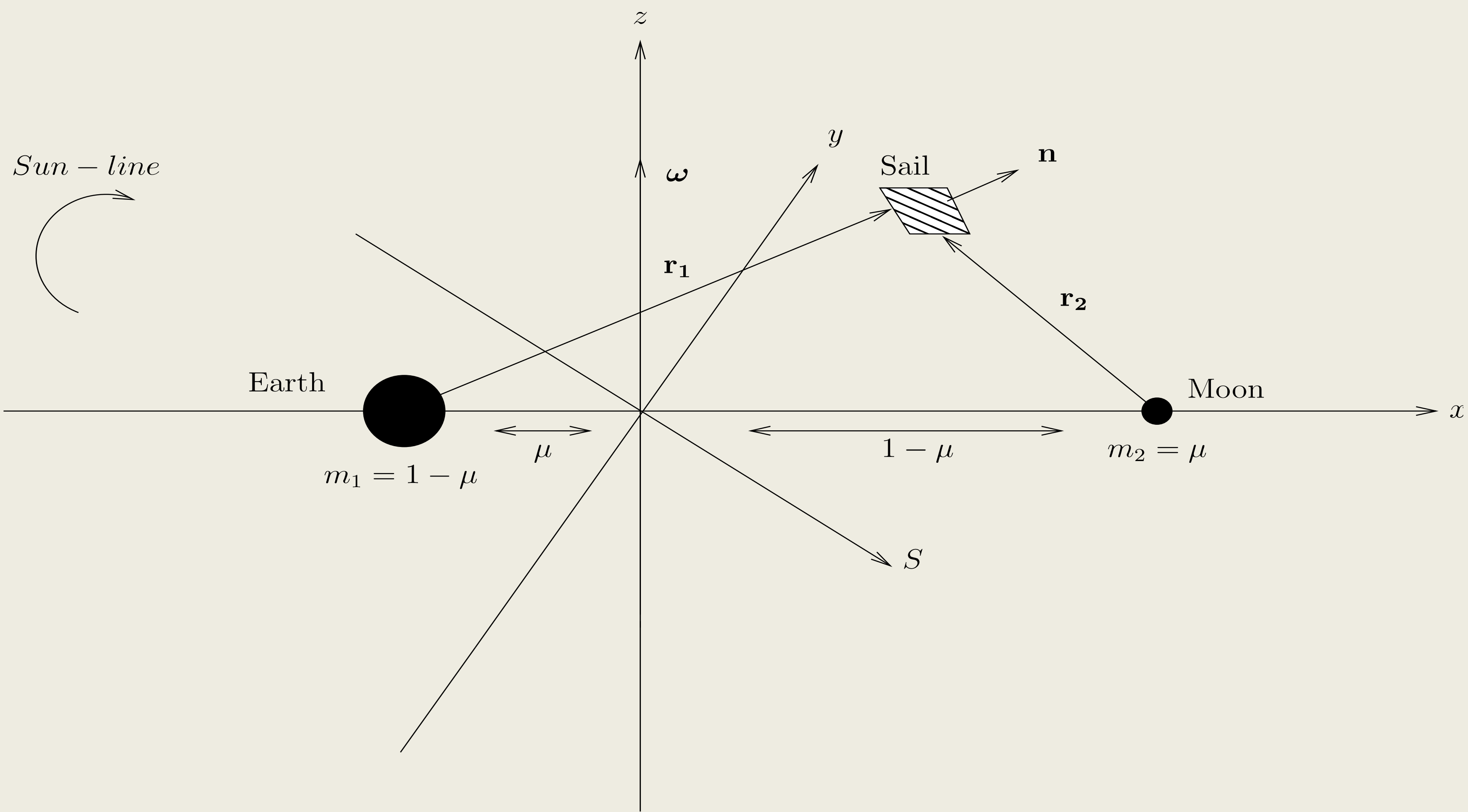
The desired periodic solutions can be written in the form

$$x_i = \sum_{l=0}^{\infty} D_{il} \sin(\omega_{il} t + \varphi_{il})$$

$$y_i = \sum_{l=0}^{\infty} E_{il} \sin(\omega_{il} t + \gamma_{il})$$

Displaced periodic orbits (1)

displaced periodic orbits describe the dynamics of the planetoid (e.g. a solar sail) in the neighborhood of the Lagrangian points.



