# Radon Transform and Light-Cone Distributions

Light Cone - 2015
Frascati, September 24, 2014

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# Main Topics

- Radon transform and Generalized Parton Distributions
- Crossing, Tomography, Holography
- Quarks in photons: limited-angle tomography
- Analyticity and real-photon limit
- Radon transform for Conditional Parton Distributions and Dihadron Fragmentation Functions
- Radon-Wigner transform: GPDs vs TMDs

# Radon Transform

- Discovered (invented) by Johann Radon in 1917 (100th Anniversary ahead!)
- Ahead of time – rediscovered than needed

(1887-1956)

- **The Nobel Prize in Physiology or Medicine 1979 was awarded jointly to Allan M. Cormack and Godfrey N. Hounsfield *"for the development of computer assisted tomography"***

# Radon transform

- Function of 2 variables <-> integrals over all the straight lines (position+slope)

$$R(p,\vec{\xi}) = \int_{-1}^{1} dx \int_{|x|-1}^{1-|x|} dy\, f(x,y)\delta(p-\vec{x}\vec{\xi})$$

# Inversion

- 1D vs 2D Fourier transform

$$F(\vec{q}) = \int d^2\vec{x} e^{i\vec{x}\vec{q}} f(\vec{x}) = \int_{-\infty}^{\infty} dt \delta(t - \vec{x}\vec{q}) \int d^2\vec{x} e^{i\vec{x}\vec{q}} f(\vec{x})$$

$$F(\vec{\xi}\lambda) = \int_{-\infty}^{\infty} dt \delta(t - \lambda\vec{x}\vec{\xi}) \int d^2\vec{x} e^{i\lambda\vec{x}\vec{\xi}} f(\vec{x}) =_{t\to\lambda t} \int_{-\infty}^{\infty} dt \delta(t - \vec{x}\vec{\xi}) \int d^2\vec{x} e^{i\lambda\vec{x}\vec{\xi}} f(\vec{x}) =$$
$$\int_{-\infty}^{\infty} dt e^{i\lambda t} \int d^2\vec{x} \delta(t - \vec{x}\vec{\xi}) f(\vec{x}) = \int_{-\infty}^{\infty} dt e^{i\lambda t} R(t, \vec{\xi})$$

- Inversion

$$f(\vec{x}) = \frac{1}{(2\pi)^2} \int d^2\vec{q} e^{-i\vec{x}\vec{q}} F(\vec{q}) = \frac{1}{(2\pi)^2} \int_0^{\infty} \lambda d\lambda \int_0^{2\pi} d\phi e^{-i\lambda\vec{x}\vec{\xi}} F(\lambda\vec{\xi}) =$$
$$\frac{1}{(2\pi)^2} \int_0^{\infty} \lambda d\lambda \int_0^{2\pi} d\phi e^{-i\lambda\vec{x}\vec{\xi}} \int_{-\infty}^{\infty} dp e^{i\lambda p} R(p, \vec{\xi})$$

# Simplification

- Average over tangent lines

$$\phi' = \phi, p' = p - \vec{x}\vec{\xi}$$

$$f(\vec{x}) = \frac{1}{(2\pi)^2}\int_0^\infty \lambda d\lambda \int_{-\infty}^\infty dp' e^{i\lambda p'} \int_0^{2\pi} d\phi' R(p' + \vec{\xi}\vec{x}, \vec{\xi}) \equiv$$

$$\frac{1}{2\pi}\int_0^\infty \lambda d\lambda \int_{-\infty}^\infty dp e^{i\lambda p} \bar{R}(p, \vec{x}).$$

$$\bar{R}(p, \vec{x}) = \frac{1}{2\pi}\int_0^{2\pi} d\phi R(p + \vec{\xi}\vec{x}, \vec{\xi})$$

