Extended Conformal Symmetry of Abelian Gauge Theory in $d \neq 4$

Kelly Yu-Ju Chiu

SLAC National Accelerator Laboratory

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with
Stanley J. Brodsky

Overview

1. Scale vs. conformal invariance
2. Conformal non-invariance of Maxwell theory in $d \neq 4$
   - *Classical*: non-invariance of the action and equation of motion (EOM)
   - *Quantum*: mixing between physical and unphysical sectors
3. Extended conformal symmetry on the physical sector of Maxwell theory
   - Extended special conformal transformation (ESCT)
   - Invariance of action and classical EOM
   - Invariance of gauge-invariant transverse two-point correlation functions
4. Discussion

Global Conformal Transformations in d spacetime dimensions

In d spacetime dimensions, conformal group is identified with the noncompact $SO(d,2)$ group. It consists of

- Poincare group : translations, Lorentz rotations
- Scaling (dilatation)
- Special conformal transformations (SCT)

Scale invariance *usually* implies conformal invariance

Scale vs. conformal invariance

For Poincare invariant unitary theories...

$d = 2$:

- Scale invariance $\Rightarrow$ conformal invariance
  (Zamolodchikov and Polchinski)

$d > 2$:

- Scale invariance *usually* implies conformal invariance
- Counterexamples
  - higher spin theory in curved spacetime (Iorio)
  - 2 real scalars + one Weyl spinor in $d = 4 - \epsilon$ (Fortin)
  - **Maxwell theory in $d \neq 4$**

Is classical Maxwell theory conformally invariant?

Maxwell theory

$$\mathcal{L}(x) = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x)$$

Under scale transformation

$$\delta x^{\mu} = x^{\mu},$$

$$\delta \mathcal{L}(x) = -\partial_{\mu}[x^{\mu} \mathcal{L}(x)] \quad \text{(total derivative)}$$

$\Rightarrow$ Maxwell theory is scale invariant in any dimension

Is classical Maxwell theory conformally invariant?

Under the special conformal transformation (SCT)

$$\delta^\sigma x^\mu = 2x^\sigma x^\mu - g^{\sigma\mu} x^2,$$

$$\delta^\sigma A^\mu(x) = \big(x^2\partial^\sigma - 2x^\sigma x\cdot\partial - (d-2)x^\sigma\big)A^\mu(x) + 2x^\mu A^\sigma(x) - 2g^{\sigma\mu} x\cdot A(x)$$

- Action is non-invariant in $d \neq 4$

$$\delta^\sigma \mathcal{L}(x) = -\partial_\mu\big[(2x^\sigma x^\mu - g^{\sigma\mu}x^2)\mathcal{L}(x)\big] + (d-4)F^{\sigma\mu}(x)A_\mu(x)$$

- Transformation of the gauge-invariant field strength is gauge dependent in $d \neq 4$

$$\begin{aligned}\delta^\sigma F^{\mu\nu}(x) = &(x^2\partial^\sigma - 2x^\sigma x\cdot\partial - dx^\sigma)F^{\mu\nu}(x) + 2x^\mu F^{\sigma\nu}(x) - 2x^\nu F^{\sigma\mu}(x)\\ &+ 2g^{\sigma\mu}x_\alpha F^{\nu\alpha}(x) - 2g^{\sigma\nu}x_\alpha F^{\mu\alpha}(x)\\ &+ (d-4)[g^{\sigma\nu}A^\mu(x) - g^{\sigma\mu}A^\nu(x)]\end{aligned}$$

Is classical Maxwell theory conformally invariant?

- Classical EOM non-invariant in $d \neq 4$

$$\begin{aligned}
&\delta^\sigma \left(\partial_\mu F^{\mu\nu}(x)\right) \\
=&\left(x^2\partial^\sigma - 2x^\sigma x\cdot\partial - (d+2)x^\sigma\right)\partial_\mu F^{\mu\nu}(x) + 2x^\nu\partial_\mu F^{\mu\sigma}(x) \\
&- 2g^{\sigma\nu}x_\alpha\partial_\mu F^{\mu\alpha}(x) + (d-4)[g^{\sigma\nu}\partial\cdot A(x) - \partial^\nu A^\sigma(x)] \\
=&(d-4)[g^{\sigma\nu}\partial\cdot A(x) - \partial^\nu A^\sigma(x)] \\
\neq& 0
\end{aligned}$$