Displaced periodic orbits (2)

Vector dynamical equation for the solar sail

$$\frac{d^2\boldsymbol{r}}{dt^2} + 2\omega \times \frac{d\boldsymbol{r}}{dt} + \nabla U(\boldsymbol{r}) = \boldsymbol{a}$$

where

$$\boldsymbol{a} \;=\; a_0(\boldsymbol{S}\cdot\boldsymbol{n})^2\boldsymbol{n}$$

$$\boldsymbol{n} = \begin{bmatrix} \cos(\gamma)\cos(\omega_\star t) & -\cos(\gamma)\sin(\omega_\star t) & \sin(\gamma) \end{bmatrix}^T,$$

$$\boldsymbol{S} = \begin{bmatrix} \cos(\omega_\star t) & -\sin(\omega_\star t) & 0 \end{bmatrix}^T,$$

$$\omega_\star \;=\; 0.923$$

Displaced periodic orbits (3)

Linear variational equations at Lagrangian points L_4 and L_5

$$\ddot{\xi} - 2\dot{\eta} = U^0_{xx}\xi + U^0_{xy}\eta + a_\xi,$$

$$\ddot{\eta} + 2\dot{\xi} = U^0_{xy}\xi + U^0_{yy}\eta + a_\eta,$$

$$\ddot{\zeta} = U^0_{zz}\zeta + a_\zeta,$$

Assume that a solution is periodic of the form

$$\xi(t) = A_\xi \cos(\omega_\star t) + B_\xi \sin(\omega_\star t),$$

$$\eta(t) = A_\eta \cos(\omega_\star t) + B_\eta \sin(\omega_\star t),$$

Displaced periodic orbits (4)

linear system for the four amplitudes

$$-(\omega_\star^2 - U_{xx}^0)B_\xi + 2\omega_\star A_\eta - U_{xy}^0 B_\eta = 0,$$

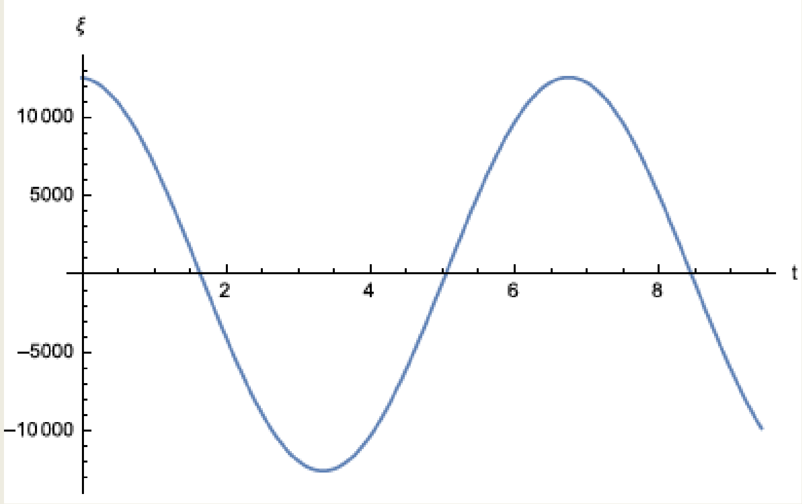
$$U_{xy}^0 A_\xi + 2\omega_\star B_\xi - (\omega_\star^2 - U_{yy}^0)A_\eta = 0,$$

$$-(\omega_\star^2 - U_{xx}^0)A_\xi + U_{xy}^0 A_\eta - 2\omega_\star B_\eta = a_0 \cos^3\gamma,$$

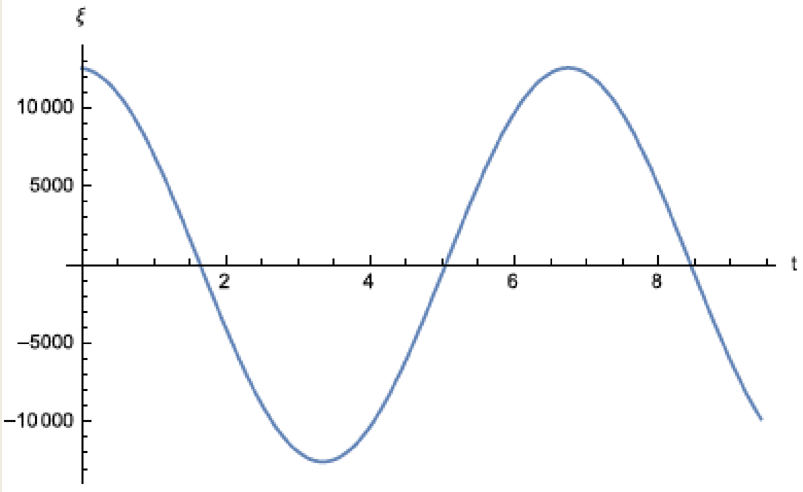
$$-2\omega_\star A_\xi + U_{xy}^0 B_\xi - (\omega_\star^2 - U_{yy}^0)B_\eta = -a_0 \cos^3\gamma$$

