Wave Functions on Space-Time

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Joint work with Roderich Tumulka

Aim: fully relativistic description of quantum state

Non-relativistic QM: ($\boldsymbol{x}_i \in \mathbb{R}^3$)

$$\psi(t, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$$

$$i\partial_t \psi = H\psi$$

Proposal: multi-time wave function ($x_i = (t_i, \boldsymbol{x}_i) \in \mathbb{R}^4$)

$$\phi(t_1, \boldsymbol{x}_1, \ldots, t_N, \boldsymbol{x}_N) = \phi(x_1, \ldots, x_N)$$

$$i\partial_{t_j}\phi = H_j \phi, \; j = 1, \ldots, N$$

[Dirac (1932), Dirac, Fock, Podolsky (1932), Bloch (1934)]

Intended:

- agreement with QM: $\phi(t, \boldsymbol{x}_1, \ldots, t, \boldsymbol{x}_N) = \psi(t, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$
  $\Rightarrow \sum_j H_j = H$ (for equal times)
- $|\phi(x_1, \ldots, x_N)|^2$ prob. dist. for spacelike $x_1, \ldots, x_N$

Main result: new type of representation for quantum state in QFT

multi-time Schrödinger-picture particle-position representation

$\hookrightarrow$ alternative to Tomonaga-Schwinger approach
$\hookrightarrow$ connected to operator-valued fields

Advantages:

- manifestly Lorentz-invariant (also gen. cov. in curved space-time)
- simple, (locally) finite dimensional equations (PDEs)
- works with cutoff
- some insight into what “good” rel. interactions in QFT are
- useful for foundations of QM

Back Story

Main novel feature of multi-time equations

$$i\partial_{t_j}\phi = H_j\phi$$

is the *consistency condition*:

$$\left[i\partial_{t_j} - H_j, i\partial_{t_k} - H_k\right] = 0 \qquad (*)$$

$\hookrightarrow$ necessary and sufficient for existence of joint solution for all initial conditions

- heuristically clear (next slide)
- in some cases rigorously proven (SP, RT (2014))

$\hookrightarrow$ challenge: interacting $H_1, \ldots, H_N$ that fulfill $(*)$

Back Story: Consistency Condition

$$\left[i\partial_{t_j} - H_j, i\partial_{t_k} - H_k\right] = 0 \qquad (*)$$

Heuristic for time-independent H_j:

- consider $\phi(t_1, t_2) \in L^2(\mathbb{R}^6)$, initial condition $\phi(0,0)$:

$$\phi(t_1, t_2) = e^{-iH_1 t_1}\phi(0, t_2) = e^{-iH_1 t_1} e^{-iH_2 t_2}\phi(0,0)$$

$$\phi(t_1, t_2) = e^{-iH_2 t_2}\phi(t_1, 0) = e^{-iH_2 t_2} e^{-iH_1 t_1}\phi(0,0)$$

- unique $\phi(t_1, t_2)$ for all possible $\phi(0,0)$ if and only if $[H_1, H_2] = 0$

More general:

- evolution operator U_γ for every path γ in the space $\mathbb{R}^N$ spanned by time axes (Dyson series, path-ordered exponential)
- path-independence $\Leftrightarrow$ consistency condition

Back Story: Interaction Potentials

$$\left[i\partial_{t_j} - H_j, i\partial_{t_k} - H_k\right] = 0 \qquad (*)$$

$H_j = H_j^{\text{free}} + V_j(x_1, \ldots, x_N)$ with $V =$ mult. op. violates $(*)$

$\hookrightarrow$ interaction potentials are not consistent!

