Light-Front Perturbation without Spurious Singularities

Jerzy A. Przeszowski

Faculty of Physics,
University of Białystok, Poland

21-25 September 2015
LightCone 2015
INFN Frascati National Laboratories

A new form of the light front Feynman propagators is proposed. It contains no energy denominators. Instead the dependence on the longitudinal subinterval $\boldsymbol{x}_L^2 = 2x^+x^-$ is explicit and a new formalism for doing the perturbative calculations is invented. These novel propagators are implemented for the one-loop effective potential and vacuum polarization for a massive scalar field. The consistency with results for the standard covariant Feynman diagrams is obtained and no spurious singularities are encountered at any step. Some remarks on the calculations with fermion and gauge fields in QED and QCD are added.

Outlook

1. Definitions and introductory remarks about the light front (LF)
   - Definitions in D = 3+1
   - LF propagator - standard approach
   - Novel LF representation of propagator
2. Convolutions of LF propagators
   - Two propagators
   - Convolution of three propagators
   - Convolution of n propagators and effective potential
3. One loop 2-point functions
   - Scalar vacuum polarization function
   - Vector vacuum polarization - I case
     - Longitudinal component
     - Transverse component
   - Vector vacuum polarization - II case
4. Conclusions and prospects

Definitions in D = 3+1

Definitions in D = 3+1

coordinates

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}, \quad x^- = \frac{x^0 - x^3}{\sqrt{2}}, \quad \boldsymbol{x}_L = (x^+, x^-) \quad \boldsymbol{x}_\perp = (x^1, x^2)$$

momenta

$$p^+ = \frac{p^0 + p^3}{\sqrt{2}}, \quad p^- = \frac{p^0 - p^3}{\sqrt{2}}, \quad \boldsymbol{p}_L = (p^+, p^-) \quad \boldsymbol{p}_\perp = (p^1, p^2)$$

longitudinal (L) and transverse (⊥) scalar subproducts

$$\boxed{\boldsymbol{x}_L{}^2 = 2x^+x^-, \quad \boldsymbol{x}_L \cdot \boldsymbol{p}_L = x^+p^- + x^-p^-,} \quad \boxed{\boldsymbol{x}_\perp{}^2 = (x^1)^2 + (x^2)^2, \quad \boldsymbol{x}_\perp \cdot \boldsymbol{p}_\perp = x^1p^1 + x^2p^2,}$$

partial derivatives

$$\partial_+ = \frac{\partial}{\partial x^+}, \quad \partial_- = \frac{\partial}{\partial x^-}, \quad \partial_i = \frac{\partial}{\partial x^i}, \quad i \in \{1, 2\}$$

Wightman functions and Feynman propagators

Wightman function for a free scalar field $\langle 0|\phi(x)\phi(0)|0\rangle = W_2(x)$

$$W_2(x^+, x^-, \boldsymbol{x}_\perp) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{-i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp} \int_0^\infty \frac{dk^+}{4\pi k^+} e^{-ik^+x^-}\, e^{-i\frac{m^2+\boldsymbol{k}_\perp^2}{2k^+}x^+}.$$

LF propagator $\Delta_{LF}(x)$ is defined as

$$\begin{aligned} \Delta_{LF}(x) = \langle 0|T^+\phi(x)\phi(0)|0\rangle \quad &:= \quad \Theta(x^+)\langle 0|\phi(x)\phi(0)|0\rangle + \Theta(-x^+)\langle 0|\phi(0)\phi(x)|0\rangle = \\ &= \quad \Theta(x^+)W_2(x^+, x^-, \boldsymbol{x}_\perp) + \Theta(-x^+)W_2(-x^+, -x^-, -\boldsymbol{x}_\perp). \end{aligned}$$

Outlook

1. Definitions and introductory remarks about the light front (LF)
   - Definitions in D = 3+1
   - LF propagator - standard approach
   - Novel LF representation of propagator
2. Convolutions of LF propagators
   - Two propagators
   - Convolution of three propagators
   - Convolution of n propagators and effective potential
3. One loop 2-point functions
   - Scalar vacuum polarization function
   - Vector vacuum polarization - I case
     - Longitudinal component
     - Transverse component
   - Vector vacuum polarization - II case
4. Conclusions and prospects

LF standard approach

$$\Theta(x^+) = \int_{\mathbb{R}} \frac{dk^-}{2\pi} e^{-ik^- x^+} \frac{i}{k^- + i0},$$

$$\int_{\mathbb{R}} \frac{dk^-}{2\pi} e^{-ik^- x^+} \frac{i}{k^- + i0} \int_0^\infty \frac{dk^+}{k^+} e^{-ik^+ x^-} e^{-ix^+ (m^2 + k_\perp^2)/(2k^+)}$$
$$\stackrel{?}{=} \int_0^\infty dk^+ e^{-ik^+ x^-} \int_{\mathbb{R}} \frac{dk^-}{2\pi} e^{-ik^- x^+} \frac{1}{k^+} \frac{i}{k^- - \frac{m^2 + k_\perp^2}{2k^+} + i0}.$$

LF standard approach

$$\Theta(x^+) = \int_{\mathbb{R}} \frac{dk^-}{2\pi} e^{-ik^-x^+} \frac{i}{k^- + i0},$$

$$\int_{\mathbb{R}} \frac{dk^-}{2\pi} e^{-ik^-x^+} \frac{i}{k^- + i0} \int_0^\infty \frac{dk^+}{k^+} e^{-ik^+x^-} e^{-ix^+(m^2+k_\perp^2)/(2k^+)}$$
$$\stackrel{?}{=} \int_0^\infty dk^+ e^{-ik^+x^-} \int_{\mathbb{R}} \frac{dk^-}{2\pi} e^{-ik^-x^+} \frac{1}{k^+} \frac{i}{k^- - \frac{m^2+k_\perp^2}{2k^+} + i0}.$$

$$\Theta(x^0) = \int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{-ik_0x^0} \frac{i}{k_0 + i0},$$

$$\int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{-ik_0x^0} \frac{i}{k_0 + i0} \int_{\boldsymbol{k}} \frac{1}{2\omega_k} e^{-ix^0\omega_k} + \int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{ik_0x^0} \frac{i}{-k_0 + i0} \int_{\boldsymbol{k}} \frac{1}{2\omega_k} e^{ix^0\omega_k}$$
$$= \int_{\boldsymbol{k}} \int_{\mathbb{R}} \frac{dk_0}{2\pi} \frac{e^{-ik_0x^0}}{2\omega_k} \left(\frac{i}{k_0 - \omega_k + i0} - \frac{i}{k_0 + \omega_k - i0} \right), \qquad \omega_k = \sqrt{m^2 + \vec{\boldsymbol{k}}^2}$$

Outlook

1. Definitions and introductory remarks about the light front (LF)
   - Definitions in D = 3+1
   - LF propagator - standard approach
   - Novel LF representation of propagator
2. Convolutions of LF propagators
   - Two propagators
   - Convolution of three propagators
   - Convolution of n propagators and effective potential
3. One loop 2-point functions
   - Scalar vacuum polarization function
   - Vector vacuum polarization - I case
     - Longitudinal component
     - Transverse component
   - Vector vacuum polarization - II case
4. Conclusions and prospects

