Invariant Gibbs measures for Hamiltonian PDEs

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Finite dimensional invariant Gibbs measures

Hamiltonian flow on $\mathbb{R}^{2n}$:

$$\dot{p}_j = \frac{\partial H}{\partial q_j}, \quad \dot{q}_j = -\frac{\partial H}{\partial p_j}$$

with Hamiltonian $H(p,q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$

- Vector field $X = (\frac{\partial H}{\partial q_j}, -\frac{\partial H}{\partial p_j})$ is divergence-free:

  By *Liouville's theorem*, Lebesgue measure $\prod_{j=1}^n dp_j dq_j$ is invariant

- Hamiltonian $H(p(t), q(t))$ is invariant under the flow

$\Longrightarrow$ *Gibbs measure*: $d\mu = Z^{-1} \exp(-\beta H(p,q)) \prod_{j=1}^n dp_j dq_j$ is **invariant**

Namely,

$$\mu(\Phi(-t)A) = \mu(A) \quad \text{for all } t \in \mathbb{R}$$

Moreover, if $F(p,q)$ is a "nice" conserved quantity, then

$$d\mu_F = Z^{-1} \exp(-F(p,q)) \prod_{j=1}^n dp_j dq_j$$

is also invariant

Q: Why do we care about *invariant measures*?

Given an invariant measure μ , we can view the system as a dynamical system with *measure-preserving* transformation T:

$$T = \text{ solution map} : (p(0), q(0)) \mapsto (p(t), q(t))\big|_{t=1}$$

We have the following theorems on recurrence properties of the dynamics:

Poincaré recurrence theorem

For any measurable A with $\mu(A) > 0$, there exists n such that

$$\mu(A \cap T^{-n} A) > 0$$

Q: Can we construct invariant measures for Hamiltonian PDEs?

Nonlinear Schrödinger equation (NLS):

$$iu_t + u_{xx} = \pm|u|^{p-2}u, \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad u, \text{ complex-valued}$$

- NLS is a Hamiltonian PDE:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |u_x|^2 dx \pm \frac{1}{p}\int_{\mathbb{T}} |u|^p dx, \quad M(u) = \int_{\mathbb{T}} |u|^2 dx,$$

- $H(u)$ is conserved under the NLS flow

Gibbs measure: "$d\mu = Z^{-1}e^{-H(u)}du$" is "expected" to be ***invariant***

- Gibbs measure as a weighted Wiener measure:

$$d\mu = Z^{-1}e^{-H(u)}du = Z^{-1}e^{\mp\frac{1}{p}\int_{\mathbb{T}}|u|^p dx} \underbrace{e^{-\frac{1}{2}\int_{\mathbb{T}}|u_x|^2 dx}du}_{\text{Wiener measure on } \mathbb{T}}$$

We actually consider

$$d\mu = Z^{-1}e^{\mp\frac{1}{p}\int_{\mathbb{T}}|u|^p dx} e^{-\frac{1}{2}\int_{\mathbb{T}}|u_x|^2 dx - \frac{1}{2}\int_{\mathbb{T}}|u|^2 dx} du$$

Gibbs measure for Hamiltonian PDEs on $\mathbb{T}$

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- Wiener measure (or rather Ornstein-Uhlenbeck process):

$$d\rho = Z^{-1} \exp\Big(-\frac{1}{2}\int_{\mathbb{T}} |u|^2 dx - \frac{1}{2}\int_{\mathbb{T}} |u_x|^2 dx\Big) du \quad \text{on } H^s(\mathbb{T}),\ s < \tfrac{1}{2}$$

Under this measure, u is represented by

$$u(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+4\pi^2 n^2}} e^{2\pi i n x} \in H^s(\mathbb{T}), s < \tfrac{1}{2}, \text{ almost surely}$$

where $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ = independent standard Gaussian r.v.'s

- Lebowitz-Rose-Speer '88, Bourgain '94: made sense of Gibbs measure

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Gibbs measure: $d\mu = Z^{-1} e^{-H(u)} du$