Displaced periodic orbits (5)

- Time evolution of ξ for L4 in Newtonian case

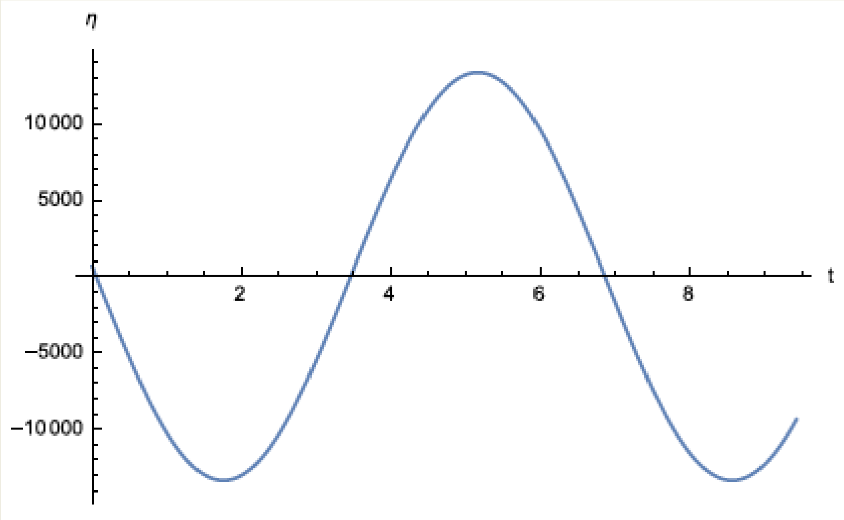


- Time evolution of ξ for L4 in quantum-corrected model

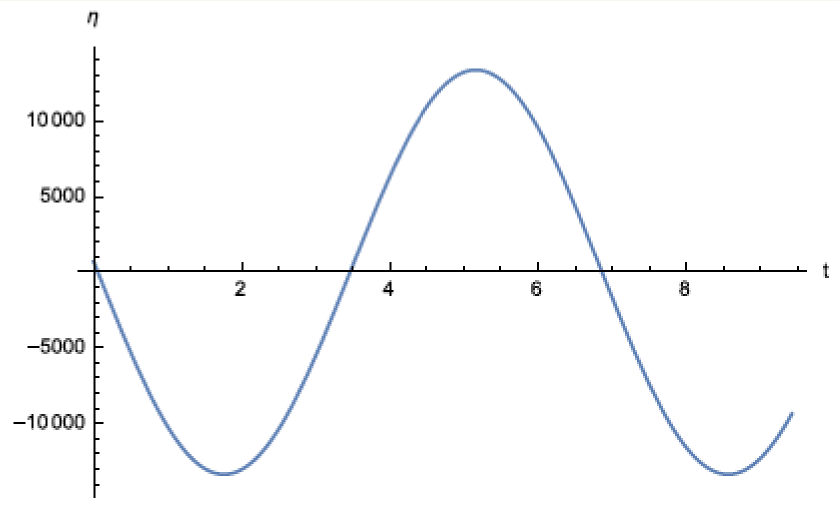


Displaced periodic orbits (6)

- Time evolution of η for L4 in Newtonian case



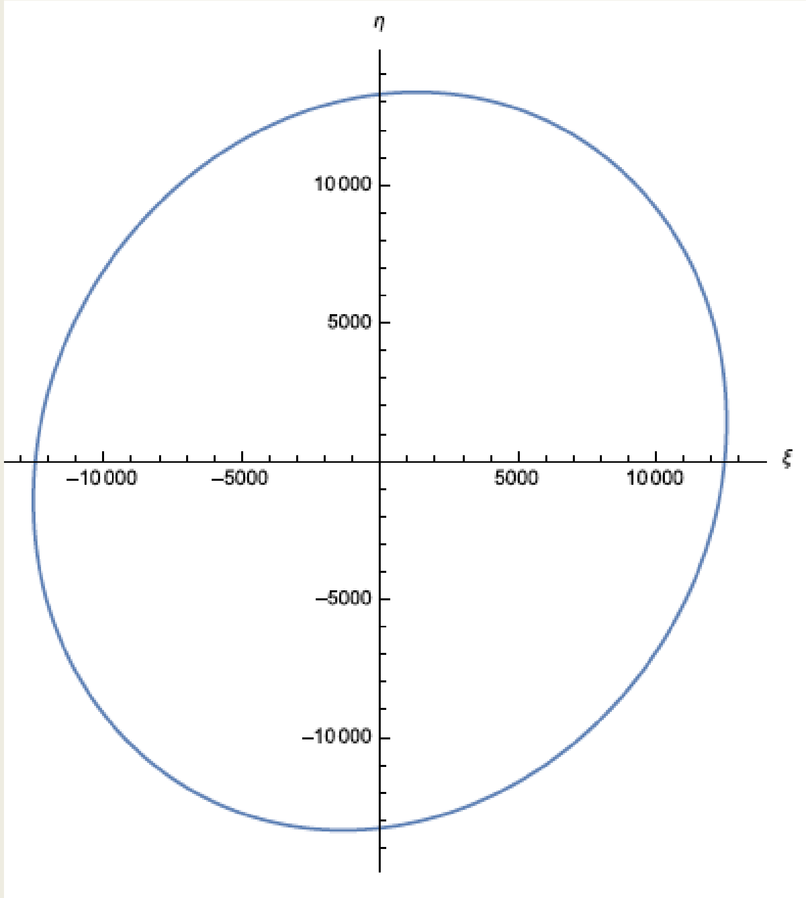
- Time evolution of η for L4 in quantum-corrected model