Novel LF representation of propagator

for $x^+ > 0$ we take $k^+ = 2\lambda x^+$

$$W_2(x) \simeq \int_0^\infty \frac{dk^+}{4\pi k^+} e^{-ik^+x^-} \, e^{-i\frac{m^2+\boldsymbol{k}_\perp^2}{2k^+}x^+} = \int_0^\infty \frac{d\lambda}{4\pi\lambda} e^{-i\lambda \boldsymbol{x}_L^2} \, e^{-i(m^2+\boldsymbol{k}_\perp^2)/(4\lambda)}$$

for $x^+ < 0$ we take $k^+ = -2\lambda x^+$

$$W_2(-x) \simeq \int_0^\infty \frac{dk^+}{4\pi k^+} e^{ik^+x^-} \, e^{i\frac{m^2+\boldsymbol{k}_\perp^2}{2k^+}x^+} = \int_0^\infty \frac{d\lambda}{4\pi\lambda} e^{-i\lambda \boldsymbol{x}_L^2} \, e^{-i(m^2+\boldsymbol{k}_\perp^2)/(4\lambda)}$$

This leads to the λ representation of the LF propagator

$$\langle 0|T^+\phi(x)\phi(0)|0\rangle = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} \int_0^\infty \frac{d\lambda}{4\pi\lambda} e^{-i\lambda \boldsymbol{x}_L^2} \, e^{-iM^2/(4\lambda)} = \Delta_{LF}(x),$$

where $\boldsymbol{x}_L^2 = 2x^+x^-$ and $M^2 = m^2 + \boldsymbol{k}_\perp^2$

Novel LF representation of propagator

Let us take the Fourier transform in the 2-dimensional longitudinal subspace

$$e^{-i\lambda \boldsymbol{x}_L^2} = \int_{\mathbb{R}^2} \frac{dk^+ dk^-}{4\pi\lambda} e^{-i(k^+x^- + k^-x^-)} e^{ik^+k^-/(2\lambda)} = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_L}{4\pi\lambda} e^{-i\boldsymbol{k}_L\cdot\boldsymbol{x}_L} e^{i\boldsymbol{k}_L^2/(4\lambda)},$$

which leads to

$$\begin{aligned}\Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} \int_0^\infty \frac{d\lambda}{4\pi\lambda} e^{-iM^2/(4\lambda)} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_L}{4\pi\lambda} e^{-i\boldsymbol{k}_L\cdot\boldsymbol{x}_L} e^{i\boldsymbol{k}_L^2/(4\lambda)} = \\ &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_L}{(4\pi)^2} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} e^{-i\boldsymbol{k}_L\cdot\boldsymbol{x}_L} \boxed{\int_0^\infty \frac{d\lambda}{\lambda^2} e^{i(\boldsymbol{k}_L^2 - M^2)/(4\lambda)}}.\end{aligned}$$

Novel LF representation of propagator

Let us take the Fourier transform in the 2-dimensional longitudinal subspace

$$e^{-i\lambda \boldsymbol{x}_L^2} = \int_{\mathbb{R}^2} \frac{dk^+dk^-}{4\pi\lambda} e^{-i(k^+x^-+k^-x^-)} e^{ik^+k^-/(2\lambda)} = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_L}{4\pi\lambda} e^{-i\boldsymbol{k}_L\cdot\boldsymbol{x}_L} e^{i\boldsymbol{k}_L^2/(4\lambda)},$$

which leads to

$$\begin{aligned}\Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} \int_0^\infty \frac{d\lambda}{4\pi\lambda} e^{-iM^2/(4\lambda)} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_L}{4\pi\lambda} e^{-i\boldsymbol{k}_L\cdot\boldsymbol{x}_L} e^{i\boldsymbol{k}_L^2/(4\lambda)} = \\ &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_L}{(4\pi)^2} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} e^{-i\boldsymbol{k}_L\cdot\boldsymbol{x}_L} \boxed{\int_0^\infty \frac{d\lambda}{\lambda^2} e^{i(\boldsymbol{k}_L^2-M^2)/(4\lambda)}.}\end{aligned}$$

The integral over λ is defined in the sense of distributions

$$\boxed{\int_0^\infty \frac{d\lambda}{\lambda^2} e^{iA/\lambda} = \frac{i}{A+i0},}$$

thus we have the correspondence with the covariant Feynman propagator

$$\begin{aligned}\Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_L}{(4\pi)^2} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} e^{-i\boldsymbol{k}_L\cdot\boldsymbol{x}_L} \frac{4i}{k_L^2 - k_\perp^2 + m^2 + i0} = \\ &= \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 - m^2 + i0} = \Delta_F(x).\end{aligned}$$

Outlook

Two propagators

We define the convolution of two LF propagators as follows

$$[\Delta_{LF} * \Delta_{LF}](x-z) = \Delta_{LF}^2(x-z) := \int_{\mathbb{R}^4} d^4y \Delta_{LF}(x-y)\Delta_{LF}(y-z).$$

Two propagators

We define the convolution of two LF propagators as follows

$$[\Delta_{LF} * \Delta_{LF}](x-z) = \Delta^2_{LF}(x-z) := \int_{\mathbb{R}^4} d^4y \Delta_{LF}(x-y)\Delta_{LF}(y-z).$$

Thus for the λ representation

$$\begin{aligned} \Delta^2_{LF}(x-z) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_\perp e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} e^{-i\boldsymbol{p}_{2\perp}\cdot(\boldsymbol{y}_\perp-\boldsymbol{z}_\perp)} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_L e^{-i2\lambda_1(x^+-y^+)(x^--y^-)} e^{-i2\lambda_2(y^+-z^+)(y^--z^-)} \times \\ &\times e^{-iM_1^2/(4\lambda_1)} e^{-iM_2^2/(4\lambda_2)}, \end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$.

Two propagators

We define the convolution of two LF propagators as follows

$$[\Delta_{LF} * \Delta_{LF}](x-z) = \Delta_{LF}^2(x-z) := \int_{\mathbb{R}^4} d^4y \Delta_{LF}(x-y)\Delta_{LF}(y-z).$$

Thus for the λ representation

$$\begin{aligned} \Delta_{LF}^2(x-z) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_\perp e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} e^{-i\boldsymbol{p}_{2\perp}\cdot(\boldsymbol{y}_\perp-\boldsymbol{z}_\perp)} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_L e^{-i2\lambda_1(x^+-y^+)(x^--y^-)} e^{-i2\lambda_2(y^+-z^+)(y^--z^-)} \times \\ &\times e^{-iM_1^2/(4\lambda_1)}\, e^{-iM_2^2/(4\lambda_2)}, \end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$.
The integration over the transverse and longitudinal coordinates give

$$\begin{aligned} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_\perp e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} e^{-i\boldsymbol{p}_{2\perp}\cdot(\boldsymbol{y}_\perp-\boldsymbol{z}_\perp)} &= (2\pi)^2\delta^2(\boldsymbol{p}_{1\perp}-\boldsymbol{p}_{2\perp}) e^{-i\boldsymbol{p}_{1\perp}\cdot\boldsymbol{x}_\perp} e^{i\boldsymbol{p}_{2\perp}\cdot\boldsymbol{z}_\perp}, \\ \int_{\mathbb{R}^2} d^2\boldsymbol{y}_L e^{-i2\lambda_1(x^+-y^+)(x^--y^-)} e^{-i2\lambda_2(y^+-z^+)(y^--z^-)} &= \frac{\pi}{\lambda_1+\lambda_2} e^{-i2\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(x^+-z^+)(x^--z^-)}. \end{aligned}$$

Two propagators

- Thus we obtain

$$\Delta^2_{LF}(x-z) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp - \boldsymbol{z}_\perp)} \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} \times \frac{\pi}{\lambda_1+\lambda_2} e^{-i2\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(x^+-z^+)(x^--z^-)} e^{-iM_1^2/(4\lambda_1)} e^{-iM_1^2/(4\lambda_2)}.$$