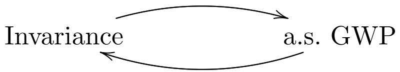
(i) Global well-posedness in the support of μ: cubic NLS ($p = 4$)

$\Longrightarrow$ Invariance of μ follows from the finite dimensional approximations

(ii) Only local well-posedness in the support of μ:

We use formal invariance of μ to construct a.s. global dynamics

- Circular argument?:



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invariance of ***finite-dimensional*** Gibbs measures

(in place of conservation laws) and approximation argument

$\Longrightarrow$ a.s. GWP on the statistical ensemble & invariance of Gibbs measure

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$\Longrightarrow$ (iii.a) *Probabilistic* local Cauchy theory & (ii)

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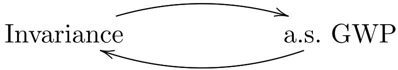

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Review of Bourgain's idea

(NLS) $$iu_t + u_{xx} = \pm |u|^{p-2}u$$

with Gibbs measure $d\mu = Z^{-1}e^{-H(u)}du$ where $H(u) = \frac{1}{2}\int |u_x|^2 \pm \frac{1}{p}\int |u|^p$

- Assume LWP in a Banach space $B \supset \text{supp}(\mu)$, e.g. $B = H^s$, $s < \frac{1}{2}$
  with local time of existence $\delta \sim \|u_0\|_B^{-\theta}$, $\theta > 0$
- $\Longrightarrow$ For $\|u_0\|_B \leq K$, consider the **finite dimensional approximation**:

  (F-NLS$_\text{N}$) $$\begin{cases} iu_t^N + u_{xx}^N = \pm \mathbf{P}_N(|u^N|^{p-1}u^N) \\ u^N\big|_{t=0} = \mathbf{P}_N u_0 = \sum_{|n|\leq N} \widehat{u}_0(n)e^{2\pi inx}, \end{cases}$$

  is LWP on $[0, \delta]$ where $\delta \sim K^{-\theta}$, independent of N
- (F-NLS$_\text{N}$) preserves $\int |u^N|^2 dx = \sum_{|n|\leq N} |\widehat{u^N}(n)|^2 =$ Euclidean distance on $\mathbb{C}^{2N+1}$

  $\Longrightarrow$ (F-NLS$_\text{N}$) is GWP for each N, but no *uniform* estimate as $N \to \infty$

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Lebesgue measure $du^N := \prod_{|n|\leq N} d\widehat{u^N}(n)$ is invariant under the flow

2. Conservation of $H(u^N) \Longrightarrow$ finite dimensional Gibbs measure

$$d\mu_N := Z_N^{-1} \exp\big(-H(u^N)\big)du^N$$

is *invariant* under the flow of (F-NLS$_\mathrm{N}$)

Proposition: Bourgain '94

Given $T < \infty, \varepsilon > 0$, there exists $\Omega_N = \Omega_N(\varepsilon, T) \subset B$ s.t.

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- for $u_0^N \in \Omega_N$, the solution u^N to (FNLS$_\mathrm{N}$) with $u^N|_{t=0} = u_0^N$ satisfies the following *growth estimate*:

$$\|u^N(t)\|_B \lesssim \left(\log\frac{T}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for } |t| \leq T$$

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Proof.

Let $\Phi_N(t)$ = flow map of (F-NLS$_N$) : $u_0^N \mapsto u^N(t)$, and define

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$\Longrightarrow$ By choosing $K \sim \big(\log \frac{T}{\varepsilon}\big)^{\frac{1}{2}}$, we have $\mu_N(\Omega_N^c) < \varepsilon$

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$\Longrightarrow$ By local theory,

