The light-front vacuum

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(Phys. Rev. D91,085043,2015)

Research Supported by the US Department of Energy under grant number No. DE-FG02-86ER40286 with the University of Iowa.

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Why is the light-front vacuum different than the canonical vacuum if both describe the same theory?

Outline

- Triviality of vacuum
- Annihilation operators / role of algebras
- Light-front Fock algebra
- Extension of algebras
- Interacting fields
- Zero modes

Notation

Light front: $x^+ = x^0 + \hat{\mathbf{n}} \cdot \mathbf{x} = 0$

Translation generators

$$P^{\pm} := P^0 \pm \hat{\mathbf{n}} \cdot \mathbf{P} \geq 0 \qquad \mathbf{P}_{\perp} := \mathbf{P} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{P})$$

$$P^- = P_0^- + V \qquad P^+ = P_0^+ = \sum_i p_i^+$$

$$p_i^+ = \omega_i(\mathbf{p}) + \hat{\mathbf{n}} \cdot \mathbf{p}_{m_i} \geq 0 \qquad \omega_{m_i}(\mathbf{p}) = \sqrt{\mathbf{p}_i^2 + m_i^2}$$

Light-front 3 vectors

$$\tilde{\mathbf{p}} = (p^+, \mathbf{p}_{\perp}) \qquad \tilde{\mathbf{x}} = (x^-, \mathbf{x}_{\perp})$$

$$\tilde{\mathbf{p}} \cdot \tilde{\mathbf{x}} = -\frac{1}{2} p^+ x^- + \mathbf{p}_{\perp} \cdot \mathbf{x}_{\perp}$$

Triviality of the light-front vacuum (spectral condition $p_i^+ \geq 0$)

$$[P^+, P^-] = [P^+, P_0^-] = 0 \qquad \Rightarrow \qquad [P^+, V] = 0$$

$$P^+ V|0\rangle = VP^+|0\rangle = 0|0\rangle$$

$$: V := \int \mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) a^\dagger(\tilde{\mathbf{p}}_1) \cdots a^\dagger(\tilde{\mathbf{p}}_n) d\tilde{\mathbf{p}}_1 \cdots d\tilde{\mathbf{p}}_n + \cdots$$

$[P^+, a^\dagger(\tilde{\mathbf{p}})] = p^+ a^\dagger(\tilde{\mathbf{p}})$ $a^\dagger(\tilde{\mathbf{p}})$ **increases** P^+ **for** $p^+ > 0$

$$\sum_i p_i^+ = 0 \qquad p_i^+ \geq 0 \qquad \Rightarrow \qquad \mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) = 0$$

unless $p_1^+ = p_2^+ \cdots p_n^+ = 0$**.**

$\therefore$ **if** $\mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n)$ **is continuous and normal ordered then** $\mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) = 0$ **and the vacuum is unchanged.**

Single oscillator

"Vacuum" determined by annihilation operator

$$\langle x|a|0\rangle = 0 \qquad \Rightarrow \qquad \langle x|0\rangle$$

Linear canonical transformation changes the vacuum

$$a' = \cosh(\eta)a + \sinh(\eta)a^\dagger$$

$$\langle x|a'|0'\rangle = 0 \qquad \Rightarrow \qquad \langle x|0'\rangle$$

Implemented by the unitary transformation

$$a' = e^{iG} a e^{-iG} \qquad |0'\rangle = e^{iG}|0\rangle$$

e^{iG} **where** $G = G^\dagger = -i\frac{\eta}{2}(aa - a^\dagger a^\dagger)$

Free fields of different mass
(simplest example)

Interaction = mass difference

$$V = \frac{1}{2}(m_1^2 - m_2^2) \int : \phi(\mathbf{x})^2 : d\mathbf{x}$$

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_m(\mathbf{p})}} (e^{ip\cdot x} a(\mathbf{p}) + e^{-ip\cdot x} a(\mathbf{p}))$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d\tilde{\mathbf{p}}}{\sqrt{2p^+}} (e^{ip\cdot x} a(\tilde{\mathbf{p}}) + e^{-ip\cdot x} a(\tilde{\mathbf{p}}))$$

Vacuum

$a|0\rangle = 0 \qquad \rightarrow \qquad a(\mathbf{p})|0\rangle = 0$ **or** $a(\tilde{\mathbf{p}})|0\rangle = 0$

Observations

- $m_1 \neq m_2$ **fields restricted to** $t = 0$ **related by linear canonical transformation:**

$$a_2(\mathbf{p}) = \underbrace{\frac{1}{2}\left(\sqrt{\frac{\omega_{m_2}}{\omega_{m_1}}} + \sqrt{\frac{\omega_{m_1}}{\omega_{m_2}}}\right)}_{\cosh(\eta(\mathbf{p}))} a_1(\mathbf{p}) + \underbrace{\frac{1}{2}\left(\sqrt{\frac{\omega_{m_2}}{\omega_{m_1}}} - \sqrt{\frac{\omega_{m_1}}{\omega_{m_2}}}\right)}_{\sinh(\eta(\mathbf{p}))} a_1^\dagger(\mathbf{p})$$

- $m_1 \neq m_2$ **fields restricted to** $x^+ = 0$ **identical:**

$$a_2(\tilde{\mathbf{p}}) = a_1(\tilde{\mathbf{p}})$$

Relations: $t = 0$ **vs** $x^+ = 0$

$$a_i(\mathbf{p}) = a_i(\tilde{\mathbf{p}})\sqrt{\frac{p^+}{\omega_{m_i}(\mathbf{p})}}$$