- $U(1)$ current conservation is NOT preserved in $d \neq 4$

$$\begin{aligned}
\delta^\sigma\partial_\mu J^\mu(x) &= \left(x^2\partial^\sigma - 2x^\sigma x\cdot\partial - (d+4)x^\sigma\right)\partial\cdot J(x) + (d-4)J^\sigma(x) \\
&= (d-4)J^\sigma(x) \\
&\neq 0\,!
\end{aligned}$$

$\Rightarrow$SCT is not compatible with gauge symmetry in $d \neq 4$

Recovering conformal symmetry in $\xi = d/(d-4)$ gauge

- Gauge-invariant classical Maxwell theory does not have the SCT invariance in $d \neq 4$
- El Showk *et al.* showed that the gauge fixed theory with

$$\mathcal{L}_\xi(x) = -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) - \frac{1}{2\xi}\big(\partial_\mu A^\mu(x)\big)^2$$

is SCT invariant in any dimension if $\xi = d/(d-4)$

Proof:
Under the SCT, the quantum two-point function is invariant

$$\delta^\sigma \langle A^\mu(x) A^\nu(0)\rangle = 0$$

Recovering conformal symmetry in $\xi = d/(d-4)$ gauge

– *Classical* : gauge-invariant and the gauge-fixed Maxwell theory are distinct

– *Quantum* : expectation values of gauge invariant operators are independent of specific gauge choices

$$\text{SCT Invariance in } \xi = d/(d-4)$$

$$\Downarrow ??$$

Quantum Maxwell theory is SCT invariant in any gauge

Is quantum Maxwell theory conformally invariant?

In QFT, a valid transformation $\mathcal{O}$ compatible with gauge symmetry should only map one physical state to another physical state.

In the BRST language, the $(d-2)$ physical/transverse modes $a^{\dagger}_{\perp_i}$ in d dimensions satisfy

$$[Q_B, a^{\dagger}_{\perp_i}(\mathbf{k})] = 0$$

$\Rightarrow$ $\mathcal{O}$ must commute with the BRST charge Q_B in the physical Hilbert space :

$$[[Q_B, \mathcal{O}], a^{\dagger}_{\perp_i}(\mathbf{k})] = 0$$

Check:
Generators of Poincare transformations and dilatation all satisfy

$$[[Q_B, \mathcal{O}], a^{\dagger}_{\perp_i}(\mathbf{k})] = 0$$

Is quantum Maxwell theory conformally invariant?

For the SCT generator δ^σ,

$$[[Q_B, \delta^\sigma], a^\dagger_{\perp_i}(\mathbf{k})] \sim (d-4)\varepsilon^\sigma_{\perp_i} c^\dagger(\mathbf{k}) \neq 0$$

SCT mixes physical modes with the unphysical sector in $d \neq 4$

$\Rightarrow$ SCT is not a physical transformation of quantum Maxwell theory in $d \neq 4$

Gauge field decomposition

To better understand how the physical and unphysical components transform under the SCT, we decompose

$$A^{\mu}(x) = A_T^{\mu}(x) + A_L^{\mu}(x),$$

$$A_L^{\mu}(x) = \frac{1}{\Box}\partial^{\mu}(\partial \cdot A(x))$$

$$A_T^{\mu}(x) = A^{\mu}(x) - \frac{1}{\Box}\partial^{\mu}(\partial \cdot A(x))$$

$$\begin{cases} A_L^{\mu}(x): \ \text{gauge dependent, and} \ \ \partial \cdot A = \partial \cdot A_L \\ \\ A_T^{\mu}(x): \ \text{gauge invariant, and} \ \ \partial \cdot A_T = 0 \end{cases}$$

SCT of A_L^μ and A_T^μ

- Under the SCT,

$$\delta^\sigma A^\mu(x) = \big(x^2\partial^\sigma - 2x^\sigma x\cdot\partial - (d-2)x^\sigma\big)A^\mu + 2x^\mu A^\sigma - 2g^{\sigma\mu}x\cdot A$$
$$\equiv K^\sigma A^\mu(x)$$