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This proposition $\|u^N(t)\|_B \lesssim \big(\log \frac{T}{\varepsilon}\big)^{\frac{1}{2}}$ along with ***uniform convergence***:

$$\|u - u^N\|_{C([-T,T];B^{s_1})} \to 0, \quad s_1 < s$$

$$\text{uniformly for } u_0 \text{ with } \|u_0\|_{B^s} \le K \text{ as } N \to \infty$$

provides an *a priori bound* on the growth of solutions

$\Longrightarrow$ **Almost a.s. GWP**: Given T and $\varepsilon > 0$ (unrelated!!), there exists $\Omega_{T,\varepsilon}$ such that

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- Then, let $\Omega_\varepsilon = \bigcap_{j=1}^\infty \Omega_j$
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- Now, let $\widetilde{\Omega} = \bigcup_{\varepsilon>0} \Omega_\varepsilon$
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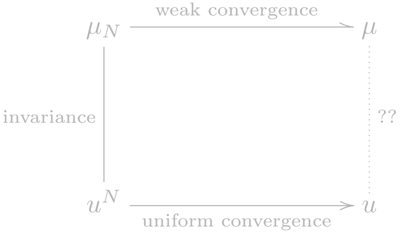

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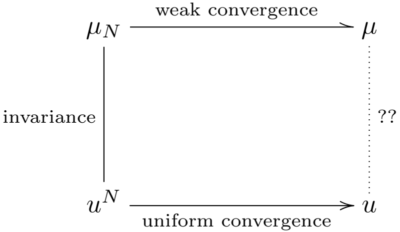

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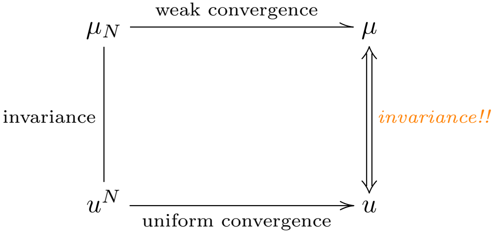

Invariance of Gibbs measure μ

Let F be a continuous and bounded function on $X = C(\mathbb{T};\mathbb{C})$

- Weak convergence of finite dimensional Gibbs measure μ_N to μ:

$$\lim_{N\to\infty} \int F(\phi)d\mu_N(\phi) = \int F(\phi)d\mu(\phi)$$

Let Φ_N and Φ = solution maps of F-NLS$_\mathrm{N}$ and NLS on $\mathbb{T}$:

- $u^N(t) := \Phi_N(t)\phi_N(\omega) \to u(t) := \Phi(t)\phi(\omega)$ a.s. in $C([0,T];X)$

  By DCT with $\mu = P \circ \phi^{-1}$ and $\mu_N = P \circ \phi_N^{-1}$,

$$\int F \circ \Phi(t)d\mu = \int F(\Phi(t)\phi)d\mu(\phi) = \int F(\Phi(t)\phi(\omega))dP(\omega)$$

$$= \lim_{N\to\infty} \int F(\Phi_N(t)\phi_N(\omega))dP(\omega) = \lim_{N\to\infty} \int F \circ \Phi_N(t)d\mu_N$$

- By invariance of μ_N under $\Phi_N(t)$, we have

$$\int F \circ \Phi_N(t)d\mu_N = \int F d\mu_N$$

- McKean '95: cubic NLS, and other equations (with Vaninsky)
- Many results on invariant measures for Hamiltonian PDEs:
  - Bourgain (in mid 90's),
  - Tzvetkov, Burq-Tzvetkov, Burq-Tzvetkov-Thomann, Oh (late 2000's) with their collaborators and students

More dynamical properties?

1. μ invariant $\implies$ $u(t) \overset{\mathcal{D}}{\sim} u(0)$ but how are $u(t)$ and $u(0)$ related?

   Can we say anything about the space-time covariance $\mathbb{E}_\mu[u(x,t)\overline{u(y,0)}]$?