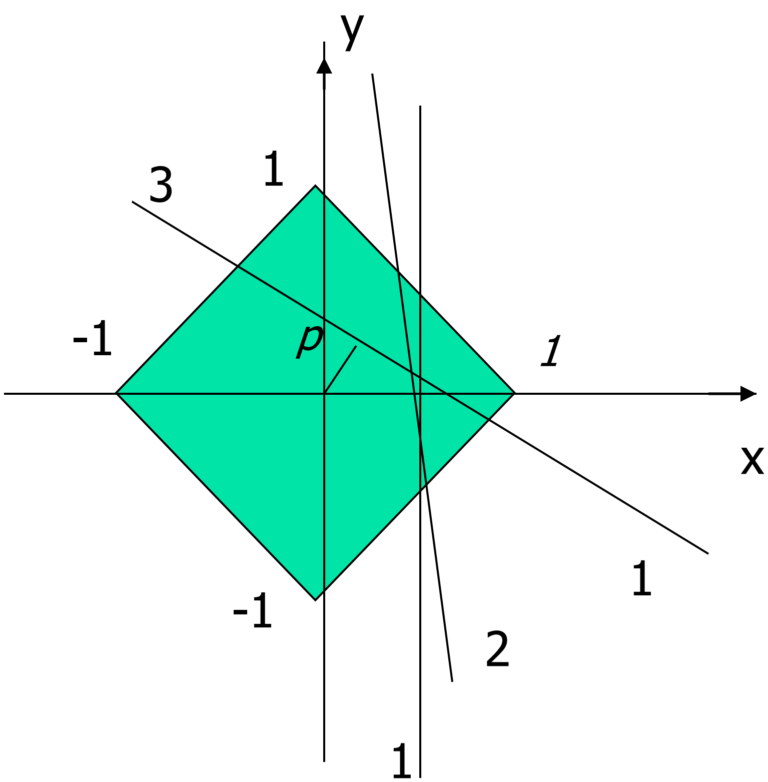
- Inverse Radon transform

$$f(\vec{x}) = \frac{1}{4\pi}\int_{-\infty}^\infty sign(\lambda)\lambda d\lambda dp e^{i\lambda p}\bar{R}(p, \vec{x}) = \frac{i}{4\pi}\int_{-\infty}^\infty sign(\lambda) d\lambda dp e^{i\lambda p}\bar{R}'_p(p, \vec{x}) =$$

$$-\frac{1}{2\pi}\int_{-\infty}^\infty \frac{dp}{p}\bar{R}'_p(p, \vec{x}) = -\frac{1}{\pi}\int_0^\infty \frac{dp}{p^2}(\bar{R}(p, \vec{x}) - \bar{R}(0, \vec{x}))$$

# (NP)QCD case: GPDs/DDs (D. Mueller et al, Radyushkin) are 1D/2D Fourier transforms of the same light-cone operators matrix elements

- Slope of the integration line- skewness
- Kinematics of DIS: $\xi = 0$ ("forward") - vertical line (1)
- Kinematics of DVCS: $\xi < 1$ - line 2
- Line 3: $\xi > 1$ unphysical region - required to restore DD by inverse Radon transform: tomography

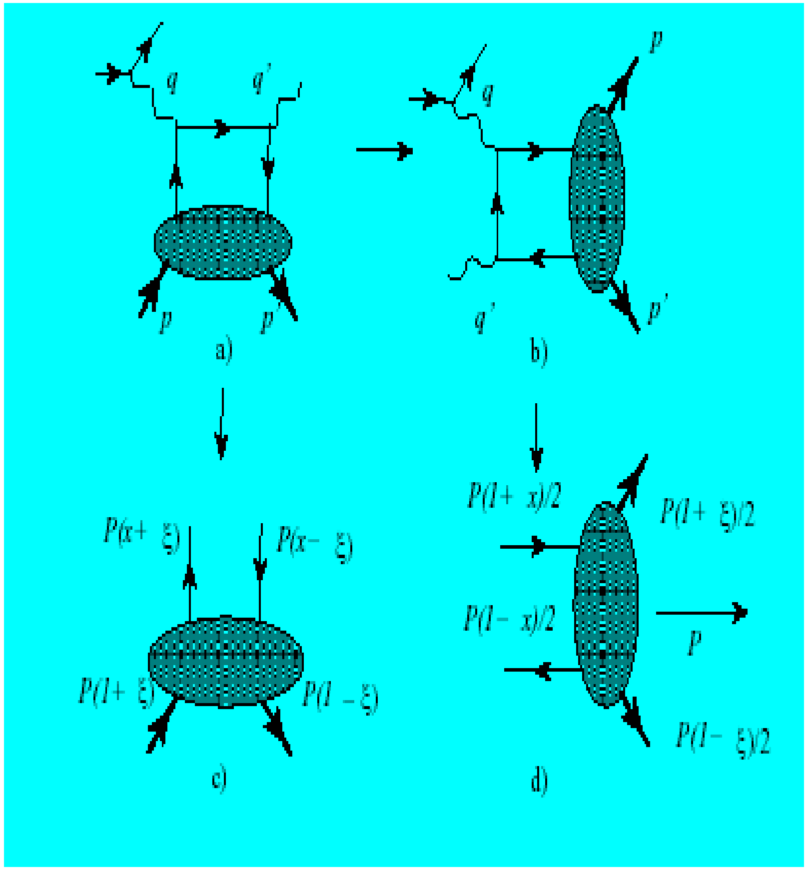



$$f(x,y) = -\frac{1}{2\pi^2}\int_0^\infty \frac{dp}{p^2}\int_0^{2\pi} d\phi|cos\phi|(H(p/cos\phi + x + ytg\phi, tg\phi) - H(x + ytg\phi, tg\phi)) =$$