More exactly:

Theorem (SP, RT (2014))

Let $H_j^{\text{free}} = -i\boldsymbol{\alpha}_j \cdot \nabla_j + \beta_j m$, $V_j : \mathbb{R}^{4N} \to \mathbb{R}$ smooth. Then $(*)$ is satisfied on $\mathbb{R}^{4N}$ if and only if the multi-time eq.s are gauge-equivalent to non-interacting eq.s, i.e., there are functions $\theta(x_1, \ldots, x_N)$ and $\tilde{V}_j(x_j)$ such that $\tilde{\phi} = e^{i\theta}\phi$ satisfies

$$i\partial_{t_j}\tilde{\phi} = \left(H_j^{\text{free}} + \tilde{V}_j(x_j)\right)\tilde{\phi}.$$

Back Story: Interaction Potentials

$$\left[i\partial_{t_j} - H_j, i\partial_{t_k} - H_k\right] = 0 \qquad (*)$$

Generalizations: Theorem still holds if

- $H_j^{\text{free}} = -i\boldsymbol{A}_j(x_j) \cdot \nabla_j + B_j(x_j)$, with $A_{j,1}, A_{j,2}, A_{j,3}, I$ lin. indep. and self-adjoint
- $V_j(x_1, \ldots, x_N)$ matrix-valued (acting on j-th spin space)
- H_j^{free} is second-order differential op.
- $(*)$ only holds for spacelike separated configurations
- $N = 2$, $V_j = \frac{1}{2}|\boldsymbol{x}_1 - \boldsymbol{x}_2|^{-1}$, i.e., Coulomb interaction (and free Dirac or Schrödinger)

We take these results to rule out interaction potentials for multi-time eq.s.

$\hookrightarrow$ next: interaction by particle creation/annihilation works!

$\hookrightarrow$ one reasoning for why we need variable particle number (and Fock-space) for relativistic interaction!

The Emission-Absorption Model

$\hookrightarrow$ a concrete example of multi-time QFT model with particle creation/annihilation
$\hookrightarrow$ M x-particles interact by emitting/absorbing y-particles

Simplifications:

- first: formal calculations, ignore UV divergence (cutoff later)
- consider Dirac particles, so no problems with part.-pos. repr.
- Dirac particles can have negative energies, so no localization problems

Definition of Emission-Absorption Model: One-Time

- one-particle Hilbert space $\mathscr{H}_1 = L^2(\mathbb{R}^3, \mathbb{C}^4)$
- full Hilbert space $\mathscr{H} = \mathscr{H}_x \otimes \mathscr{F}_y$, $\mathscr{H}_x = S_- \mathscr{H}_1^{\otimes M}$,
  $\mathscr{F}_y = \bigoplus_{N=0}^{\infty} S_+ \mathscr{H}_1^{\otimes N}$

  (S_-: anti-sym. op., S_+: sym. op.)
- (M, N)-sector: $\psi_t(x^{3M}, y^{3N}) = \psi_t(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_N)$
- let $a^\dagger(\boldsymbol{x})/a(\boldsymbol{x})$ = fermionic creation/annihilation op.s
- let $b^\dagger(\boldsymbol{x})/b(\boldsymbol{x})$ = bosonic creation/annihilation op.s
- let $H^{\text{free}}_{x/y} = -i\boldsymbol{\alpha} \cdot \nabla + \beta m_{x/y}$
- $g \in \mathbb{C}^4$ some fixed spinor

$$i\partial_t \psi_t(x^{3M}, y^{3N}) = H\psi_t(x^{3M}, y^{3N}) \quad \forall\, N,$$

$$H = \int d^3\boldsymbol{x} \left(a^\dagger(\boldsymbol{x}) H^{\text{free}}_x a(\boldsymbol{x}) + b^\dagger(\boldsymbol{x}) H^{\text{free}}_y b(\boldsymbol{x}) \right) + \int d^3\boldsymbol{x} \left(a^\dagger(\boldsymbol{x}) \left(g^\dagger b(\boldsymbol{x}) + b^\dagger(\boldsymbol{x}) g \right) a(\boldsymbol{x}) \right)$$

Definition of Emission-Absorption Model: One-Time

$$H = \int d^3\boldsymbol{x} \left(a^\dagger(\boldsymbol{x}) H_x^{\text{free}} a(\boldsymbol{x}) + b^\dagger(\boldsymbol{x}) H_y^{\text{free}} b(\boldsymbol{x}) \right) + \int d^3\boldsymbol{x} \left(a^\dagger(\boldsymbol{x}) \left(g^\dagger b(\boldsymbol{x}) + b^\dagger(\boldsymbol{x}) g \right) a(\boldsymbol{x}) \right)$$