   - Lukkarinen-Spohn, '11: weakly nonlinear & large box limit of lattice NLS

2. Ergodicity and 'asymptotic stability' of μ?
   - Mass M and momentum P:
     - are conserved for (NLS)
     - are finite a.s. with respect to Gibbs measure
   - Oh-Quastel '13: invariant Gibbs measures with *prescribed* M and P
   - These questions have been answered for some stochastic PDEs. This is mainly due to *uniqueness* of invariant measures. However, for Hamiltonian PDEs, there are more than one (formally) invariant measures and such questions are out of reach at this point...

Gibbs measures on $\mathbb{T}^2$

Goal: Construct invariant Gibbs measures: $d\mu = Z^{-1} e^{-\frac{1}{p}\int_{\mathbb{T}^2} |u|^p} d\rho$
for the **defocusing NLS on $\mathbb{T}^2$:**

$$iu_t + \Delta u = |u|^{p-2}u$$

Difficulty: The Gaussian masure

$$d\rho = Z^{-1} \exp\Big(-\frac{1}{2}\int_{\mathbb{T}^2} |u|^2 dx - \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 dx\Big) du$$

is support on $H^s(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, $s < 0$. They are not even functions!!

In particular, $\int_{\mathbb{T}^2} |u|^p = \infty$ a.s.

Two problems:

- Construction of the Gibbs measure: *Wick renormalization*
- Construction of the global-in-time dynamics:

(iii.a) probabilistic Cauchy theory

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Wick ordering

Given $u(x;\omega) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x}$ under ρ, we have

$$\sigma_N = \mathbb{E}\bigg[\int_{\mathbb{T}^2} |\mathbf{P}_N u|^2 dx\bigg] = \sum_{|n|\le N} \frac{1}{1+|n|^2} \sim \log N \to \infty$$

Wick ordered monomial: $:\!|\mathbf{P}_N u|^2\!: \stackrel{\text{def}}{=} |\mathbf{P}_N u|^2 - \sigma_N$

$\Longrightarrow$ For any $q < \infty$,

$$\int_{\mathbb{T}^2} :\!|\mathbf{P}_N u|^2\!: dx \in L^q(\rho) \qquad \big(\text{with a uniform bound in } N \text{ for each } q\big)$$

Hence, we can define the limit in $L^q(\rho)$:

$$\int_{\mathbb{T}^2} :\!|u|^2\!: dx \stackrel{\text{def}}{=} \lim_{N\to\infty} \int_{\mathbb{T}^2} :\!|\mathbf{P}_N u|^2\!: dx$$

Similarly, for any *even* $p = 2m$, we can define the **Wick ordered monomial**:

$$:|\mathbf{P}_N u|^p: \stackrel{\text{def}}{=} (-1)^m m! \cdot \underbrace{L_m(|\mathbf{P}_N u|^2; \sigma_N)}_{\text{Laguerre polynomial}}$$

In the real-valued setting, $:(\mathbf{P}_N u)^p:$ can be defined for any $p \geq 2$ by Hermite polynomials

- $\displaystyle\int_{\mathbb{T}^2} :|u|^p: dx \stackrel{\text{def}}{=} \lim_{N\to\infty} \int_{\mathbb{T}^2} :|\mathbf{P}_N u|^p: dx$ exists in $L^q(\rho)$ for any $q < \infty$

  $\Longrightarrow$ Wick ordered Hamiltonian: $\displaystyle H(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 dx + \frac{1}{p}\int_{\mathbb{T}^2} :|u|^p: dx$
- Main tool: hypercontractivity/Wiener chaos estimate by Nelson '73

Theorem: Gibbs measure for the Wick ordered NLS on $\mathbb{T}^2$

Let $p \geq 4$ be an even integer. Then, the Gibbs measure

$$d\mu = Z^{-1} e^{-\frac{1}{p}\int_{\mathbb{T}^2} :|u|^p: dx} d\rho$$

is a probability measure on $H^s(\mathbb{T}^2)$, $s < 0$

- Euclidean quantum field theory: Nelson, Simon, Glimm-Jaffe...
- **No** Gibbs measure in the focusing case: Brydges-Slade '96