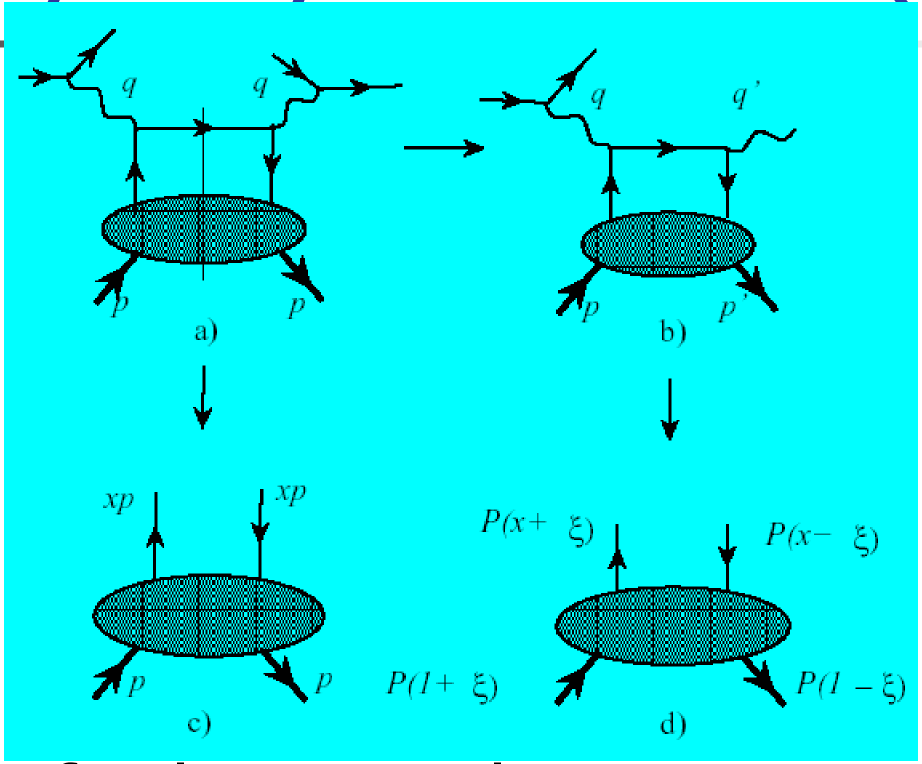
$$= -\frac{1}{2\pi^2}\int_{-\infty}^\infty \frac{dz}{z^2}\int_{-\infty}^\infty d\xi(H(z + x + y\xi, \xi) - H(x + y\xi, \xi))$$

# Crossing for DVCS and GPD

- DVCS -> hadron pair production in the collisions of real and virtual photons
- GPD -> Generalized Distribution Amplitudes (Diehl, Gousset, Pire, OT '98,…
- Also t -> s



# Radon transform and analyticity for DVCS (cf DIS)




- Manifestly spectral

$$\mathcal{H}(x_B) = \int_{-1}^{1} dx \frac{H(x)}{x - x_B + i\epsilon}$$

- Extra dependence on $\xi$

$$\mathcal{H}(\xi) = \int_{-1}^{1} dx \frac{H(x,\xi)}{x - \xi + i\epsilon},$$

# Analytic continuation

- DIS : Analytical function – if $1 \leq |x_B|$ polynomial in $1/x_B$

$$H(x_B) = -\int_{-1}^{1} dx \sum_{n=0}^{\infty} H(x) \frac{x^n}{x_B^{n+1}}$$

- DVCS – additional problem of analytical continuation of $H(x,\xi)$
- Solved by using of Double Distributions Radon transform

$$H(z,\xi) = \int_{-1}^{1} dx \int_{|x|-1}^{1-|x|} dy (F(x,y) + \xi G(x,y)) \delta(z - x - \xi y)$$