Two propagators

- Thus we obtain

$$\Delta^2_{LF}(x-z) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp - \boldsymbol{z}_\perp)} \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} \times \frac{\pi}{\lambda_1+\lambda_2} e^{-i2\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(x^+-z^+)(x^--z^-)} e^{-iM_1^2/(4\lambda_1)} e^{-iM_1^2/(4\lambda_2)}.$$

- For evaluating the integrals over $\lambda_{1,2}$ we introduce new parameters $\lambda \in \langle 0, \infty)$ and $\xi \in \langle 0, 1\rangle$ defined as

$$\boxed{\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}, \quad \xi = \frac{\lambda_2}{\lambda_1+\lambda_2}} \quad \lambda_1 = \frac{\lambda}{\xi}, \quad \lambda_2 = \frac{\lambda}{1-\xi}, \quad \mathcal{J} = \frac{\lambda}{\xi^2(1-\xi)^2}$$

thus we have

$$\int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} \frac{\pi}{\lambda_1+\lambda_2} e^{-i2\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(x^+-z^+)(x^--z^-)} e^{-iM_1^2/(4\lambda_1)} e^{-iM_1^2/(4\lambda_2)} =$$

$$= \frac{1}{16\pi} \int_0^\infty \frac{d\lambda}{\lambda^2} \boxed{\int_0^1 d\xi}\, e^{-i2\lambda(x^+-z^+)(x^--z^-)} e^{-i\frac{M_1^2}{4\lambda}} = \frac{1}{16\pi} \int_0^\infty \frac{d\lambda}{\lambda^2} e^{-i\lambda(\boldsymbol{x}-\boldsymbol{z})^2_L} e^{-i\frac{M_1^2}{4\lambda}}.$$

Two propagators

- Thus we obtain

$$\Delta^2_{LF}(x-z) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp - \boldsymbol{z}_\perp)} \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} \times \frac{\pi}{\lambda_1+\lambda_2} e^{-i2\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(x^+-z^+)(x^--z^-)} e^{-iM_1^2/(4\lambda_1)} e^{-iM_1^2/(4\lambda_2)}.$$

- For evaluating the integrals over $\lambda_{1,2}$ we introduce new parameters $\lambda \in \langle 0, \infty)$ and $\xi \in \langle 0, 1\rangle$ defined as

$$\boxed{\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}, \quad \xi = \frac{\lambda_2}{\lambda_1+\lambda_2}} \quad \lambda_1 = \frac{\lambda}{\xi}, \quad \lambda_2 = \frac{\lambda}{1-\xi}, \quad \mathcal{J} = \frac{\lambda}{\xi^2(1-\xi)^2}$$

thus we have

$$\int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} \frac{\pi}{\lambda_1+\lambda_2} e^{-i2\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(x^+-z^+)(x^--z^-)} e^{-iM_1^2/(4\lambda_1)} e^{-iM_1^2/(4\lambda_2)} =$$

$$= \frac{1}{16\pi} \int_0^\infty \frac{d\lambda}{\lambda^2} \boxed{\int_0^1 d\xi}\, e^{-i2\lambda(x^+-z^+)(x^--z^-)} e^{-i\frac{M_1^2}{4\lambda}} = \frac{1}{16\pi} \int_0^\infty \frac{d\lambda}{\lambda^2} e^{-i\lambda(\boldsymbol{x}-\boldsymbol{z})_L^2} e^{-i\frac{M_1^2}{4\lambda}}.$$

- λ representation for the convolution of two LF propagators

$$\boxed{\Delta^2_{LF}(x-z) = \frac{1}{4} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{z}_\perp)} \int_0^\infty \frac{d\lambda}{4\pi\lambda^2} e^{-i\lambda(\boldsymbol{x}_L-\boldsymbol{z}_L)^2} e^{-i\frac{m^2+\boldsymbol{p}_\perp^2}{4\lambda}}.}$$

Two propagators

- For comparing LF and covariant results we introduce the longitudinal momenta as

$$e^{-i\lambda(\boldsymbol{x}_L-\boldsymbol{z}_L)^2} = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{4\pi\lambda} e^{-i\boldsymbol{p}_L\cdot(\boldsymbol{x}_L-\boldsymbol{z}_L)} e^{i\boldsymbol{p}_L^2/(4\lambda)},$$

thus we have

$$\begin{aligned}\Delta_{LF}^2(x-z) &= \frac{1}{64\pi^2}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{z}_\perp)}\int_{\mathbb{R}^2}d^2\boldsymbol{p}_L\int_0^\infty\frac{d\lambda}{\lambda^3}e^{-i\boldsymbol{p}_L\cdot(\boldsymbol{x}_L-\boldsymbol{z}_L)}e^{i(\boldsymbol{p}_L^2-m^2-\boldsymbol{p}_\perp^2)/(4\lambda)} = \\ &= \frac{1}{16}\int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-z)}\int_0^\infty\frac{d\lambda}{\lambda^3}e^{i(p^2-m^2)/(4\lambda)}.\end{aligned}$$

The integral over λ can be performed explicitly according to

$$\int_0^\infty \frac{d\lambda}{\lambda^3}e^{iA/\lambda} = -\frac{1}{(A+\mathrm{i}0)^2},$$

leading to

$$\Delta_{LF}^2(x-z) = \int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-z)}\frac{i^2}{(p^2-m^2+\mathrm{i}0)^2},$$

which agrees with the covariant result.

Outlook

Convolution of three propagators

- We define the convolution of three LF propagators as follows

$$[\Delta_{LF} * \Delta_{LF} * \Delta_{LF}](x-z) = \Delta_{LF}^3(x-z) := \int_{\mathbb{R}^4} d^4y \Delta_{LF}(x-y)\Delta_{LF}^2(y-z).$$

Convolution of three propagators

- We define the convolution of three LF propagators as follows

$$[\Delta_{LF} * \Delta_{LF} * \Delta_{LF}](x-z) = \Delta^3_{LF}(x-z) := \int_{\mathbb{R}^4} d^4y \Delta_{LF}(x-y)\Delta^2_{LF}(y-z).$$

- Thus we have

$$\begin{aligned}\Delta^3_{LF}(x-z) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_\perp e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} e^{-i\boldsymbol{p}_{2\perp}\cdot(\boldsymbol{y}_\perp-\boldsymbol{z}_\perp)} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{16\pi\lambda_2^2} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_L e^{-i2\lambda_1(x^+-y^+)(x^--y^-)} e^{-i2\lambda_2(y^+-z^+)(y^--z^-)} \times \\ &\times e^{-iM_1^2/(4\lambda_1)} e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$.

Convolution of three propagators

- We define the convolution of three LF propagators as follows

$$[\Delta_{LF} * \Delta_{LF} * \Delta_{LF}](x-z) = \Delta^3_{LF}(x-z) := \int_{\mathbb{R}^4} d^4y \Delta_{LF}(x-y)\Delta^2_{LF}(y-z).$$

- Thus we have

$$\begin{aligned}\Delta^3_{LF}(x-z) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_\perp e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} e^{-i\boldsymbol{p}_{2\perp}\cdot(\boldsymbol{y}_\perp-\boldsymbol{z}_\perp)} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{16\pi\lambda_2^2} \int_{\mathbb{R}^2} d^2\boldsymbol{y}_L e^{-i2\lambda_1(x^+-y^+)(x^--y^-)} e^{-i2\lambda_2(y^+-z^+)(y^--z^-)} \times \\ &\times e^{-iM_1^2/(4\lambda_1)}\, e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$.