Similarly, for any *even* $p = 2m$, we can define the **Wick ordered monomial**:

$$:|\mathbf{P}_N u|^p: \stackrel{\text{def}}{=} (-1)^m m! \cdot \underbrace{L_m(|\mathbf{P}_N u|^2; \sigma_N)}_{\text{Laguerre polynomial}}$$

In the real-valued setting, $:(\mathbf{P}_N u)^p:$ can be defined for any $p \geq 2$ by Hermite polynomials

- $\displaystyle\int_{\mathbb{T}^2} :|u|^p: dx \stackrel{\text{def}}{=} \lim_{N\to\infty} \int_{\mathbb{T}^2} :|\mathbf{P}_N u|^p: dx$ exists in $L^q(\rho)$ for any $q < \infty$

  $\Longrightarrow$ Wick ordered Hamiltonian: $\displaystyle H(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 dx + \frac{1}{p}\int_{\mathbb{T}^2} :|u|^p: dx$
- Main tool: hypercontractivity/Wiener chaos estimate by Nelson '73

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(iii.a) Probabilistic Cauchy theory

Defocusing Wick ordered NLS on $\mathbb{T}^2$:

(WNLS) $$iu_t + \Delta u = :|u|^{p-2}u: \quad (= \tfrac{\partial}{\partial \overline{u}} :|u|^p:)$$

- Gibbs measure on $H^s(\mathbb{T}^2)$, $s < 0$
- ill-posed for $s < s_{\text{crit}} = 1 - \frac{2}{p-2}$: $s_{\text{crit}} = 0$ if $p = 4$, $s_{\text{crit}} = \frac{1}{2}$ if $p = 6$, ...

Probabilistic Cauchy theory:

- construct (local) solutions a.s. with respect to $u|_{t=0} = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x}$
- gain of integrability of linear solution under randomization
- $p = 4$ (cubic NLS): the regularity gap is small, i.e. any $\varepsilon > 0$

  Bourgain '96 constructed local solutions a.s. & (ii)

  $\Longrightarrow$ a.s. global dynamics and invariance of the Gibbs measure

For $p \geq 6$, the regularity gap $> s_{\text{crit}} > \frac{1}{2}$ is too large...

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(iii.b) Compactness argument

μ_N = finite dimensional *invariant* Gibbs measure for

(F-WNLS$_N$) $$i\partial_t u^N + \Delta u^N = \mathbf{P}_N\big(:|\mathbf{P}_N u^N|^{p-2}\mathbf{P}_N u^N: \big)$$

Let $\Phi_N : u_0^N \in H^s \mapsto u^N \in C(\mathbb{R}; H^s)$ be the solution map

1. extend μ_N to ν_N = measure on *space-time* functions:
$$\nu_N \stackrel{\text{def}}{=} \mu_N \circ \Phi_N^{-1}$$
2. show $\{\nu_N\}_{N\in\mathbb{N}}$ is tight (= compact) $\stackrel{\text{Prokhorov}}{\Longrightarrow}$ weak convergence
3. Skorokhod's theorem: $\nu_N \Longrightarrow \nu$ and
$$u^N \text{ converges to some } u \ (= \text{global-in-time weak solution to WNLS}) \text{ a.s.}$$

Theorem: Oh-Thomann '15

There exists a set Σ of μ-full measure such that for every ϕ, there exists a solution $u \in C(\mathbb{R} : H^s)$ with $u|_{t=0} = \phi$. Moreover, the law $\mathcal{L}(u(t))$ is the same as μ for any $t \in \mathbb{R}$

Generalized KdV equation

Generalized KdV equation (gKdV):

$$\partial_t u + \partial_x^3 u = \pm \frac{1}{k} \partial_x (u^k), \qquad (x,t) \in \mathbb{T} \times \mathbb{R}$$