# Unphysical regions for DIS and DVCS

- Recall DIS

$$H(x_B) = -\int_{-1}^{1} dx \sum_{n=0}^{\infty} H(x) \frac{x^n}{x_B^{n+1}}$$

- Non-positive powers of $x_B$

- DVCS

$$H(\xi) = -\int_{-1}^{1} dx \sum_{n=0}^{\infty} H(x,\xi) \frac{x^n}{\xi^{n+1}}$$

- Polynomiality (general property of Radon transforms!*): moments - integrals in $x$ weighted with $x^n$ - are polynomials in $1/\xi$ of power $n+1$
- As a result, analyticity is preserved: only non-positive powers of $\xi$ appear
- *Cavalieri conditions

# Holographic property (OT'05)

- Factorization Formula ->

$$\mathcal{H}(\xi) = \int_{-1}^{1} dx \frac{H(x,\xi)}{x-\xi+i\epsilon}$$

- Analyticity ("dynamical") -> Imaginary part -> Dispersion relation:

$$\mathcal{H}(\xi) = \int_{-1}^{1} dx \frac{H(x,x)}{x-\xi+i\epsilon}$$

$$\Delta\mathcal{H}(\xi) \equiv \int_{-1}^{1} dx \frac{H(x,x)-H(x,\xi)}{x-\xi+i\epsilon}$$

- "Holographic" equation (DVCS AND VM)

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \xi^n} \int_{-1}^{1} H(x,\xi) dx (x-\xi)^{n-1} = const$$

# Holographic property - II

- Directly follows from double distributions

$$H(z,\xi) = \int_{-1}^{1} dx \int_{|x|-1}^{1-|x|} dy (F(x,y) + \xi G(x,y)) \delta(z - x - \xi y)$$

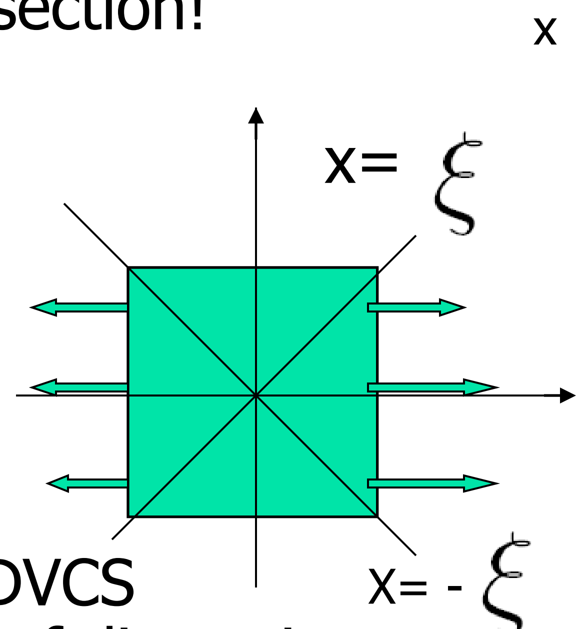
- Constant is the SUBTRACTION one - due to the (generalized) Polyakov-Weiss term G(x,y)

$$\Delta \mathcal{H}(\xi) = \int_{-1}^{1} dx \int_{|x|-1}^{1-|x|} dy \frac{G(x,y)}{1-y}$$

$$= \int_{-\xi}^{\xi} dx \frac{D(x/\xi)}{x - \xi + i\epsilon} = \int_{-1}^{1} dz \frac{D(z)}{z-1} = const$$

# Holographic property - III

- 2-dimensional space -> 1-dimensional section!
- Momentum space: any relation to holography in coordinate space ?!




- ERBL $\longrightarrow$ "GDA" region
- Strategy (now adopted) of GPD's

studies: start at diagonals

(through SSA due to imaginary part of DVCS amplitude ) and restore by making use of dispersion relations + subtraction constants

# Holographic property - IV

- Follows directly from DD -> preserved by (LO) evolution; NLO –Diehl, D.Ivanov'07
- Asymptotic GPD -> Pure real DVCS Amplitude (=subtraction term) growing like $\xi^{-2}$
- Direct consequence of finite asymptotic value of the quark momentum fraction

# Radon Tomography for Photons

- Require 2 channels (calls for universal description of GPDs and GDAs)
- Performed (Gabdrakhmanov,OT'12) for photon (Pire, Szymanowski, Wallon, Friot, El Beiyad) GPDs/GDAs