- These integrals are very similar to those for the convolution of two propagators. The integration over the transverse coordinates and the integration over the longitudinal coordinates is unchanged, thus we have

$$\begin{aligned}\Delta^3_{LF}(x-z) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} e^{-i\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{x}_\perp-\boldsymbol{z}_\perp)} \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{16\pi\lambda_2^2} \\ &\times \frac{\pi}{\lambda_1+\lambda_2} e^{-i2\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(x^+-z^+)(x^--z^-)}\, e^{-iM_1^2/(4\lambda_1)} e^{-iM_1^2/(4\lambda_2)}.\end{aligned}$$

Three propagators

For evaluating the integrals over $\lambda_{1,2}$ we again introduce the parameters

$$\boxed{\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}, \quad \xi = \frac{\lambda_2}{\lambda_1 + \lambda_2}} \lambda_1 = \frac{\lambda}{\xi}, \qquad \lambda_2 = \frac{\lambda}{1-\xi}, \quad \mathcal{J} = \frac{\lambda}{\xi^2(1-\xi)^2}$$

thus we have

$$\int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{16\pi\lambda_2^2} \frac{\pi}{\lambda_1+\lambda_2} e^{-i2\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(x^+-z^+)(x^--z^-)} e^{-iM_1^2/(4\lambda_1)} e^{-iM_1^2/(4\lambda_2)} =$$

$$= \frac{1}{64\pi} \int_0^\infty \frac{d\lambda}{\lambda^3} \boxed{\int_0^1 d\xi(1-\xi)} e^{-i2\lambda(x^+-z^+)(x^--z^-)} e^{-i\frac{M_1^2}{4\lambda}} = \frac{1}{128\pi} \int_0^\infty \frac{d\lambda}{\lambda^3} e^{-i\lambda(\boldsymbol{x}-\boldsymbol{z})_L^2} e^{-i\frac{M_1^2}{4\lambda}}.$$

Thus the convolution of three LF propagators in λ representation is given by

$$\boxed{\Delta_{LF}^3(x-z) = \frac{1}{4^2}\frac{1}{2!} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{z}_\perp)} \int_0^\infty \frac{d\lambda}{4\pi\lambda^3} e^{-i\lambda(\boldsymbol{x}_L-\boldsymbol{z}_L)^2} e^{-i\frac{m^2+\boldsymbol{p}_\perp^2}{4\lambda}}.}$$

Three propagators

- For comparing it with the covariant result we introduce the longitudinal momenta as

$$\boxed{e^{-i\lambda(\boldsymbol{x}_L - \boldsymbol{z}_L)^2} = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{4\pi\lambda} e^{-i\boldsymbol{p}_L\cdot(\boldsymbol{x}_L - \boldsymbol{z}_L)} e^{i\boldsymbol{p}_L^2/(4\lambda)},}$$

Three propagators

- For comparing it with the covariant result we introduce the longitudinal momenta as

$$\boxed{e^{-i\lambda(\boldsymbol{x}_L - \boldsymbol{z}_L)^2} = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{4\pi\lambda} e^{-i\boldsymbol{p}_L\cdot(\boldsymbol{x}_L - \boldsymbol{z}_L)} e^{i\boldsymbol{p}_L^2/(4\lambda)},}$$

Three propagators

- For comparing it with the covariant result we introduce the longitudinal momenta as

$$e^{-i\lambda(\boldsymbol{x}_L-\boldsymbol{z}_L)^2} = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{4\pi\lambda} e^{-i\boldsymbol{p}_L\cdot(\boldsymbol{x}_L-\boldsymbol{z}_L)} e^{i\boldsymbol{p}_L^2/(4\lambda)},$$

  thus we have

$$\begin{aligned}\Delta^2_{LF}(x-z) &= \frac{1}{128}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{z}_\perp)}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_L}{(2\pi)^2}\int_0^\infty\frac{d\lambda}{\lambda^4}e^{-i\boldsymbol{p}_L\cdot(\boldsymbol{x}_L-\boldsymbol{z}_L)}e^{i(\boldsymbol{p}_L^2-m^2-\boldsymbol{p}_\perp^2)/(4\lambda)} = \\ &= \frac{1}{128}\int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-z)}\int_0^\infty\frac{d\lambda}{\lambda^3}e^{i(p^2-m^2)/(4\lambda)}.\end{aligned}$$

- The integral over λ can be performed explicitly according to

$$\int_0^\infty \frac{d\lambda}{\lambda^4}e^{iA/\lambda} = \frac{2!\,i^3}{(A+\mathrm{i}0)^3},$$

  leading to

$$\Delta^3_{LF}(x-z) = \int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-z)}\frac{i^3}{(p^2-m^2+\mathrm{i}0)^3},$$

  which agrees with the covariant result.

Outlook

Convolution of n propagators and effective potential

- The convolution for generic n propagators is given by

$$\boxed{\Delta^n_{LF}(x-z) = \frac{1}{4^{n-1}}\frac{1}{(n-1)!}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{z}_\perp)}\int_0^\infty\frac{d\lambda}{4\pi\lambda^n}e^{-i\lambda(\boldsymbol{x}_L-\boldsymbol{z}_L)^2}e^{-i\frac{m^2+\boldsymbol{p}_\perp^2}{4\lambda}}.}$$

Convolution of n propagators and effective potential

- The convolution for generic n propagators is given by

$$\Delta_{LF}^n(x-z) = \frac{1}{4^{n-1}} \frac{1}{(n-1)!} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp \cdot (\boldsymbol{x}_\perp - \boldsymbol{z}_\perp)} \int_0^\infty \frac{d\lambda}{4\pi\lambda^n} e^{-i\lambda(\boldsymbol{x}_L - \boldsymbol{z}_L)^2} e^{-i\frac{m^2+\boldsymbol{p}_\perp^2}{4\lambda}}.$$

- The 1-loop effective potential (for the $\Lambda\phi^4$ theory) is given by

$$V_{eff}^{(1)}[\phi_c] = \frac{i}{2} \sum_{n=1} \frac{(-i)^n}{n} \left(\frac{\Lambda}{2}\phi_c^2\right)^n \Delta_{LF}^n(0),$$

Convolution of n propagators and effective potential

- The convolution for generic n propagators is given by

$$\Delta^n_{LF}(x-z) = \frac{1}{4^{n-1}}\frac{1}{(n-1)!}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{z}_\perp)}\int_0^\infty\frac{d\lambda}{4\pi\lambda^n}e^{-i\lambda(\boldsymbol{x}_L-z_L)^2}e^{-i\frac{m^2+\boldsymbol{p}_\perp^2}{4\lambda}}.$$

- The 1-loop effective potential (for the $\Lambda\phi^4$ theory) is given by

$$V^{(1)}_{eff}[\phi_c] = \frac{i}{2}\sum_{n=1}\frac{(-i)^n}{n}\left(\frac{\Lambda}{2}\phi_c^2\right)^n\Delta^n_{LF}(0),$$

- thus we need to evaluate $\Delta^n_{LF}(0)$

$$\Delta^n_{LF}(0) = \frac{1}{4^{n-1}}\frac{1}{(n-1)!}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}\int_0^\infty\frac{d\lambda}{4\pi\lambda^n}e^{-i\frac{m^2+\boldsymbol{p}_\perp^2}{4\lambda}}.$$

Convolution of n propagators and effective potential

- The convolution for generic n propagators is given by

$$\Delta_{LF}^{n}(x-z)=\frac{1}{4^{n-1}}\frac{1}{(n-1)!}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{z}_\perp)}\int_0^\infty\frac{d\lambda}{4\pi\lambda^n}e^{-i\lambda(\boldsymbol{x}_L-z_L)^2}e^{-i\frac{m^2+\boldsymbol{p}_\perp^2}{4\lambda}}.$$