- $k = 2$: Korteweg-de Vries equation (KdV)
  $k = 3$: modified KdV equation (mKdV)
- Hamiltonian: $H(u) = \frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2 dx \pm \frac{1}{k(k+1)} \int_{\mathbb{T}} u^{k+1} dx$
- invariance of Gibbs measure μ:
  - Bourgain '94: $k = 2, 3$
  - Richards '12: $k = 4$ (probabilistic Cauchy theory)

Theorem: Oh-Richards-Thomann '15 (compactness argument)

Let k be an odd integer. Then, there exists a set Σ of μ-full measure such that for every ϕ, there exists a solution $u \in C(\mathbb{R} : H^s)$, $s < \frac{1}{2}$, to the *defocusing* gKdV with $u|_{t=0} = \phi$. Moreover, the law $\mathcal{L}(u(t))$ is the same as μ for any $t \in \mathbb{R}$

- Focusing case: up to $k = 6$ $\big($with an L^2-cutoff in Gibbs measure$\big)$

Gibbs measure on $\mathbb{R}$

Defocusing NLS on $\mathbb{R}$: $iu_t + u_{xx} = |u|^{p-2}u$

- Constructed invariant Gibbs measures for NLS on $\mathbb{T}$.
- This construction applies to NLS on $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$ of any finite period L

Goal: Take $L \to \infty$

- Gibbs measure μ_L on $\mathbb{T}_L$:

$$d\mu_L = Z_L^{-1} e^{-\frac{1}{p}\int_0^L |u|^p dx} e^{-\frac{1}{2}\int_0^L |u_x|^2 - \frac{1}{2}\int_0^L |u|^2} du$$

Free measure ρ_L: $d\rho_L = Z_L^{-1} e^{-\frac{1}{2}\int_0^L |u_x|^2 - \frac{1}{2}\int_0^L |u|^2} du$

- For finite L, $\mu_L \ll \rho_L$ but $\mu_\infty \not\ll \rho_\infty$ = Ornstein-Uhlenbeck:

$$\int_0^L |\phi|^p dx \sim L \quad \text{as } L \to \infty$$

- Under ρ_L, we have

$$\phi(x;\omega) = \sum_{n\in\mathbb{Z}} \frac{\sqrt{L}}{\sqrt{n^2 + L^2}} g_n(\omega) e^{\frac{2\pi i n x}{L}} \longrightarrow \text{OU on } \mathbb{R}$$

Gibbs measure on $\mathbb{R}$

Defocusing NLS on $\mathbb{R}$: $iu_t + u_{xx} = |u|^{p-2}u$

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Bourgain '00:

- Gaussian domination (Brascamp-Lieb inequality): For $I \subset [-\frac{L}{2}, \frac{L}{2}]$,

$$\mathbb{E}_{\mu_L} \|\phi_L\|_{L^\infty(I)} \lesssim \big[\log(1+|I|)\big]^{\frac{1}{2}} = \text{ growth bound on OU}$$

uniformly in $L \gg 1$

- Invariance of μ_L and the Duhamel formulation:

$$\text{(growth)} \qquad \mathbb{E}_{\mu_L}\Big[\sup_{|t|\le T} \|u_L(t)\|_{L^\infty(I)}\Big] \lesssim \big[\log(T+|I|)\big]^{\frac{1}{2}}$$

Theorem: Bourgain '00

(i) Let $p > 2$. There exists a *subsequence* $\{L_j\}_{j=1}^\infty$ such that $L_j \to \infty$ and

$$\phi_{L_j} \to \phi \qquad \text{and} \qquad u_{L_j} \to u, \quad \text{almost surely, where}$$

(i.a) convergence is uniform on bounded space-time regions,

(i.b) u is a distributional solution to NLS

(ii) (sub-)cubic NLS ($p \le 4$): uniqueness and continuous dependence

- No mention of the limiting Gibbs measure $\mu := \mu_\infty$:

  weak convergence of μ_L to μ, invariance of μ, etc.