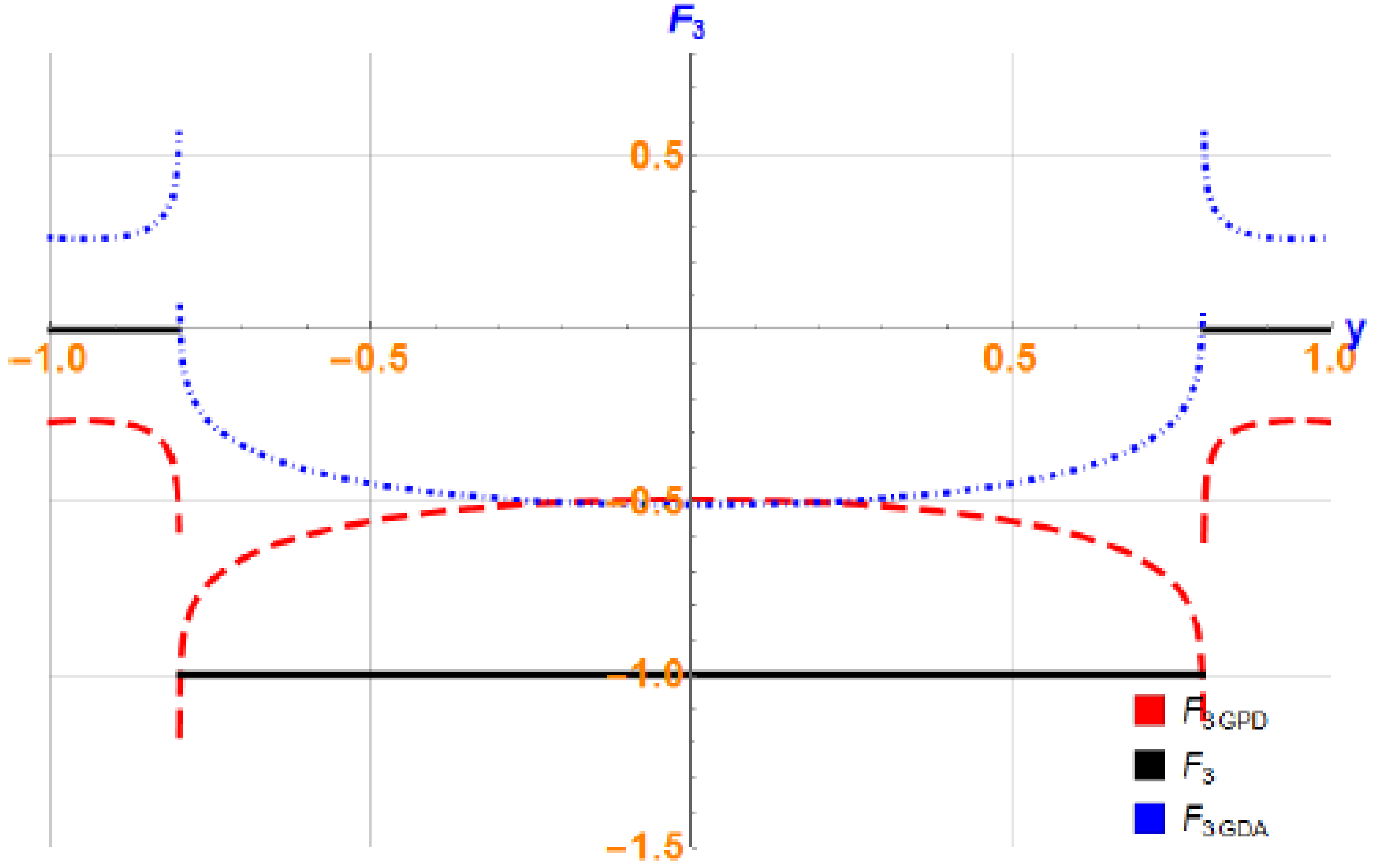
$F_{1D}(\beta,\alpha) = [2(1-|\beta|-|\alpha|)-1+\delta(\alpha)]sgn(\beta)$. $\quad D_1(\alpha) \;=\; (|\alpha|-1)(2|\alpha|+1)sgn(\alpha)$

- Realistic case – very difficult numerically
- Limited angle tomography?