- The 1-loop effective potential (for the $\Lambda\phi^4$ theory) is given by

$$V_{eff}^{(1)}[\phi_c]=\frac{i}{2}\sum_{n=1}\frac{(-i)^n}{n}\left(\frac{\Lambda}{2}\phi_c^2\right)^n\Delta_{LF}^{n}(0),$$

- thus we need to evaluate $\Delta_{LF}^{n}(0)$

$$\Delta_{LF}^{n}(0)=\frac{1}{4^{n-1}}\frac{1}{(n-1)!}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}\int_0^\infty\frac{d\lambda}{4\pi\lambda^n}e^{-i\frac{m^2+\boldsymbol{p}_\perp^2}{4\lambda}}.$$

- But we have

$$\int_0^\infty\frac{d\lambda}{\lambda^{n+1}}e^{iA/\lambda}=\frac{i^n(n-1)!}{(A+\mathrm{i}0)^n},$$

Convolution of n propagators and effective potential

- Thus using

$$\int_0^\infty \frac{d\lambda}{\lambda^{n+1}} e^{iA/\lambda} = \frac{i^n (n-1)!}{(A + \mathrm{i}0)^n},$$

we directly find

$$\Delta_{LF}^n(0) = \frac{i^{n-1}}{4\pi(n-1)} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} \frac{1}{(m^2 + \boldsymbol{p}_\perp^2)^{n-1}}.$$

Convolution of n propagators and effective potential

- Thus using

$$\boxed{\int_0^\infty \frac{d\lambda}{\lambda^{n+1}} e^{iA/\lambda} = \frac{i^n (n-1)!}{(A+\mathrm{i}0)^n},}$$

we directly find

$$\Delta^n_{LF}(0) = \frac{i^{n-1}}{4\pi(n-1)} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} \frac{1}{(m^2+\boldsymbol{p}_\perp^2)^{n-1}}.$$

- For the standard LF propagators one obtains

$$\bar{\Delta}^n_{LF}(0) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} \int_{\mathbb{R}} \frac{dp^+}{4\pi(p^+)^n} \int_{\mathbb{R}} dp^- \frac{1}{\left[p^- - \frac{m^2+\boldsymbol{p}_\perp^2}{2p^+} + i\mathrm{sgn}(p^+)0\right]^n}$$

Convolution of n propagators and effective potential

- Thus using

$$\int_0^\infty \frac{d\lambda}{\lambda^{n+1}} e^{iA/\lambda} = \frac{i^n (n-1)!}{(A + \mathrm{i}0)^n},$$

  we directly find

$$\Delta^n_{LF}(0) = \frac{i^{n-1}}{4\pi(n-1)} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} \frac{1}{(m^2 + \boldsymbol{p}_\perp^2)^{n-1}}.$$

- For the standard LF propagators one obtains

$$\bar{\Delta}^n_{LF}(0) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} \int_{\mathbb{R}} \frac{dp^+}{4\pi(p^+)^n} \int_{\mathbb{R}} dp^- \frac{1}{\left[p^- - \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+} + i\mathrm{sgn}(p^+)0\right]^n}$$

- by residua we find

$$\int_{\mathbb{R}} dp^- \frac{1}{[p^- - a \pm \mathrm{i}0]^n} = 0, \qquad n > 1,$$

  for any finite value of a.

Convolution of n propagators and effective potential

- Thus using

$$\boxed{\int_0^\infty \frac{d\lambda}{\lambda^{n+1}} e^{iA/\lambda} = \frac{i^n(n-1)!}{(A+\mathrm{i}0)^n},}$$

we directly find

$$\Delta^n_{LF}(0) = \frac{i^{n-1}}{4\pi(n-1)} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} \frac{1}{(m^2+\boldsymbol{p}_\perp^2)^{n-1}}.$$

- For the standard LF propagators one obtains

$$\bar{\Delta}^n_{LF}(0) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} \int_{\mathbb{R}} \frac{dp^+}{4\pi(p^+)^n} \int_{\mathbb{R}} dp^- \frac{1}{\left[p^- - \frac{m^2+\boldsymbol{p}_\perp^2}{2p^+} + i\,\mathrm{sgn}(p^+)0\right]^n}$$

- by residua we find

$$\boxed{\int_{\mathbb{R}} dp^- \frac{1}{[p^- - a \pm \mathrm{i}0]^n} = 0, \qquad n > 1,}$$

for any finite value of a.
- One needs to look for the large circle contribution etc.

Outlook

1. Definitions and introductory remarks about the light front (LF)
   - Definitions in D = 3+1
   - LF propagator - standard approach
   - Novel LF representation of propagator
2. Convolutions of LF propagators
   - Two propagators
   - Convolution of three propagators
   - Convolution of n propagators and effective potential
3. One loop 2-point functions
   - Scalar vacuum polarization function
   - Vector vacuum polarization - I case
     - Longitudinal component
     - Transverse component
   - Vector vacuum polarization - II case
4. Conclusions and prospects

Scalar vacuum polarization function

- Let us consider the product of two covariant Feynman propagators

$$\Delta_F(x)\Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0}.$$

Thus the 2-point scalar vacuum polarization function $\Sigma(p)$ is the inverse Fourier transformation of the product of two Feynman propagators.

Scalar vacuum polarization function

- Let us consider the product of two covariant Feynman propagators

$$\Delta_F(x)\Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0}.$$

Thus the 2-point scalar vacuum polarization function $\Sigma(p)$ is the inverse Fourier transformation of the product of two Feynman propagators.

Scalar vacuum polarization function

- Let us consider the product of two covariant Feynman propagators

$$\Delta_F(x)\Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0}.$$

  Thus the 2-point scalar vacuum polarization function $\Sigma(p)$ is the inverse Fourier transformation of the product of two Feynman propagators.
- Therefore we will consider $\Delta_{LF}(x)\Delta_{LF}(x)$

$$\begin{aligned}\Delta_{LF}(x)\Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} e^{-i(\lambda_1+\lambda_2)(\boldsymbol{x}_L)^2} e^{-iM_1^2/(4\lambda_1)} e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

  where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$.

Scalar vacuum polarization function

- Let us consider the product of two covariant Feynman propagators

$$\Delta_F(x)\Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0}.$$

  Thus the 2-point scalar vacuum polarization function $\Sigma(p)$ is the inverse Fourier transformation of the product of two Feynman propagators.
- Therefore we will consider $\Delta_{LF}(x)\Delta_{LF}(x)$

$$\begin{aligned}\Delta_{LF}(x)\Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} e^{-i(\lambda_1+\lambda_2)(\boldsymbol{x}_L)^2}\, e^{-iM_1^2/(4\lambda_1)}\, e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

  where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$.
- For evaluating the integrals over $\lambda_{1,2}$ we introduce

$$\lambda_1 = \lambda\xi, \qquad \lambda_2 = \lambda(1-\xi), \quad \mathcal{J} = \lambda$$

Scalar vacuum polarization function

For new integral variables

$$\lambda_1 = \lambda\xi, \qquad \lambda_2 = \lambda(1-\xi), \quad \mathcal{J} = \lambda$$

we find

$$\begin{aligned} I_\lambda &= \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} e^{-i(\lambda_1+\lambda_2)\boldsymbol{x}_L^2}\, e^{-iM_1^2/(4\lambda_1)}\, e^{-iM_2^2/(4\lambda_2)} = \\ &= \frac{1}{16\pi^2} \int_0^\infty \frac{d\lambda}{\lambda} \int_0^1 \frac{d\xi}{\xi(1-\xi)} e^{-i\lambda\boldsymbol{x}_L^2}\, e^{-iM_1^2/(4\xi\lambda)}\, e^{-iM_2^2/(4(1-\xi)\lambda)}. \end{aligned}$$

