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Approach to Equilibrium for Lattice Nonlinear Schrödinger Equation NLS

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- interacting bosons on $\mathbb{Z}$ cold atoms

$\Rightarrow$ classical field theory

$$H = \sum_{j \in \mathbb{Z}} \left\{ \frac{1}{2} |\psi_{j+1} - \psi_j|^2 + \frac{1}{2} g\, |\psi_j|^4 \right\} \qquad g > 0$$

defocusing

$$\rightsquigarrow \quad \left\| \; i \partial_t \psi_j = -\frac{1}{2} \Delta \psi_j + g |\psi_j|^2 \psi_j \; \right\| , \quad j \in \mathbb{Z} , \quad \psi_j \in \mathbb{C}$$

continuum NLS dynamics is integrable

different !

- finite temperature, approach to equilibrium

- conserved fields

$$\rho_j = |\psi_j|^2 \qquad \sum_j \rho_j = N$$

$$e_j = \tfrac{1}{2}|\psi_{j+1} - \psi_j|^2 + \tfrac{1}{2} g |\psi_j|^4$$

no other

- for local observables

limit $t \to \infty$: $\quad \frac{1}{Z} e^{-\beta(H - \mu N)} \prod_j d\psi_j \, d\psi_j^*$

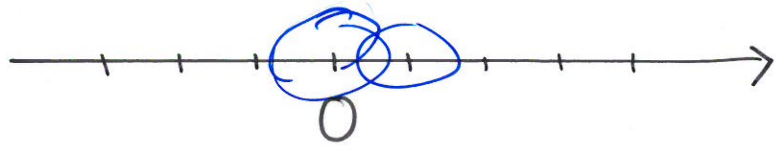
initial conditions

HERE

late stage

return to equilibrium

local perturbation



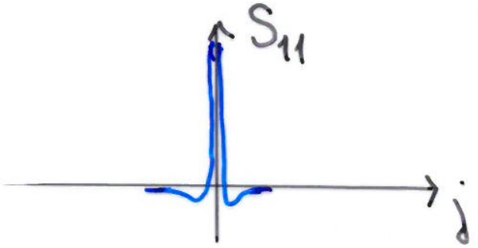
$e^{-\beta(H-\mu N)}$

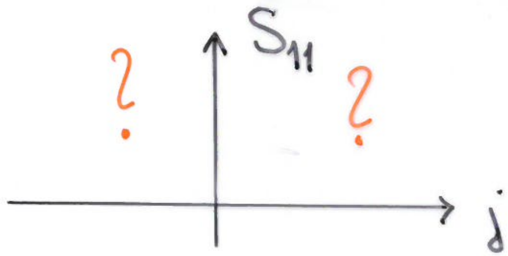
given

problem:

$$\langle \underbrace{|\psi_j(t)|^2}_{\text{response}} \; \underbrace{|\psi_0(0)|^2}_{\text{perturbation}} \rangle - \langle |\psi_0(0)|^2 \rangle^2 = S_{11}(j,t)$$