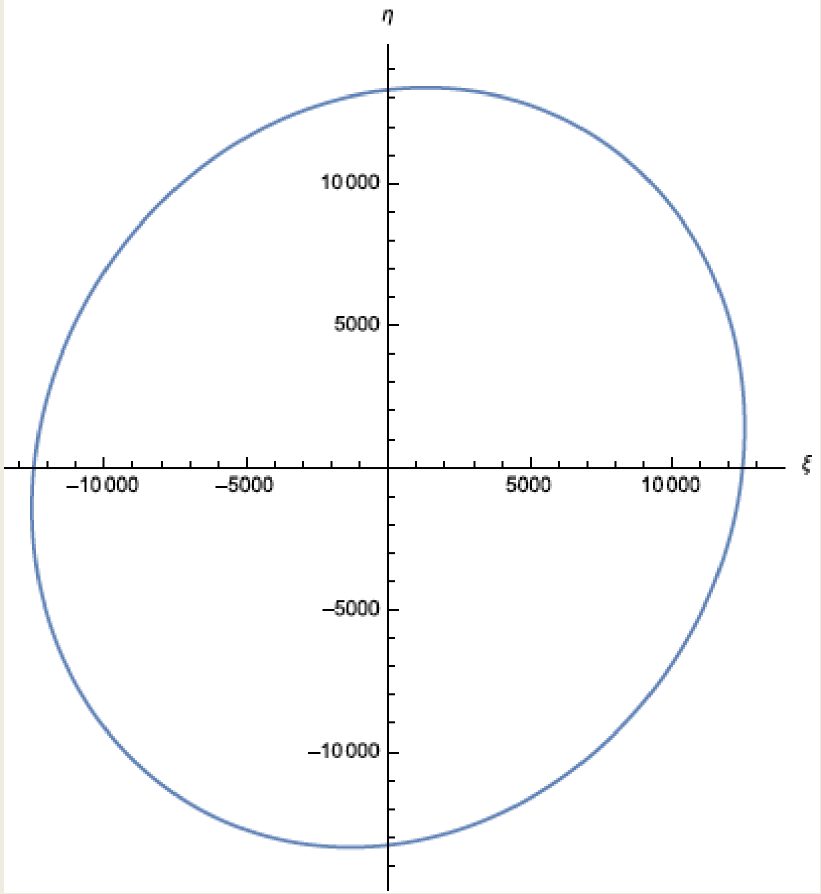


Displaced periodic orbits (7)

Periodic orbits at linear order

around L4 in Newtonian theory





Periodic orbits at linear order

around L4 in the quantum case

Displaced periodic orbits (8)