Next we take the Fourier transform in the 2-dimensional longitudinal subspace

$$\boxed{e^{-i\lambda\boldsymbol{x}_L^2} = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{4\pi\lambda} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} e^{i\boldsymbol{p}_L^2/(4\lambda)},}$$

which leads to

$$I_\lambda = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^\infty \frac{d\lambda}{\lambda^2} \int_0^1 \frac{d\xi}{\xi(1-\xi)}\, e^{i\boldsymbol{p}_L^2/(4\lambda)} e^{-iM_1^2/(4\xi\lambda)}\, e^{-iM_2^2/(4(1-\xi)\lambda)}.$$

Scalar vacuum polarization function

The integral over λ can be performed explicitly according to

$$\boxed{\int_0^\infty \frac{d\lambda}{\lambda^2} e^{iA/\lambda} = \frac{i}{A + \mathrm{i}0},}$$

thus we obtain

$$\begin{aligned} I_\lambda &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \int_0^\infty \frac{d\lambda}{\lambda^2}\, e^{i\boldsymbol{p}_L^2/(4\lambda)} e^{-iM_1^2/(4\xi\lambda)}\, e^{-iM_2^2/(4(1-\xi)\lambda)} = \\ &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \frac{4i}{\boldsymbol{p}_L^2 - M_1^2/\xi - M_2^2/(1-\xi) + \mathrm{i}0}. \end{aligned}$$

This leads to

$$\begin{aligned} \Delta_{LF}(x)\Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} I_\lambda = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \times \\ &\times \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} \frac{4i}{\boldsymbol{p}_L^2 - M_1^2/\xi - M_2^2/(1-\xi) + \mathrm{i}0}. \end{aligned}$$

Scalar vacuum polarization

- We parameterize the transverse momenta as

$$\boldsymbol{p}_{1\perp} = \boldsymbol{p}_\perp \xi + \boldsymbol{q}_\perp, \quad \boldsymbol{p}_{2\perp} = \boldsymbol{p}_\perp (1-\xi) - \boldsymbol{q}_\perp, \qquad \mathcal{J} = 1$$

with the property

$$\boxed{\frac{\boldsymbol{p}_{1\perp}^2}{\xi} + \frac{\boldsymbol{p}_{2\perp}^2}{1-\xi} = \boldsymbol{p}_\perp{}^2 + \frac{\boldsymbol{q}_\perp^2}{\xi(1-\xi)},}$$

Accordingly we find

$$\Delta_{LF}(x)\Delta_{LF}(x) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(2\pi)^2} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(4\pi)^3} \frac{4i}{\boldsymbol{p}_L^2 - \boldsymbol{p}_\perp^2 - \frac{m^2+\boldsymbol{q}_\perp^2}{\xi(1-\xi)} + \mathrm{i}0}.$$

and finally

$$\boxed{\Delta_{LF}(x)\Delta_{LF}(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{1}{p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i\epsilon} = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \Sigma(p).}$$

Scalar vacuum polarization

- We parameterize the transverse momenta as

$$\boldsymbol{p}_{1\perp} = \boldsymbol{p}_\perp \xi + \boldsymbol{q}_\perp, \quad \boldsymbol{p}_{2\perp} = \boldsymbol{p}_\perp (1-\xi) - \boldsymbol{q}_\perp, \qquad \mathcal{J} = 1$$

with the property

$$\boxed{\frac{\boldsymbol{p}_{1\perp}^2}{\xi} + \frac{\boldsymbol{p}_{2\perp}^2}{1-\xi} = \boldsymbol{p}_\perp{}^2 + \frac{\boldsymbol{q}_\perp^2}{\xi(1-\xi)},}$$

Accordingly we find

$$\Delta_{LF}(x)\Delta_{LF}(x) = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(2\pi)^2} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(4\pi)^3} \frac{4i}{\boldsymbol{p}_L^2 - \boldsymbol{p}_\perp^2 - \frac{m^2+\boldsymbol{q}_\perp^2}{\xi(1-\xi)} + i0}.$$

and finally

$$\boxed{\Delta_{LF}(x)\Delta_{LF}(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{1}{p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i\epsilon} = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\Sigma(p).}$$

- Thus the scalar vacuum polarization is given by

$$\Sigma(p) = \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0} \qquad (1)$$

$$= \boxed{\frac{i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{1}{p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i0}}. \qquad (2)$$

Outlook

Vector vacuum polarization - I case

Vector vacuum polarization - I case

- Let us consider another product of two Feynman propagators

$$\partial_\mu \Delta_F(x)\Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \boxed{\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0}} = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \Sigma_\mu(p),$$

where the expression in the box is the vector vacuum polarization function.

Vector vacuum polarization - I case

- Let us consider another product of two Feynman propagators

$$\partial_\mu \Delta_F(x)\Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \boxed{\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0}} = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \Sigma_\mu(p),$$

  where the expression in the box is the vector vacuum polarization function.
- First we consider the longitudinal partial derivative, say ∂_+, which leads to

$$\begin{aligned} \partial_+ \Delta_{LF}(x)\Delta_{LF}(x) &= (-2ix^-) \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot \boldsymbol{x}_\perp} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \lambda_1 \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} e^{-i(\lambda_1+\lambda_2)\boldsymbol{x}_L^2}\, e^{-iM_1^2/(4\lambda_1)}\, e^{-iM_2^2/(4\lambda_2)}, \end{aligned}$$

  where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$.

Vector vacuum polarization - I case

- Let us consider another product of two Feynman propagators

$$\partial_\mu \Delta_F(x)\Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \boxed{\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0}} = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \Sigma_\mu(p),$$

where the expression in the box is the vector vacuum polarization function.

- First we consider the longitudinal partial derivative, say ∂_+, which leads to

$$\begin{aligned} \partial_+ \Delta_{LF}(x)\Delta_{LF}(x) &= (-2ix^-) \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot \boldsymbol{x}_\perp} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \lambda_1 \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} e^{-i(\lambda_1+\lambda_2)\boldsymbol{x}_L^2} e^{-iM_1^2/(4\lambda_1)} e^{-iM_2^2/(4\lambda_2)}, \end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$.