t = 0

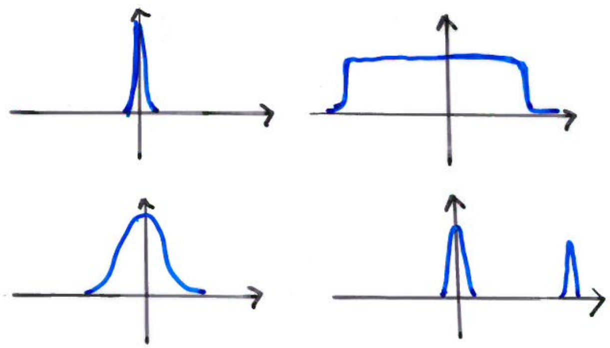




$t > 0$

A □   □ B

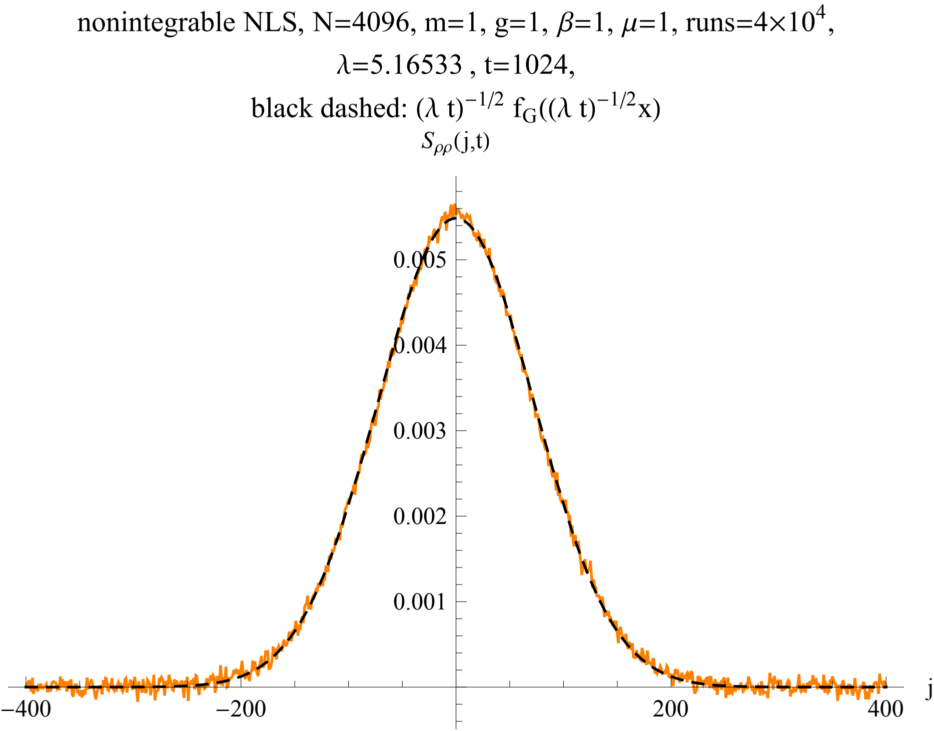
C □   □ D

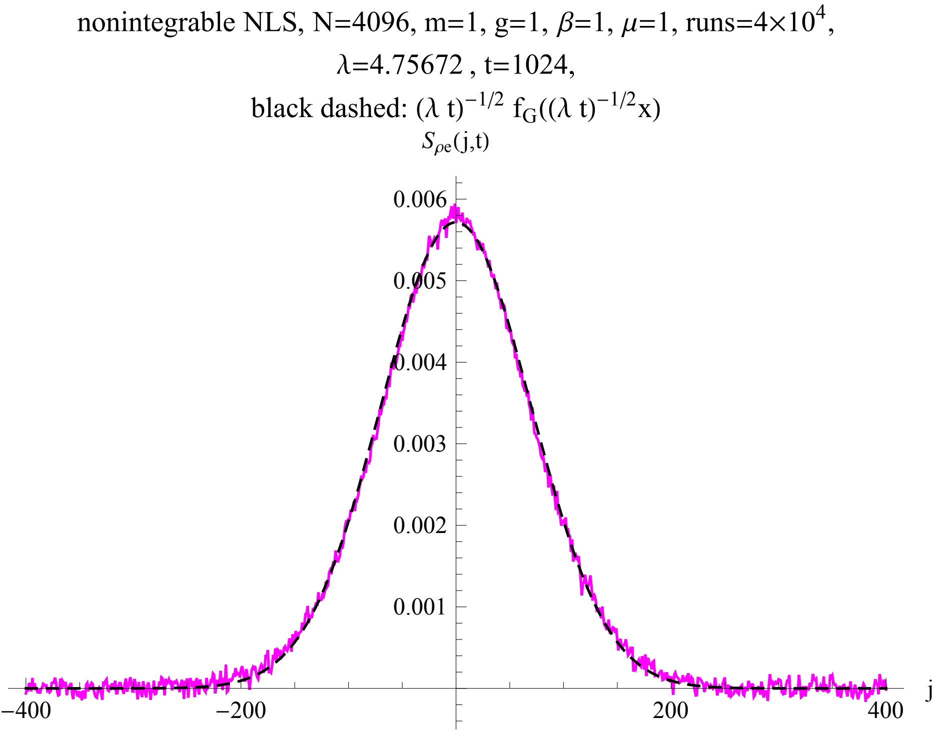


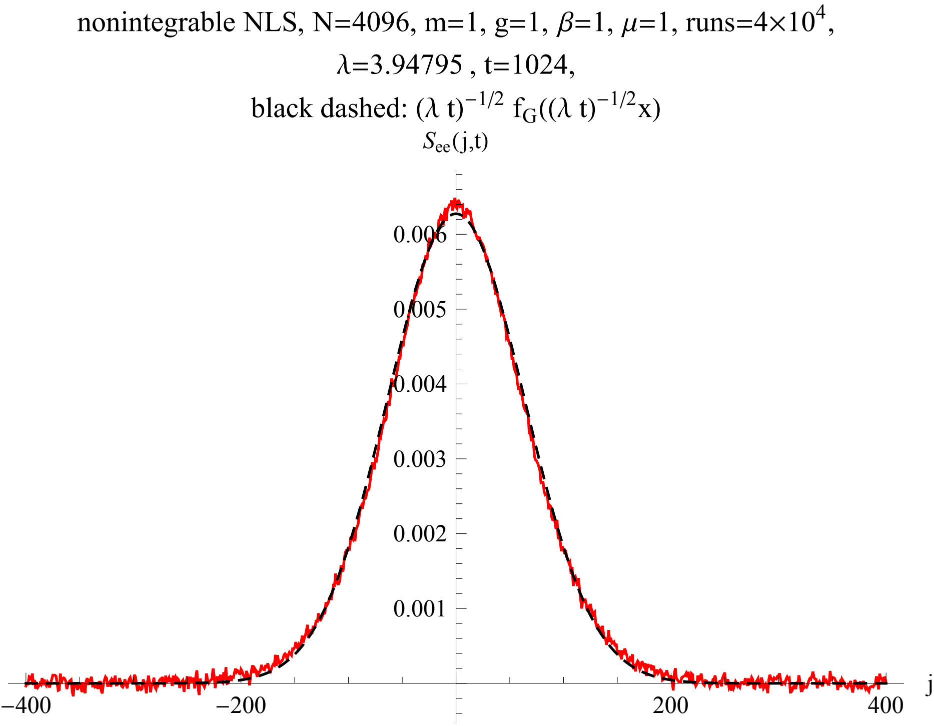
MD simulations

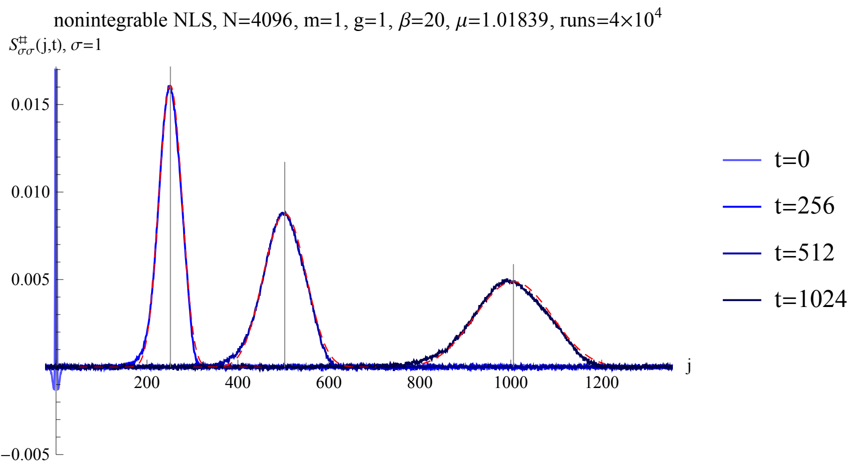
size $N = 4096$, periodic b.c., $t_{max} = 1024$

- $\beta = 1$, $\mu = 1$
- $\beta = 20$, $\mu = 1$


nonintegrable NLS, N=4096, m=1, g=1, β=1, μ=1, runs=4×10^4,
λ=5.16533 , t=1024,
black dashed: (λ t)^−1/2 f_G((λ t)^−1/2 x)
S_ρρ(j,t)
0.005
0.004
0.003
0.002
0.001
j
−400
−200
200
400


nonintegrable NLS, N=4096, m=1, g=1, β=1, μ=1, runs=4×10^4,
λ=4.75672 , t=1024,
black dashed: (λ t)^−1/2 f_G((λ t)^−1/2 x)
S_ρe(j,t)
0.006
0.005
0.004
0.003
0.002
0.001
−400
−200
200
400
j