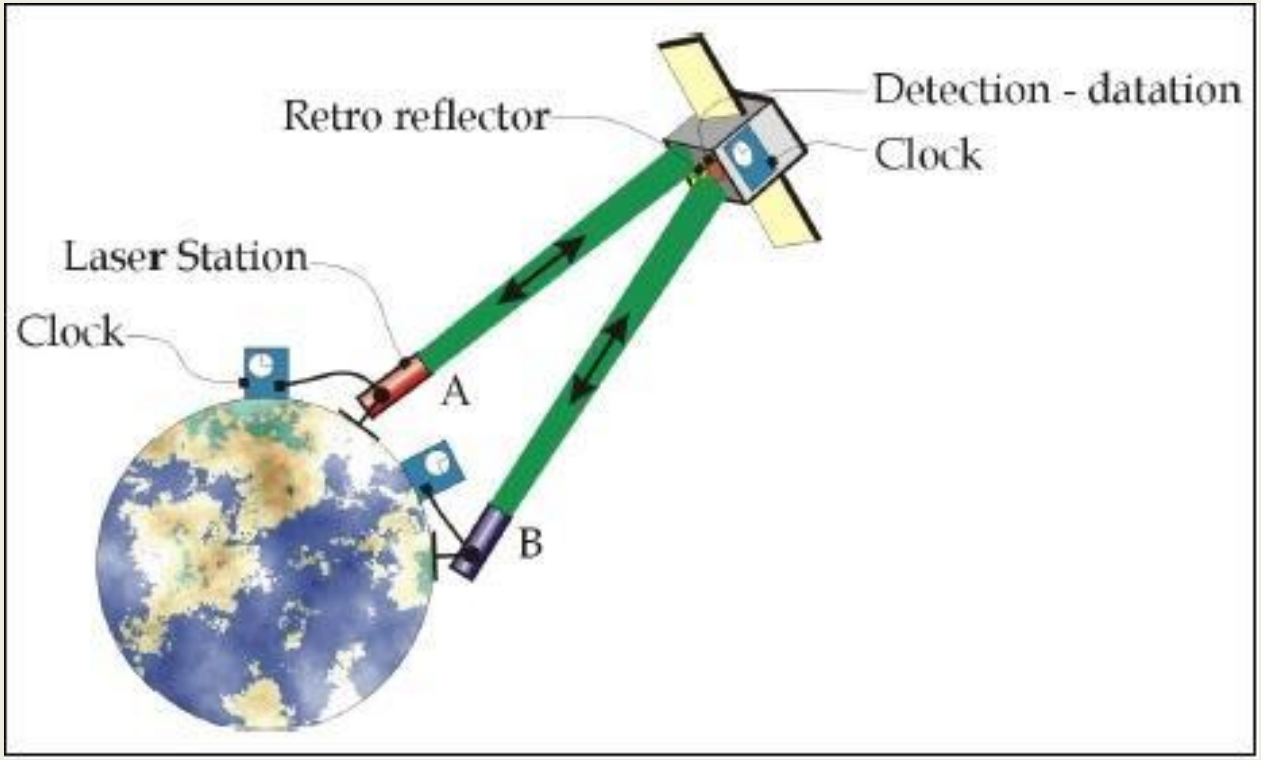
Linear variational equations at Lagrangian points L_1, $L_2$and L_3

$$
\begin{aligned}
\ddot{\xi} - 2\dot{\eta} - U^o_{xx}\xi &= a_\xi\,, \\
\ddot{\eta} + 2\dot{\xi} - U^o_{yy}\eta &= a_\eta\,, \\
\ddot{\zeta} - U^o_{zz}\zeta &= a_\zeta\,,
\end{aligned}
$$

periodic solution

$$
\begin{aligned}
\xi(t) &= \xi_0 \cos(\omega_\star t), \\
\eta(t) &= \eta_0 \sin(\omega_\star t).
\end{aligned}
$$

Laser ranging technique (1)



- Very short pulse of light is fired towards satellites equipped with cube corner retro-reflectors.
- The round trip time of flight is measured.

Laser ranging technique (2)

Key Performance Indicators to design an appropriate laser ranging test mass:

- Adequate laser return signal.
- Acceptable rejection of the unavoidable nongravitational perturbations.
- Optimization/minimization of S/M.
- Time-durability of the test mass to prolonged measurements.

Conclusions and open problems (1)

- We have seen some numerical values that give an idea about the tiny quantum corrections on the equilibrium points of the planetoid. In the Earth-Moon system these corrections are of 8.8 mm on the abscissa and of 4 mm on the ordinate, whereas for Sun-Earth system there are larger corrections.
- We have derived a recursive scheme for the analysis of variational equations of the full three-body problem, although we have not solved them explicitly.
- The evaluation of periodic solutions of the full three-body problem within the post-Newtonian regime is still in its infancy: only results for the circular restricted three-body problem are available so far.
- We have shown that displaced periodic orbits around unperturbed circular motion exist at all libration points also in the quantum-corrected case.

Conclusions and open problems (2)

- Satellite laser ranging/lunar laser ranging represent valid techiques to measure the quantum effects. Detecting tiny departures from classical gravity is a challenging task and the years to come will hopefully tell us whether the scheme described may have observational consequences in orbital motion physics and in the experimental search for quantum gravity effects.
- The precise characterization of regions of stability and instability of displaced periodic orbits is a fascinating problem for the years to come.
- The evaluation of the amount of perturbation of a fourth body on the position of Lagrangian points (in a purely classical scheme) is an open problem.