- For evaluating the integrals over λ's we parameterize them as

$$\lambda_1 = \lambda\xi, \quad \lambda_2 = \lambda(1-\xi), \quad \mathcal{J} = \lambda,$$

Also we take the Fourier transform in the 2-dimensional longitudinal subspace

$$\boxed{(-2ix^-)e^{-i\lambda \boldsymbol{x}_L^2} = \frac{1}{\lambda}\partial_+ e^{-i\lambda \boldsymbol{x}_L^2} = \partial_+ \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{4\pi\lambda^2} e^{-i\boldsymbol{p}_L\cdot \boldsymbol{x}_L} e^{i\boldsymbol{p}_L^2/(4\lambda)},}$$

Vector vacuum polarization - I case

This gives

$$
\begin{aligned}
I'_\lambda &= (-2ix^-)\int_0^\infty \frac{d\lambda_1}{4\pi}\int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} e^{-i(\lambda_1+\lambda_2)\boldsymbol{x}_L^2}\, e^{-iM_1^2/(4\lambda_1)}\, e^{-iM_2^2/(4\lambda_2)} = \\
&= \partial_+ \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^\infty \frac{d\lambda}{\lambda^2}\int_0^1 \frac{d\xi}{1-\xi}\, e^{i\boldsymbol{k}_L^2/(4\lambda)} e^{-iM_1^2/(4\xi\lambda)}\, e^{-iM_2^2/(4(1-\xi)\lambda)}.
\end{aligned}
$$

The integral over λ can be performed explicitly according to

$$
\boxed{\int_0^\infty \frac{d\lambda}{\lambda^2} e^{iA/\lambda} = \frac{i}{A+\mathrm{i}0},}
$$

thus we obtain

$$
I'_\lambda = \partial_+ \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^1 \frac{d\xi}{1-\xi} \frac{4i}{\boldsymbol{k}_L^2 - M_1^2/\xi - M_2^2/(1-\xi) + \mathrm{i}0}.
$$

This leads to

$$
\begin{aligned}
\partial_+\Delta_F(x)\Delta_F(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} I'_\lambda = \partial_+ \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^1 \frac{d\xi}{1-\xi} \times \\
&\times \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} \frac{4i}{\boldsymbol{k}_L^2 - M_1^2/\xi - M_2^2/(1-\xi) + \mathrm{i}0}.
\end{aligned}
$$

Vector vacuum polarization - I case

- We parameterize the transverse momenta as

$$\boxed{\boldsymbol{p}_{1\perp} = \boldsymbol{p}_{\perp}\xi + \boldsymbol{q}_{\perp}, \qquad \boldsymbol{p}_{2\perp} = \boldsymbol{p}_{\perp}(1-\xi) - \boldsymbol{q}_{\perp},}$$

thus

$$\boxed{\frac{\boldsymbol{p}_{1\perp}^2}{\xi} + \frac{\boldsymbol{p}_{2\perp}^2}{1-\xi} = {\boldsymbol{p}_{\perp}}^2 + \frac{\boldsymbol{q}_{\perp}^2}{\xi(1-\xi)},}$$

which leads to

$$\partial_+ \Delta_F(x)\Delta_F(x) = \partial_+ \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(2\pi)^2} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp} \int_0^1 \frac{d\xi}{1-\xi} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(4\pi)^3} \frac{4i}{\boldsymbol{p}_L^2 - \boldsymbol{p}_\perp^2 - \frac{m^2+\boldsymbol{q}_\perp^2}{\xi(1-\xi)} + \mathrm{i}0}.$$

and finally

$$\partial_+ \Delta_F(x)\Delta_F(x) = \boxed{\partial_+ \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi}{p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i\epsilon}}$$

$$= \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\, \Sigma_{\pm}(p).$$

Vector vacuum polarization - I case

- We parameterize the transverse momenta as

$$\boxed{\boldsymbol{p}_{1\perp} = \boldsymbol{p}_\perp \xi + \boldsymbol{q}_\perp, \qquad \boldsymbol{p}_{2\perp} = \boldsymbol{p}_\perp (1-\xi) - \boldsymbol{q}_\perp,}$$

thus

$$\boxed{\frac{\boldsymbol{p}_{1\perp}^2}{\xi} + \frac{\boldsymbol{p}_{2\perp}^2}{1-\xi} = {\boldsymbol{p}_\perp}^2 + \frac{\boldsymbol{q}_\perp^2}{\xi(1-\xi)},}$$

which leads to

$$\partial_+ \Delta_F(x)\Delta_F(x) \;=\; \partial_+ \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(2\pi)^2} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp} \int_0^1 \frac{d\xi}{1-\xi} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(4\pi)^3} \frac{4i}{\boldsymbol{p}_L^2 - \boldsymbol{p}_\perp^2 - \frac{m^2+\boldsymbol{q}_\perp^2}{\xi(1-\xi)} + i0}.$$

and finally

$$\partial_+ \Delta_F(x)\Delta_F(x) = \boxed{\partial_+ \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi}{p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i\epsilon}}$$

$$= \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\, \Sigma_\pm(p).$$

- Thus the vector vacuum polarization is given by

$$\Sigma_\pm(p) \;=\; \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\pm}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0} = \boxed{\frac{p_\pm}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi}{p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i0}}.$$

Vector vacuum polarization - I case

- Then we consider the transverse partial derivative, say ∂_i, which leads to

$$\begin{aligned}\partial_i \Delta_{LF}(x)\Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} (-i\boldsymbol{p}_{1i}) e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi\lambda_2} e^{-i(\lambda_1+\lambda_2)(\boldsymbol{x}_L)^2}\, e^{-iM_1^2/(4\lambda_1)}\, e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$. The calculation with λ's is exactly that from the scalar case thus we may take

$$\boxed{I_\lambda = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \frac{4i}{\boldsymbol{p}_L^2 - M_1^2/\xi - M_2^2/(1-\xi) + i0},}$$

which leads to

$$\begin{aligned}\partial_i \Delta_{LF}(x)\Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} (-i\boldsymbol{p}_{1i}) e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} \\ &\times = \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(4\pi)^3} e^{-i\boldsymbol{p}_L\cdot\boldsymbol{x}_L} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \frac{4i}{\boldsymbol{p}_L^2 - M_1^2/\xi - M_2^2/(1-\xi) + i0},\end{aligned}$$

Vector vacuum polarization - I case

- Then we parameterize the transverse momenta as

$$\boxed{\boldsymbol{p}_{1\perp} = \boldsymbol{p}_\perp \xi + \boldsymbol{q}_\perp, \qquad \boldsymbol{p}_{2\perp} = \boldsymbol{p}_\perp (1-\xi) - \boldsymbol{q}_\perp,}$$

which leads to

$$\begin{aligned} \partial_i \Delta_{LF}(x) \Delta_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp} \\ &\times \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_L}{(2\pi)^2} e^{-i\boldsymbol{p}_L \cdot \boldsymbol{x}_L} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \frac{\boldsymbol{p}_i \xi + \boldsymbol{q}_i}{\boldsymbol{p}_L^2 - \boldsymbol{p}_\perp^2 - \frac{m^2 + \boldsymbol{q}_\perp^2}{\xi(1-\xi)} + i0}, \end{aligned}$$

or finally

$$\boxed{\partial_i \Delta_{LF}(x) \Delta_{LF}(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{\boldsymbol{p}_i}{4\pi} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \int_0^1 d\xi \frac{\xi}{\boldsymbol{p}^2 \xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i0}} = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \Sigma_i(p).$$

Thus the vector vacuum polarization is given by

$$\begin{aligned} \Sigma_i(p) &= \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{\boldsymbol{k}_i}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0} \\ &= \boxed{\frac{p_i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi}{p^2 \xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i0}}. \end{aligned}$$

Outlook

Vector vacuum polarization - II case

- Let us consider another product of two Feynman propagators

$$\partial_\mu \Delta_F(x)\Delta_F^2(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \boxed{\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2 + i0} \frac{i^2}{[(k+p)^2 - m^2 + i0]^2}},$$

where the expression in the box is another vector vacuum polarization function $\Sigma'(p)$.

Vector vacuum polarization - II case

- Let us consider another product of two Feynman propagators

$$\partial_\mu \Delta_F(x)\Delta_F^2(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \boxed{\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2 + i0} \frac{i^2}{[(k+p)^2 - m^2 + i0]^2}},$$

where the expression in the box is another vector vacuum polarization function $\Sigma'(p)$.