In more detail:

$$(H\psi)(x^{3M}, y^{3N}) = \sum_{j=1}^{N} H_{x_j}^{\text{free}} \psi(x^{3M}, y^{3N}) + \sum_{k=1}^{N} H_{y_k}^{\text{free}} \psi(x^{3M}, y^{3N}) + \sqrt{N+1} \sum_{j=1}^{M} g^\dagger \psi\big(x^{3M}, (y^{3N}, \boldsymbol{x}_j)\big) + \frac{1}{\sqrt{N}} \sum_{j=1}^{M} \sum_{k=1}^{N} g \delta^3(\boldsymbol{y}_k - \boldsymbol{x}_j) \psi\big(x^{3M}, y^{3N} \backslash \boldsymbol{y}_k\big)$$

Rem.: fixed g breaks Lorentz-invariance (artifact of simplifications)

Definition of Emission-Absorption Model: Multi-Time

(N, M)-sector of multi-time wave function:

$$\phi(x^{4M}, y^{4N}) = \phi(x_1, \ldots, x_M, y_1, \ldots, y_N)$$

Multi-time equations $i\partial_{x_j^0}\phi = H_{x_j}\phi$, $i\partial_{y_k^0}\phi = H_{y_k}\phi$, with

$$\begin{aligned} H_{x_j}\phi(x^{4M}, y^{4N}) =& H_{x_j}^{\text{free}}\phi(x^{4M}, y^{4N}) + \sqrt{N+1}g^\dagger\phi(x^{4M}, (y^{4N}, x_j)) \\ &+ \frac{1}{\sqrt{N}}\sum_{k=1}^{N} G(y_k - x_j)\phi(x^{4M}, y^{4N}\backslash y_k) \\ H_{y_k}\phi(x^{4M}, y^{4N}) =& H_{y_k}^{\text{free}}\phi(x^{4M}, y^{4N}) \end{aligned}$$

G Green function: $i\partial_{y^0}G = H_y^{\text{free}}G$, $G(0, \boldsymbol{y}) = g\delta^3(\boldsymbol{y})$

Emission-Absorption Model: Multi-Time

Assertion (SP, RT (2014))

On the set $\mathscr{S}$ of all spacelike configurations, these multi-time eq.s are consistent, i.e., they have a unique solution ϕ for all initial conditions. (Ignoring UV divergence.)

Remarks:

- permutation symmetry for space-time points:
  $\phi(\ldots, x_i, \ldots, x_j, \ldots, y^{4N}) = (-1)\phi(\ldots, x_j, \ldots, x_i, \ldots, y^{4N})$
  $\phi(x^{4M}, \ldots, y_k, \ldots, y_\ell, \ldots,) = \phi(x^{4M}, \ldots, y_\ell, \ldots, y_k, \ldots,)$
- UV cutoff: replace δ-fct. by smeared-out φ and modify $\mathscr{S}$
  $\hookrightarrow$ both modifications break Lorentz-invariance
  $\hookrightarrow$ but rigorous consistency proof possible

Other Multi-Time QFTs

Instead of particle reaction $x \leftrightarrows x + y$, consider now pair creation $y \leftrightarrows x + \overline{x}$, or generally $a \leftrightarrows b + c$.

$\hookrightarrow$ natural multi-time eq.s.

Assertion (SP, RT (2014))

On the set $\mathscr{S}$ of all spacelike configurations, the corresponding multi-time eq.s are consistent if and only if 0 or 2 of the particle species a, b, c are fermions. (Ignoring UV divergence.)