# Channels separation (Gabdrakhmanov, D. Mueller, OT, in preparation)

- Unintegrated inverse double distribution – integratiion only over line position – slope dependent
- Reduced angle tomography – separate contributions for GPD/GDA channels
- Tests for photon GPDs/GDAs

# Channels separation for quarks in photons



# GDAs: Angular distribution in hadron pairs production

- Holographic equation –valid also in GDA region
- Moments of H(x,x) - define the coefficients of powers of cosine!– $1/\xi$
- Higher powers of cosine in t-channel – threshold in s - channel
- Larger for pion than for nucleon pairs because of less fast decrease at x ->1
- Continuation of D-term from t to s channel – dispersion relation in t (Pasquini, Vanderhaegen)

$$\mathscr{H}(\xi) = -\int_{-1/\xi}^{1/\xi} dx \sum_{n=0}^{\infty} H(x,\xi) \frac{x^n}{\xi^{n+1}}$$

$$= -\int_{-1/\xi}^{1/\xi} dx \sum_{n=0}^{\infty} H(x,x) \frac{x^n}{\xi^{n+1}} + \Delta\mathscr{H}.$$

# Analyticity of Compton amplitudes in energy plane (Anikin,OT'07)

- Finite subtraction implied

$$\mathrm{Re}\mathcal{A}(\nu,Q^2)=\frac{\nu^2}{\pi}\mathcal{P}\int_{\nu_0}^{\infty}\frac{d\nu'^2}{\nu'^2}\frac{\mathrm{Im}\mathcal{A}(\nu',Q^2)}{(\nu'^2-\nu^2)}+\Delta \qquad \Delta=2\int_{-1}^{1}d\beta\frac{D(\beta)}{\beta-1}$$

$$\Delta^p_{\mathrm{CQM}}(2)=\Delta^n_{\mathrm{CQM}}(2)\approx 4.4, \qquad \Delta^p_{\mathrm{latt}}\approx\Delta^n_{\mathrm{latt}}\approx 1.1$$

- Numerically close to Thomson term for REAL proton (but NOT neutron) Compton Scattering!
- Duality (sum of squares vs square of sum; proton: 4/9+4/9+1/9=1)?!
- Stability of subtraction against NPQCD?

# Generalization: GDA channel

- Real photons limit

$$\mathrm{Re}\mathcal{A}(\nu, Q^2) = \frac{\nu^2}{\pi}\mathcal{P}\int_{\nu_0}^{\infty}\frac{d\nu'^2}{\nu'^2}\frac{\mathrm{Im}\mathcal{A}(\nu', Q^2)}{(\nu'^2-\nu^2)} + \Delta$$

- v=(s-u)/4M -> (t-u)/4M
- Scattering at $90^0$ in c.m. is defined by subtractiion constant
- Dominance of Thomson term (better for proton-antiproton – sum of charges squared argument)

# Is D-term independent?

- Fast enough decrease at large energy -> $$\mathrm{Re}\,\mathcal{A}(\nu) = \frac{\mathcal{P}}{\pi}\int_{\nu_0}^{\infty} d\nu'^2 \frac{\mathrm{Im}\,\mathcal{A}(\nu')}{\nu'^2 - \nu^2} + \mathrm{C}_0$$

$$\mathrm{C}_0 = \Delta - \frac{\mathcal{P}}{\pi}\int_{\nu_0}^{\infty} d\nu'^2 \frac{\mathrm{Im}\,\mathcal{A}(\nu')}{\nu'^2}$$
$$= \Delta + \mathcal{P}\int_{-1}^{1} dx \frac{H^{(+)}(x,x)}{x}.$$

- FORWARD limit of Holographic equation

$$\Delta = \mathcal{P}\int_{-1}^{1} dx \frac{H^{(+)}(x,0) - H^{(+)}(x,x)}{x}$$
$$= 2\mathcal{P}\int_{-1}^{1} dx \frac{H(x,0) - H(x,x)}{x},$$