($K^\sigma A^\mu$ denotes the action of SCT on a primary vector field)

- SCT on A_L^μ and A_T^μ are defined as

$$\delta^\sigma A_L^\mu(x) \equiv \frac{1}{\Box}\partial^\mu\partial_\alpha[\delta^\sigma A^\alpha(x)]$$
$$\delta^\sigma A_T^\mu(x) \equiv \delta^\sigma A^\mu(x) - \frac{1}{\Box}\partial^\mu\partial_\alpha[\delta^\sigma A^\alpha(x)]$$

(SCT on A^μ followed by the projection into A_T^μ and A_L^μ)

SCT of A_L^μ and A_T^μ

$$\delta^\sigma A_T^\mu(x) = K^\sigma A_T^\mu(x) - d\frac{1}{\Box}\partial^\mu A_T^\sigma(x) - (d-4)\frac{1}{\Box}[\partial^\mu A_L^\sigma(x) - g^{\sigma\mu}\partial\cdot A_L(x)]$$

$$\delta^\sigma A_L^\mu(x) = K^\sigma A_L^\mu(x) + d\frac{1}{\Box}\partial^\mu A_T^\sigma(x) + (d-4)\frac{1}{\Box}[\partial^\mu A_L^\sigma(x) - g^{\sigma\mu}\partial\cdot A_L(x)]$$

A_T^μ and A_L^μ

- Under Poincare transformations and dilatation, transform as vectors
- Under the SCT, do NOT transform as conformal primary vectors

$$\delta^\sigma A_T^\mu(x) \neq K^\sigma A_T^\mu(x)$$

$$\delta^\sigma A_L^\mu(x) \neq K^\sigma A_L^\mu(x)$$

- SCT mixes physical and unphysical degrees of freedom in $d \neq 4$

Extended special conformal transformation (ESCT)

Define an *extended special conformal transformation* (ESCT)

$$\widetilde{\delta}^{\sigma} A_T^{\mu}(x) \equiv K^{\sigma} A_T^{\mu}(x) - d\frac{1}{\Box}\partial^{\mu} A_T^{\sigma}(x)$$

$$\widetilde{\delta}^{\sigma} A_L^{\mu}(x) \equiv K^{\sigma} A_L^{\mu}(x) + (d-4)\frac{1}{\Box}[\partial^{\mu} A_L^{\sigma}(x) - g^{\sigma\mu}\partial \cdot A_L(x)]$$

- ESCT does not mix physical and unphysical physical degrees of freedom
- A_T^{μ} is mapped only within the physical Hilbert space under the ESCT
- ESCT of the field strength is gauge invariant in any dimension

$$\begin{aligned}\widetilde{\delta}^{\sigma} F^{\mu\nu}(x) = &(x^2\partial^{\sigma} - 2x^{\sigma} x \cdot \partial - dx^{\sigma})F^{\mu\nu}(x) + 2x^{\mu}F^{\sigma\nu}(x) - 2x^{\nu}F^{\sigma\mu}(x)\\ &+ 2g^{\sigma\mu}x_{\alpha}F^{\nu\alpha}(x) - 2g^{\sigma\nu}x_{\alpha}F^{\mu\alpha}(x)\\ &+ (d-4)[g^{\sigma\nu}A_T^{\mu}(x) - g^{\sigma\mu}A_T^{\nu}(x)]\end{aligned}$$

$\Rightarrow$ ESCT is compatible with of gauge symmetry

ESCT invariance of classical Maxwell theory

- ESCT invariance of the action :

$$\widetilde{\delta}^{\sigma}\mathcal{L}(x) = -\partial_{\mu}\left[(2x^{\sigma}x^{\mu} - g^{\sigma\mu}x^{2})\mathcal{L}(x)\right] + (d-4)F^{\sigma}{}_{\mu}(x)A_{T}^{\mu}(x)$$

and

$$F^{\sigma}{}_{\mu}A_{T}^{\mu} = \partial^{\sigma}(\frac{A_{T}^{2}}{2}) - \partial^{\mu}(A_{T}^{\sigma}A_{T\mu}) + A_{T}^{\sigma}(\partial\cdot A_{T})$$