- e.g., decay of one fermion into two bosons not consistent
- also follows from spin-statistics theorem but here we only need to use statistics (sym. or antisym.)
- presumably there are multi-time equations for all fundamental processes of particle creation/annihilation

Relation to Tomonaga-Schwinger

Tomonaga-Schwinger:

- let Σ be a spacelike hypersurface, $\mathscr{H}_\Sigma$ the corresponding Hilbert space
- choose fixed $\tilde{\mathscr{H}}$, identify $\mathscr{H}_\Sigma \to \tilde{\mathscr{H}}$ along free evolution
- then $\tilde{\psi}_\Sigma \in \tilde{\mathscr{H}}$ represents *interaction picture*
- Tomonaga-Schwinger equation:

$$i(\tilde{\psi}_{\Sigma'} - \tilde{\psi}_\Sigma) = \left(\int_\Sigma^{\Sigma'} d^4x \, \mathcal{H}_I(x) \right) \tilde{\psi}_\Sigma$$

  for infinitesimally neighboring Σ, Σ',
  with $\mathcal{H}_I(x) =$ int. Hamiltonian density in int. pict.

- functional differential eq. on ∞-dim. space
- consistency cond. $[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0$ for spacelike sep. x, y

Relation to Tomonaga-Schwinger

From multi-time to TS: for $x_1, \ldots, x_N \in \Sigma$, define

$$\psi_\Sigma(x_1, \ldots, x_N) = \phi(x_1, \ldots, x_N)$$

Assertion (SP, RT (2014))

For the emission-absorption model, this ψ_Σ, translated into the interaction picture, satisfies the Tomonaga-Schwinger equation.

From TS to multi-time: given $(\psi_\Sigma)_\Sigma$, only if for all $x_1, \ldots, x_N \in \Sigma, \Sigma'$,

$$\psi_\Sigma(x_1, \ldots, x_N) = \psi_{\Sigma'}(x_1, \ldots, x_N), \qquad (**)$$

then $(\psi_\Sigma)_\Sigma$ defines multi-time wave function ϕ.

Assertion (SP, RT (2014))

Eq. $(**)$ holds for the Tomonaga-Schwinger evolution.

Relation to Operator-valued Fields

Heisenberg picture: state vector Ψ fixed, dynamics in the operators

$$a^{(\dagger)}(t, \boldsymbol{x}) = e^{iHt} a^{(\dagger)}(\boldsymbol{x}) e^{-iHt}$$

$\hookrightarrow$ for the emission-absorption model, we can define

$$\phi(x^{4M}, y^{4N}) = \left\langle \emptyset \middle| a(x_1) \cdots a(x_M) b(y_1) \cdots b(y_N) \middle| \Psi \right\rangle, \qquad (***)$$

with $|\emptyset\rangle$ = vacuum state
(similar expressions for other models, e.g., pair creation model)

Assertion (SP, RT (2014))

For the emission-absorption model, the wave function defined by $(***)$ indeed satisfies the multi-time equations. (Similar for pair creation model.)

Relevance for Foundations of QM

Bohmian Mechanics:

- in the law of motion, the velocity of any particle depends on the positions of all other particles
- in a relativistic version: need to evaluate wave function along spacelike hypersurface
- multi-time wave function good way to define ψ_Σ
- yields relativistic Bohmian Mechanics (with preferred foliation which can follow from Lorentz-invariant law)
- note: particle creation/annihilation possible by jump process
- ϕ technical tool, e.g., to show that foliation is not detectable

GRW:

- relativistic collapse along Σ
- again: ϕ nicely defines unitary part of evolution of ψ_Σ and useful technical tool

Many-worlds:

- only based on wave function, so multi-time important for relativistic invariance

Perspective

Problems we neglected that need to be solved:

- **UV divergence**; here: need well-defined relativistic Hamiltonians with creation/annihilation $\Rightarrow$ Tumulka's idea of interior-boundary conditions (IBCs); see Lampart, Schmidt, Teufel, Tumulka
- **negative energies**; "standard" (textbook) QFT treatment leads to divergences and problems in curved space-time $\Rightarrow$ e.g., Deckert et al
- **position representation of photon wave function**

solution to these problems + multi-time
$\Rightarrow$ fully relativistic well-defined QED

Alternatively:

- **Dirac Sea** picture for pair creation (problems: stability? fluctuations?) $\Rightarrow$ Colin, Struyve; Dürr group; Finster, ...
- **direct interaction** instead of photons (quantum Wheeler-Feynman?)

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Thank you for your attention!