$$\mathrm{C}_0(t) = 2\mathcal{P}\int_{-1}^{1} dx \frac{H(x,0,t)}{x}$$

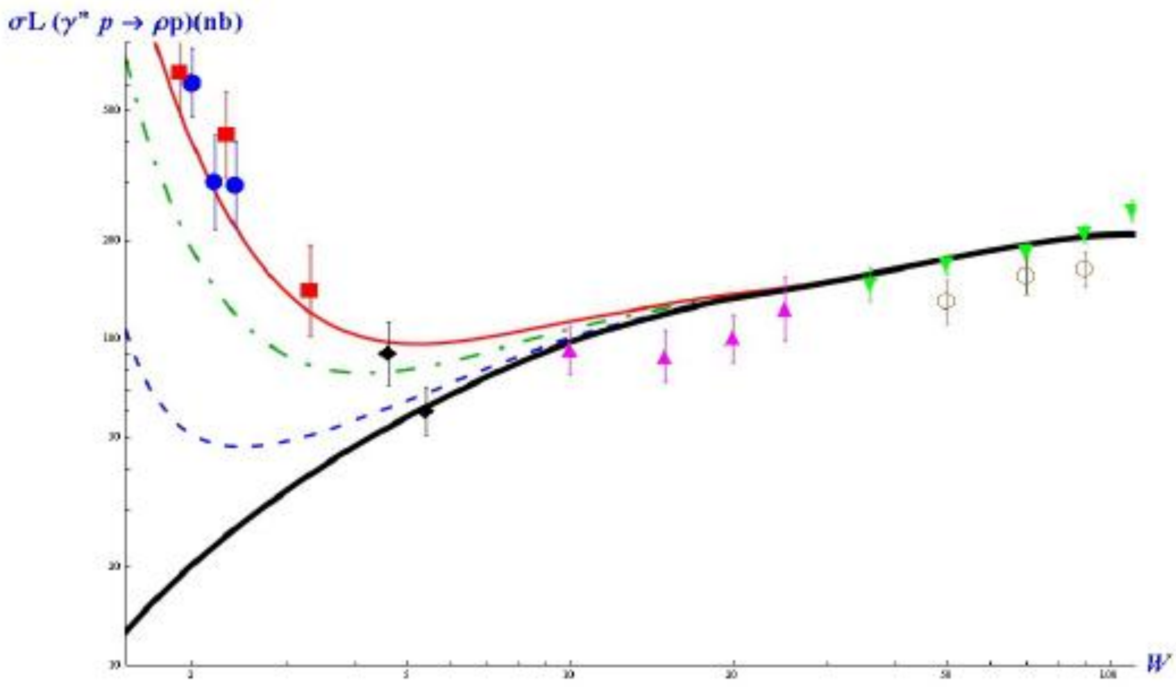
# "D – term" 30 years before…

- Cf Brodsky, Close, Gunion'72
- Divergence of inverse moment: D-term – a sort of renormalization constant?!
- Recover through special regularization procedure (D. Mueller, K. Semenov-Tyan-Shansky)?
- Cf physical (e.g. BLM) and MS renormalization schemes
- D-term: appear also for vector mesons

# Subtraction in exclusive electroproduction (absent in Brodsky et al. approach)

- May qualitatively explain the low energy enhancement (Gabdrakhmanov,OT'12)
- GK model

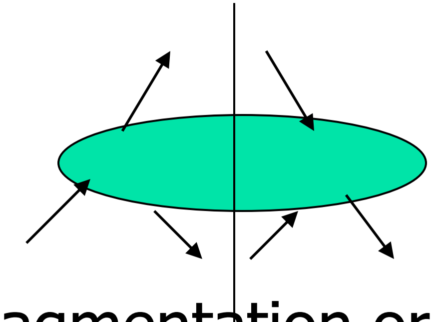
$$\sigma(W) \approx \sigma_0(W) \left| \frac{A_{Collinear}(W) + a \cdot \Delta}{A_{Collinear}(W)} \right|^2$$




- a=3,4.8
- Large – HT stability of D-term?

# Radon transform for semi-inclusive processes (in preparation)

- Consider semi-inclusive extension of DVMP in target fragmentation (TDA) region: γ*N -> NMX



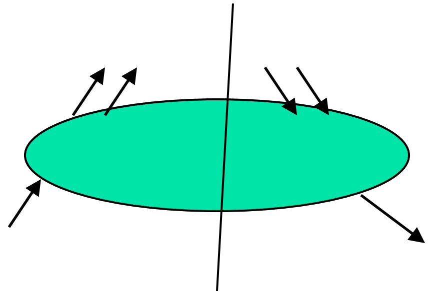
- Proceeds via fragmentation or fracture (Conditional Parton Distributions) functions (Trentadue, Veneziano)

# Radon transform for CPDs

- Key observations
- CPDs depend on the same variables as GPDs (x, ξ, t)
- Lorentz invariance: CPDs obey polynomiality
- Naturally reproduced (Cavalieri conditions) by $DD_{\_CPD}$!
- Neither relation between $x_B$ and ξ in hard kernel  nor holographic equation

# Inversion of Radon transform: crossing for CPDs?

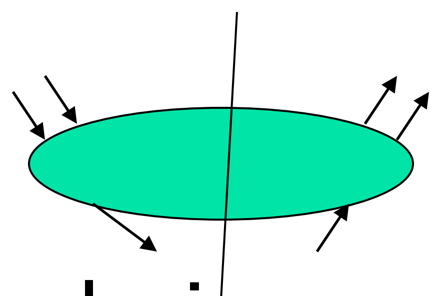
- If $|\xi|>1$ : GPDs -> GDAs; CPDs -> Dihadron Fragmentation functions