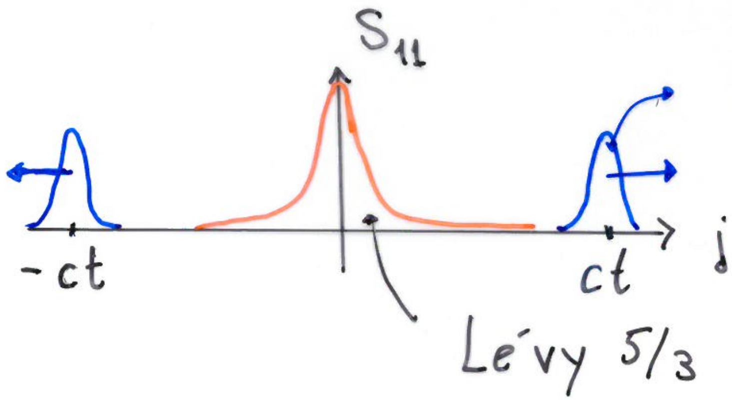

nonintegrable NLS, N=4096, m=1, g=1, β=1, μ=1, runs=4×10^4,
λ=3.94795 , t=1024,
black dashed: $(\lambda\ t)^{-1/2}\ f_G((\lambda\ t)^{-1/2}x)$
$S_{ee}(j,t)$
0.006
0.005
0.004
0.003
0.002
0.001
−400
−200
200
400
j



nonintegrable NLS, N=4096, m=1, g=1, β=20, μ=1.01839, runs=4×10^4
S^♯_σσ(j,t), σ=1
0.015
0.010
0.005
−0.005
200
400
600
800
1000
1200
j
t=0
t=256
t=512
t=1024

- high temperatures diffusive $\sqrt{t}$ spread

- low temperatures



Lévy: $e^{-|k|^{5/3} t}$

Prähofer, HS, 2003

KPZ: width $t^{2/3}$ scaling function tabulated

Kardar, Parisi, Zhang, 1986

$e^{-|x|^3}, \ |x| \to \infty$

// Can this be understood? //

ALSO for bosons

fluctuating nonlinear hydrodynamics

conservation laws $\qquad\qquad \partial f_j = f_{j+1} - f_j$

- $\partial_t \rho_j + \mathcal{J}^{\rho}_{j+1} - \mathcal{J}^{\rho}_{j} = 0 \quad , \quad \mathcal{J}^{\rho}_{j} = \frac{1}{2} i \left(\psi_{j-1} \partial \psi^*_{j-1} - \psi^*_{j-1} \partial \psi_{j-1} \right)$

- $\partial_t e_j + \mathcal{J}^{e}_{j+1} - \mathcal{J}^{e}_{j} = 0 \quad , \quad \mathcal{J}^{e}_{j} = \frac{1}{4} i \left(\Delta \psi^*_j \partial \psi_{j-1} - \Delta \psi_j \partial \psi^*_{j-1} \right) + \frac{1}{2} g \mathcal{J}^{\rho}_{j} |\psi_j|^2$

$$\langle \mathcal{J}^{\rho}_{j} \rangle = 0 \qquad \langle \mathcal{J}^{e}_{j} \rangle = 0$$

$\rightsquigarrow$ fluctuating hydrodynamics

$$\partial_t u_\alpha + \partial_x \left(- \partial_x D \vec{u}_\alpha + B \vec{\xi}_\alpha \right) = 0 \qquad \alpha = 1,2 \qquad \begin{aligned} u_1 &= \rho(x,t) \\ u_2 &= e(x,t) \end{aligned}$$

space-time white noise

$\rightsquigarrow$ $\sqrt{t}$ diffusive

• low temperatures

$\frac{1}{Z} e^{-\beta(H-\mu N)}$, $\mu > 0$, $\beta \to \infty$ $\qquad \psi_j = \sqrt{\bar\rho}\, e^{i\vartheta}$ $\qquad \bar\rho = \frac{2\mu}{g}$