Since $\partial\cdot A_{T} = 0$,

$$\begin{aligned}\widetilde{\delta}^{\sigma}\mathcal{L}(x) &= -\partial_{\mu}\left[(2x^{\sigma}x^{\mu} - g^{\sigma\mu}x^{2})\mathcal{L}(x)\right] \\ &\quad + \partial_{\mu}\left[(d-4)\Big(\frac{g^{\sigma\mu}A_{T}^{2}(x)}{2} - A_{T}^{\sigma}(x)A_{T}^{\mu}(x)\Big)\right]\end{aligned}$$

$\Rightarrow$ Maxwell action is invariant under the ESCT

ESCT invariance of quantum Maxwell theory

- ESCT invariance of $\langle A_T^\mu(x) A_T^\nu(y) \rangle$:

Gauge-invariant transverse two-point function

$$\langle A_T^\mu(k) A_T^\nu(-k) \rangle \equiv \frac{1}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)$$

In position space,

$$\langle A_T^\mu(x) A_T^\nu(0) \rangle = \frac{1}{(x^2)^\triangle} \left(g^{\mu\nu} + 2 \triangle \frac{x^\mu x^\nu}{x^2} \right), \quad \triangle = \frac{d-2}{2}$$

Under the ESCT,

$$\begin{aligned}
&\widetilde{\delta}^\sigma \langle A_T^\mu(x) A_T^\nu(0) \rangle \\
&= \left\langle \left(\widetilde{\delta}^\sigma A_T^\mu(x)\right) A_T^\nu(0) \right\rangle + \left\langle A_T^\mu(x) \left(\widetilde{\delta}^\sigma A_T^\nu(0)\right) \right\rangle \\
&= K^\sigma \langle A_T^\mu(x) A_T^\nu(0) \rangle - d \frac{1}{\Box}\left(\partial^\mu \langle A_T^\sigma(x) A_T^\nu(0) \rangle - \partial^\nu \langle A_T^\mu(x) A_T^\sigma(0) \rangle \right)
\end{aligned}$$

ESCT invariance of quantum Maxwell theory

Since

$$K^{\sigma} \langle A_T^{\mu}(x) A_T^{\nu}(0) \rangle = \frac{d}{(x^2)^{\triangle}} \left(g^{\sigma\nu} x^{\mu} - g^{\sigma\mu} x^{\nu} \right)$$

and

$$- d \frac{1}{\Box} \left(\partial^{\mu} \langle A_T^{\sigma}(x) A_T^{\nu}(0) \rangle - \partial^{\nu} \langle A_T^{\mu}(x) A_T^{\sigma}(0) \rangle \right)$$
$$= -d \frac{1}{(x^2)^{\triangle}} \left(g^{\sigma\nu} x^{\mu} - g^{\sigma\mu} x^{\nu} \right)$$

$$\Rightarrow \widetilde{\delta}^{\sigma} \langle A_T^{\mu}(x) A_T^{\nu}(0) \rangle = \frac{d}{(x^2)^{\triangle}} \left(g^{\sigma\nu} x^{\mu} - g^{\sigma\mu} x^{\nu} \right) - \frac{d}{(x^2)^{\triangle}} \left(g^{\sigma\nu} x^{\mu} - g^{\sigma\mu} x^{\nu} \right)$$
$$= 0$$

ESCT invariance of quantum Maxwell theory

Since $U(1)$ gauge theory is a non-interacting QFT,

Invariance of the gauge-invariant two-point function

$$\Downarrow$$

Invariance of all gauge-invariant n-point functions

$$\widetilde{\delta}^{\sigma} \langle A_T^{\mu}(x) A_T^{\nu}(0) \rangle = 0$$

$$\Downarrow$$

ESCT is a symmetry of the quantum Maxwell theory

Discussion

Summary

- Studied why the SCT is not a symmetry of Maxwell theory in $d \neq 4$ – SCT is not compatible with gauge symmetry
- Constructed an *extended special conformal transformation*(ESCT) which only maps physical states within the physical Hilbert space
- Unveiled a new symmetry of the gauge invariant physical sector of both classical and quantum Maxwell theory in $d \neq 4$

Future work

- Non-Abelian theories
- Abelian gauge theory in curved spacetime