- (Chiral /T)-odd case – used to measure transversity (Bachetta,Courtoy,Radici)

# Extra difficulties for CPD -> 2hFF

- t (CPD) -> s (2hFF), like for GPD -> GDA



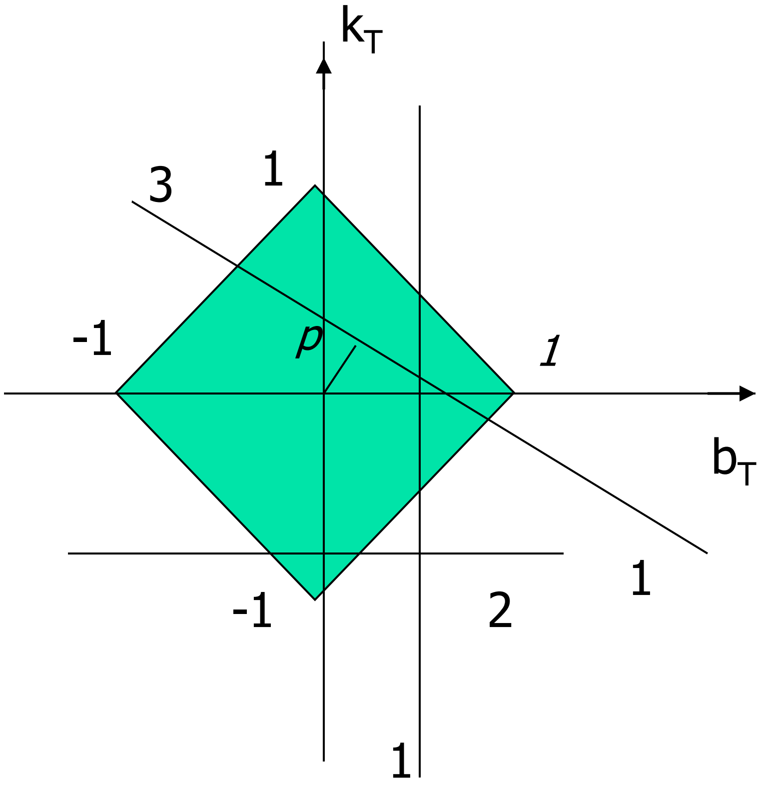
- For inclusive case – extra crossing x -> 1/z from “2 hadron pds’s (cf double parton pdf’s; are they observable?!)
- Large x,z – Gribov-Lipatov-type relations

# Radon-Wigner transform for GPDs and TMDs

- Radon transform of Wigner – positive - used in optics to recover the latter
- Suggestion: straight lines in $b_T$, $k_T$ plane
- Integration over $b_T$ : TMDs (Pasquini et al)
- Integration over $k_T$ : GPDs (-------//-------)
- Intermediate case: positive probability distribution, Wigner function may be recovered by inverse Radon transform

# TMDs/GPDs from Radon-Wigner transform

- GPDs( vertical lines 1) / TMDs( horizontal line 2): explored by B.Pasquini, C.Lorce, T.Liu
- Suggestion (Lines 3): "mixed" case – positive distribution required to restore Wigner function by inverse Radon transform $\zeta > 1$



# Complications

- QCD Wigner distributions - Operator matrix elements

- Radon transform of Wigner operator – Hermitian factorized operator ~ $VV^+$

- LC V - transform (completely similar to QM transform of WF) of large component of quark field

- Radon transform of TMD light-cone distribution of any states (e.g. wave packets – momentum eigentsates case - trivial) – positive

# Conclusions

- Radon transform combined with analyticity – powerful method for exploration of NPQCD matrix elements
- Crossing channels required,  limited angle tomography – rough estimate
- Semi-inclusive case – tomography for fracture and dihadron fragmentation functions
- Radon-Wigner transform – way to combine GPDs and TMDs ?

# Radon (OT’01) and Abel (Moiseeva, Polyakov’08) Transforms: even vs odd-dimensional spaces

- Even (integrals over lines in plane): integral (global) inversion formula
- Odd (integrals over planes in space) – differential (local) inversion formula – Huygens principle
- Triple distributions – THREE pions production (Pire, OT’01) or (deuteron) Decay PD. Relation to nuclei breakup in studies of SRC?!