- Not efficient:

  Gaussian bound without use of the potential part "$-\frac{1}{p}\int |u|^p$"

Theorem: On-Quastel-Sosoe '13

(i) For all $L \gg 1$,

$$\mathbb{E}_{\mu_L}\Big[\sup_{|t|\leq T} \|u_L(t)\|_{L^\infty(I)}\Big] \lesssim \big[\log(T+|I|)\big]^{\frac{2}{p+2}}$$

(ii) The periodic Gibbs measures μ_L converge weakly to $\mu := \mu_\infty$ on $\mathbb{R}$

(iii) μ is invariant under the (sub-)quintic NLS flow ($p \leq 6$)

Idea: view u under μ_L as a diffusion

- Focusing case (Rider '02): For cubic NLS ($p = 4$),

  Gibbs measure concentrates on the trivial (i.e. zero) function

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I: Probabilistic description of μ_L, $L \gg 1$, and μ_∞

- Dirichlet Gibbs measures: Feynman-Kac formula
- Ground state substitution: Ito's formula, Girsanov theorem
- Construct μ_∞ as a stationary diffusion process (in x) with values in $\mathbb{C}$

II: Tightness (= compactness) of $\{\mu_L\}$ as probability measures on $C(\mathbb{R};\mathbb{C})$

- Kolmogorov's continuity criterion ($\frac{1}{2} - \varepsilon$ Hölder regularity of BM/OU)
- **I** and **II**: $\mu_L \rightharpoonup \mu_\infty$

III: Improved growth bounds:

$$\mathbb{E}_{\mu_L}\Big[\sup_{|t|\le T} \|u_L(t)\|_{L^\infty(I)}\Big] \lesssim \big[\log(T+|I|)\big]^{\frac{2}{p+2}}$$

- Invariance of μ_∞ under (sub-)quintic NLS on $\mathbb{R}$: Skorohod theorem

  "μ_∞ is an invariant measure (in t) of an invariant measure in x"

- New class of *non-decaying, rough* solutions (in x) to NLS on $\mathbb{R}$
- also on $\mathbb{R}_+ = [0,\infty)$ with $u(0) = 0$

Defocusing NLS on $\mathbb{R}$:

- What about $p > 6$?

Theorem: Oh-Quastel-Sosoe '15 (compactness argument)

Let $p > 6$. Then, there exists a set Σ of μ-full measure such that for every ϕ, there exists a solution $u \in C(\mathbb{R} : H^s_{\text{loc}})$ to the defocusing NLS on $\mathbb{R}$ with $u|_{t=0} = \phi$. Moreover, the law $\mathcal{L}(u(t))$ is the same as μ for any $t \in \mathbb{R}$

generalized KdV on $\mathbb{R}$:

- Gibbs measure μ_L converges to a Dirac's δ-measure on the trivial function for
  - KdV $(k = 2)$
  - focusing modified KdV $(k = 3)$

Theorem: Oh-Quastel-Sosoe '15 (compactness argument)

There exists a set Σ of μ-full measure such that for every ϕ, there exists a solution $u \in C(\mathbb{R} : H^s_{\text{loc}})$ to the *defocusing* mKdV on $\mathbb{R}$ with $u|_{t=0} = \phi$. Moreover, the law $\mathcal{L}(u(t))$ is the same as μ for any $t \in \mathbb{R}$

White noise on $\mathbb{T}$

White noise: $d\mu_0 = Z^{-1} \exp(-\frac{1}{2}\int |u|^2 dx) du$

$$u(x;\omega) = \sum_{n\in\mathbb{Z}} g_n(\omega) e^{inx} \in H^s, \quad s < -\tfrac{1}{2}$$

Q: Invariance of white noise?

Difficulty: Very rough!!

- KdV: Quastel-Valkó '08, Oh '09, Oh-Quastel-Valkó '12
- cubic NLS? Oh-Quastel-Valkó '12:
  white noise is a *weak limit* of invariant measures for cubic NLS
  but no well-defined dynamics...
- This problem is of particular interest in *nonlinear optics*. In particular, in the context of the stochastic cubic NLS with random forcing by the space-time white noise