ϑ uniform on $[-\pi,\pi]$

$$\psi_j = \frac{1}{\sqrt{2}}(q_j + i p_j)$$

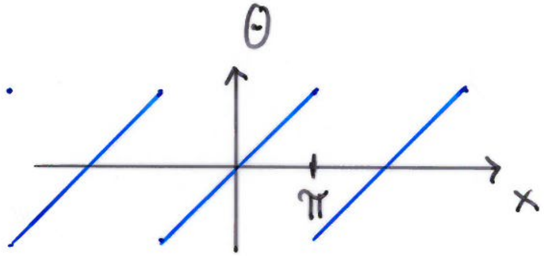
• canonical transformation $\quad \psi_j = \sqrt{\rho_j}\, e^{i\varphi_j}$

phase space

$(\mathbb{R}_+ \times [-\pi,\pi])^N$ + b.c. at $\rho_j = 0$

superfluid velocity

$$\tilde v_j = \Theta(\varphi_{j+1} - \varphi_j)$$



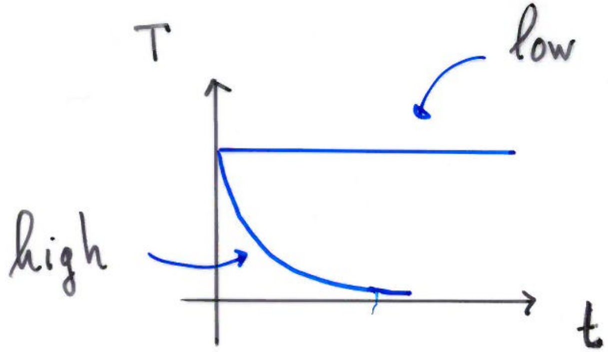
NOT conserved $\qquad$ umklapp processes high temperatures noisy

low temperatures $j \mapsto \varphi_j$ random walk on S^1

jump size $1/\sqrt{\beta}$

almost conserved

$$T(t) = \sum_j \langle \tilde{v}_j(t)\, \tilde{v}_0(0) \rangle$$



$e^{-t/\tau}$

$\tau = e^{\beta}$

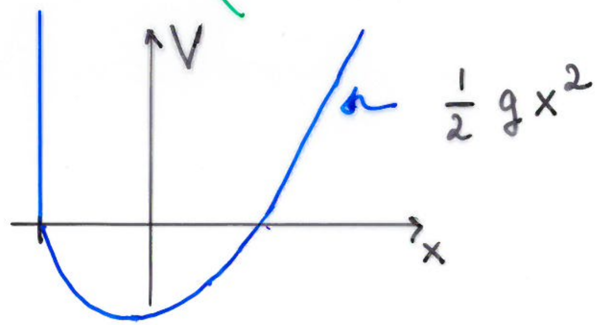
expand H at $\bar{\rho}, \vartheta$ observables do not depend on ϑ

small $\tilde{v}_j$, $z_j = \rho_j - \bar{\rho}$

$\rightsquigarrow$ effective low temperature $H_{\ell t}$ only for theory !!

quantum pressure

$$H_{\ell t} = \sum_j \left\{ \frac{1}{2} v_j^2 \left(1 + \frac{1}{2}(z_j + z_{j-1})\right) + \frac{1}{8\bar\rho} \left(z_{j+1} - z_j\right)^2 + V(z_j) \right.$$



$\tilde v_j \in [-\pi, \pi] \rightsquigarrow v_j \in \mathbb{R}$

$v_j = \varphi_{j+1} - \varphi_j$, canonical pair φ_j, z_j

$\in \mathbb{R} \times [-\bar\rho, \infty)$

3 conserved fields v_j, z_j, e_j

$\Rightarrow$ non zero Euler currents

$\approx>$ 3 peaks + spreading

- nonlinear fluctuating hydrodynamics mesoscopic scale

$$\partial_t u_\alpha + \partial_x \Big(\underbrace{j_\alpha(\vec{u})} - \partial_x D \vec{u}_\alpha + B \vec{\xi}_\alpha \Big) = 0 \qquad \alpha = 1, 2, 3$$