- First we consider the longitudinal partial derivative, say ∂_+, which leads to

$$\begin{aligned}\partial_+ \Delta_{LF}(x)\Delta_{LF}^2(x) &= (-2ix^-) \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} \times \\ &\times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1}\lambda_1 \int_0^\infty \frac{d\lambda_2}{16\pi\lambda_2^2} e^{-i(\lambda_1+\lambda_2)(\boldsymbol{x}_L)^2}\, e^{-iM_1^2/(4\lambda_1)}\, e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$. Proceeding analogously to the I case we find

$$\boxed{\partial_+ \Delta_{LF}(x)\Delta_{LF}^2(x) = \partial_+ \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{-1}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi^2}{[p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i\epsilon]^2}.}$$

Thus the vector vacuum polarization is given by

$$\begin{aligned}\Sigma'_\pm(p) &= \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\pm}{k^2 - m^2 + i0} \frac{i^2}{[(k+p)^2 - m^2 + i0]^2} \\ &= \frac{ip_\pm}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi^2}{[p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + i0]^2}.\end{aligned}$$

Vector vacuum polarization - II case

- Then we consider the transverse partial derivative, say ∂_i, which leads to

$$\begin{aligned}\partial_i \Delta_{LF}(x)\Delta_{LF}^2(x) &= \int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2}(-i\boldsymbol{p}_{1i})e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp}\times\\ &\times\int_0^\infty\frac{d\lambda_1}{4\pi\lambda_1}\int_0^\infty\frac{d\lambda_2}{16\pi\lambda_2^2}e^{-i(\lambda_1+\lambda_2)(\boldsymbol{x}_L)^2}\,e^{-iM_1^2/(4\lambda_1)}\,e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$. Proceeding analogously to the case I we find the vector vacuum polarization

$$\begin{aligned}\Sigma_i'(p) &= \int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\frac{\boldsymbol{k}_i}{k^2-m^2+i0}\frac{i}{(k+p)^2-m^2+i0}\\ &= \frac{ip_i}{4\pi}\int_0^1 d\xi\int_{\mathbb{R}^2}\frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2}\frac{\xi^2}{[p^2\xi(1-\xi)-m^2-\boldsymbol{q}_\perp^2+\mathrm{i}0]^2}.\end{aligned}$$

Vector vacuum polarization - II case

- Then we consider the transverse partial derivative, say ∂_i, which leads to

$$\begin{aligned}\partial_i \Delta_{LF}(x)\Delta_{LF}^2(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} (-i\boldsymbol{p}_{1i}) e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot \boldsymbol{x}_\perp} \times \\ &\quad \times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{16\pi\lambda_2^2} e^{-i(\lambda_1+\lambda_2)(\boldsymbol{x}_L)^2} e^{-iM_1^2/(4\lambda_1)} e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$. Proceeding analogously to the case I we find the vector vacuum polarization

$$\begin{aligned}\Sigma_i'(p) &= \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{\boldsymbol{k}_i}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0} \\ &= \frac{ip_i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi^2}{[p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + \mathrm{i}0]^2}.\end{aligned}$$

- Accordingly we have the general result

$$\begin{aligned}\Sigma_\mu'(p) &= \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2 + i0} \frac{i^2}{[(k+p)^2 - m^2 + i0]^2} \\ &= \frac{ip_\mu}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi^2}{[p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + \mathrm{i}0]^2}.\end{aligned}$$

Vector vacuum polarization - II case

- Then we consider the transverse partial derivative, say ∂_i, which leads to

$$\begin{aligned}\partial_i \Delta_{LF}(x)\Delta^2_{LF}(x) &= \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{p}_{2\perp}}{(2\pi)^2} (-i\boldsymbol{p}_{1i}) e^{-i(\boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp})\cdot\boldsymbol{x}_\perp} \times \\ &\quad \times \int_0^\infty \frac{d\lambda_1}{4\pi\lambda_1} \int_0^\infty \frac{d\lambda_2}{16\pi\lambda_2^2} e^{-i(\lambda_1+\lambda_2)(\boldsymbol{x}_L)^2} e^{-iM_1^2/(4\lambda_1)} e^{-iM_2^2/(4\lambda_2)},\end{aligned}$$

where $M_1^2 = m^2 + \boldsymbol{p}_{1\perp}^2$ and $M_2^2 = m^2 + \boldsymbol{p}_{2\perp}^2$. Proceeding analogously to the case I we find the vector vacuum polarization

$$\begin{aligned}\Sigma'_i(p) &= \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{\boldsymbol{k}_i}{k^2 - m^2 + i0} \frac{i}{(k+p)^2 - m^2 + i0} \\ &= \frac{ip_i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi^2}{[p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + \mathrm{i}0]^2}.\end{aligned}$$

- Accordingly we have the general result

$$\begin{aligned}\Sigma'_\mu(p) &= \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2 + i0} \frac{i^2}{[(k+p)^2 - m^2 + i0]^2} \\ &= \frac{ip_\mu}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} \frac{\xi^2}{[p^2\xi(1-\xi) - m^2 - \boldsymbol{q}_\perp^2 + \mathrm{i}0]^2}.\end{aligned}$$

- D.Melikhov, S.Simula, Phys.Let. B 556 (2003) 135-141, *End-point singularities of Feynman graphs on the light cone*

Conclusions and prospects

Conclusions and prospects

First

the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Conclusions and prospects

First
: the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Second
: new rules for calculating Feynman diagrams are needed

Conclusions and prospects

First
: the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Second
: new rules for calculating Feynman diagrams are needed

Third
: the effective potential is nontrivial at the 1-loop level

Conclusions and prospects

First

the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Second

new rules for calculating Feynman diagrams are needed

Third

the effective potential is nontrivial at the 1-loop level

Fourth

the vacuum polarization functions are expressed in terms of integrals over transverse momentum and ξ parameter

Conclusions and prospects

First
: the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Second
: new rules for calculating Feynman diagrams are needed

Third
: the effective potential is nontrivial at the 1-loop level

Fourth
: the vacuum polarization functions are expressed in terms of integrals over transverse momentum and ξ parameter

Conclusions and prospects

First
: the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Second
: new rules for calculating Feynman diagrams are needed

Third
: the effective potential is nontrivial at the 1-loop level

Fourth
: the vacuum polarization functions are expressed in terms of integrals over transverse momentum and ξ parameter

First
: similar calculations for fermion and gauge propagators should be done for QED and QCD

Conclusions and prospects

First
: the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Second
: new rules for calculating Feynman diagrams are needed

Third
: the effective potential is nontrivial at the 1-loop level

Fourth
: the vacuum polarization functions are expressed in terms of integrals over transverse momentum and ξ parameter

First
: similar calculations for fermion and gauge propagators should be done for QED and QCD

Second
: is the λ representation applicable for the Schwinger-Dyson equations?

Conclusions and prospects

First

the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Second

new rules for calculating Feynman diagrams are needed

Third

the effective potential is nontrivial at the 1-loop level

Fourth

the vacuum polarization functions are expressed in terms of integrals over transverse momentum and ξ parameter

First

similar calculations for fermion and gauge propagators should be done for QED and QCD

Second

is the λ representation applicable for the Schwinger-Dyson equations?

Conclusions and prospects

First

the λ representation for the scalar field LF propagator is free of any problems which usually arise for the k^- integration

Second

new rules for calculating Feynman diagrams are needed

Third

the effective potential is nontrivial at the 1-loop level

Fourth

the vacuum polarization functions are expressed in terms of integrals over transverse momentum and ξ parameter

First

similar calculations for fermion and gauge propagators should be done for QED and QCD

Second

is the λ representation applicable for the Schwinger-Dyson equations?

Thank you for your attention