KPZ is determined by stochastic Burgers alias KPZ equation

decoupling

$$\partial_t u + \partial_x \big(-c u + \underbrace{u^2} - \partial_x u + \xi \big) = 0$$

stationary stochastic process $u(x,t)$

$$\mathbb{E}\big(u(x,t)\, u(0,0)\big) \cong t^{-2/3} f_{KPZ}\big(t^{-2/3}(x-ct)\big)$$

large x, t

- central peak

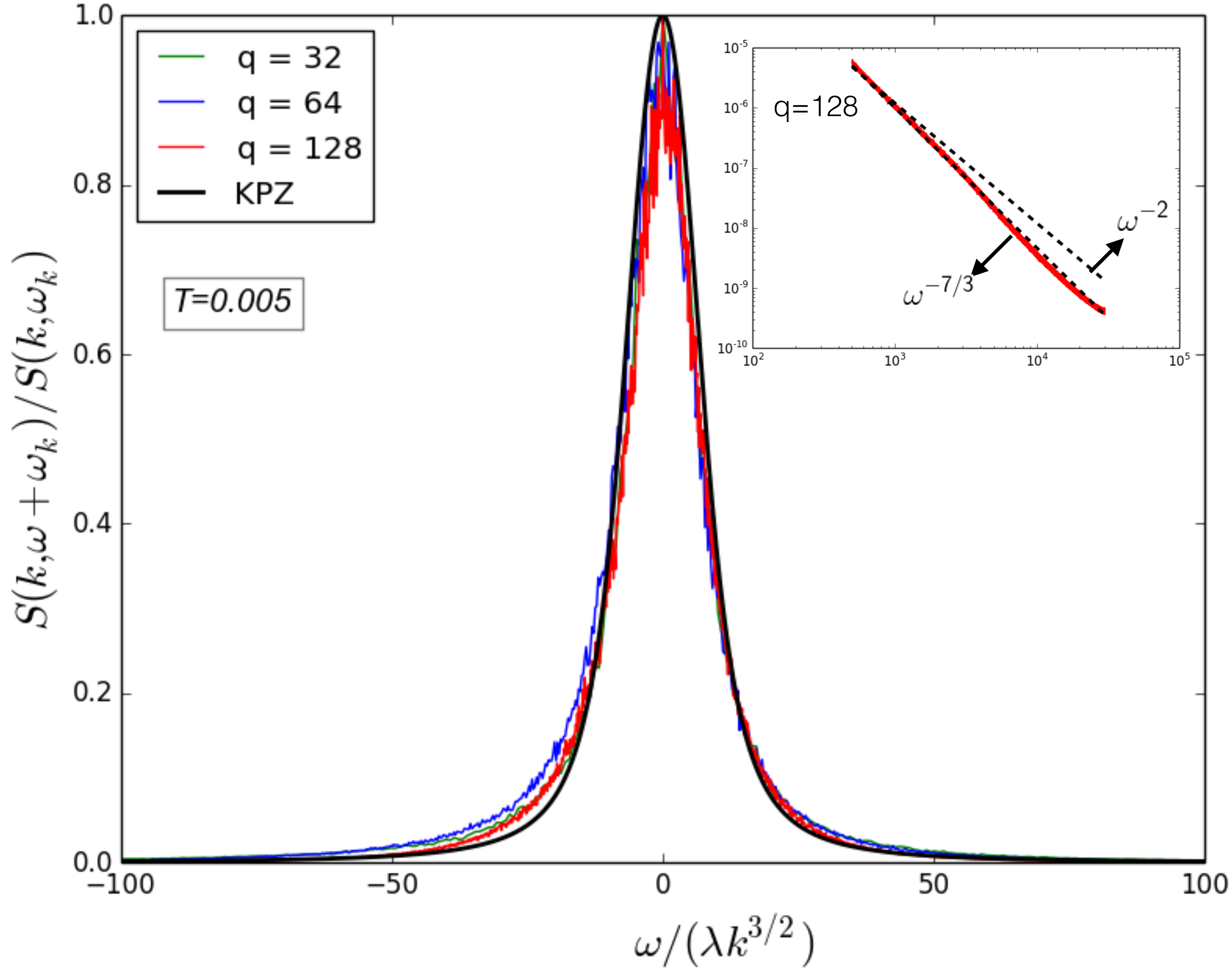
  no self-interaction

  coupling to moving peaks

⇝ tails as $t\,|x|^{-8/3}$

with cut-off at $x = \pm ct$

⇒ no signal beyond sound cone ⇐



q = 32
q = 64
q = 128
KPZ
T=0.005
q=128
ω^{-2}
$\omega^{-7/3}$
$S(k,\omega+\omega_k)/S(k,\omega_k)$
$\omega/(\lambda k^{3/2})$

Conclusions / Outlook

- should be tested in experiment
- temperature

  2 distinct scenarios for the late stage

  time scales