Observations

Relations: $x^+ = 0 \;\; m_1 \neq m_2$

Free fields of different mass restricted to the light front are unitarily equivalent

$$ {}_1\langle 0|\phi_1(\tilde{\mathbf{p}}_1)\cdots\phi_1(\tilde{\mathbf{p}}_n)|0\rangle_1 = {}_2\langle 0|\phi_2(\tilde{\mathbf{p}}_1)\cdots\phi_2(\tilde{\mathbf{p}}_n)|0\rangle_2 $$

$$\Downarrow$$

- **The correspondence** $U|0\rangle_1 = |0\rangle_2$ **and** $U\phi_1(\tilde{\mathbf{x}})U^\dagger = \phi_2(\tilde{\mathbf{x}})$ **preserves all Hilbert space inner products.**

Observations

Relations: $t = 0 \;\; m_1 \neq m_2$

Comparison with the single oscillator suggests

$$|0_2\rangle = U|0_1\rangle \qquad a_2(\mathbf{p})|0_2\rangle = 0 \qquad a_1(\mathbf{p})|0_1\rangle = 0$$

$$a_2(\mathbf{p}) = U a_1(\mathbf{p}) U^\dagger \qquad U = e^{iG}$$

$$G = -i\frac{\int d\mathbf{p}\eta(\mathbf{p})}{2}(a_1(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_1^\dagger(\mathbf{p}))$$

but

$$\|G|0\rangle\|^2 = \frac{1}{4}\int \eta(\mathbf{p})^2 d\mathbf{p}\delta(0) = \infty$$

Domain of G is empty!

- $|0_1\rangle$ **and** $|0_2\rangle$ **not in the same Hilbert space in this representation (Haag 1955).**

Characterization of vacuum by an annihilation operator?

$$0 = a_1(\mathbf{p})|0\rangle_1 = \sqrt{\frac{p^+}{\omega_{m_1}(\mathbf{p})}} a_1(\tilde{\mathbf{p}})|0\rangle_1$$

$$= \sqrt{\frac{p^+}{\omega_{m_1}(\mathbf{p})}} a_2(\tilde{\mathbf{p}})|0\rangle_1 = \sqrt{\frac{\omega_{m_2}(\mathbf{p})}{\omega_{m_1}(\mathbf{p})}} a_2(\mathbf{p})|0\rangle_1$$

contradicts

$$a_2(\mathbf{p}) = \cosh(\eta(\mathbf{p}))a_1(\mathbf{p}) + \sinh(\eta(\mathbf{p}))a_1^\dagger(\mathbf{p})$$

$$\Downarrow$$

$$a_2(\mathbf{p})|0\rangle_1 \neq 0$$

???

How do we reconcile this apparent contradiction?

- **The annihilation operators do not define the vacuum!**
- **The vacuum is an invariant linear functional on an algebra of operators.**
- **The algebra matters!**
  - $a(\mathbf{p})$: **algebra = field and time derivatives restricted to** $t = 0$ **(canonical algebra).**
  - $a(\tilde{\mathbf{p}})$: **algebra = fields restricted to** $x^+ = 0$ **(light-front algebra).**
  - $\phi(x)$: **algebra = fields smeared against functions of four space-time variables (local algebra).**

- **The local algebra is invariant under Poincaré transformations and includes all local observables.**

- **Schlieder and Seiler give an example of sub-algebras of the local algebras of different mass local free-field algebras that are (1) irreducible and (2) unitarily equivalent. This illustrates the essential role of the algebra in defining the vacuum**

Schlieder-Seiler example

Linear subspace ($\mathcal{L}$) of test functions satisfying

$$\frac{f(\sqrt{m_1^2+\mathbf{p}^2},\mathbf{p})}{(m_1^2+\mathbf{p}^2)^{1/4}} = \frac{f(\sqrt{m_2^2+\mathbf{p}^2},\mathbf{p})}{(m_2^2+\mathbf{p}^2)^{1/4}}.$$

$$\langle 0_1|\phi_1(f_1)\cdots\phi_1(f_n)|0_1\rangle = \langle 0_2|\phi_2(f_1)\cdots\phi_2(f_n)|0_2\rangle$$

$$U|0_1\rangle := |0_2\rangle \qquad U\phi_1(f)U^\dagger := \phi_2(f)$$

U **preserves all scalar products** $\rightarrow$ U **is unitary**

Irreducibility: For any $g\,\exists\, f \in \mathcal{L}$ **satisfying** $f = g$ **on mass shell.**

$$\|(\phi(f)-\phi(g))|A\rangle\| = 0.$$

The light-front Fock algebra

Generated by

$$e^{i\phi(\tilde{f})} = e^{i\int d\tilde{\mathbf{x}}\phi(\tilde{\mathbf{x}},x^+=0)f(\tilde{\mathbf{x}})}$$

Operator products

$$e^{i\phi(\tilde{f})}e^{i\phi(\tilde{g})} = e^{i\phi(\tilde{f}+\tilde{g})}e^{-\frac{1}{2}((\tilde{f},\tilde{g})-(\tilde{g},\tilde{f}))}$$

Light-front inner product

$$(\tilde{f},\tilde{g}) = \int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{p^+}f(-\tilde{\mathbf{p}})g(\tilde{\mathbf{p}})$$

- $(\tilde{f},\tilde{g})$ **is log divergent if** $\tilde{f}(\tilde{\mathbf{p}}) \neq 0$ **for** $p^+ = 0$**; however** $((\tilde{f},\tilde{g}) - (\tilde{g},\tilde{f}))$ **is defined**

Irreducibility

$$\tilde{f}(\tilde{\mathbf{p}}) = \tilde{f}_r(\tilde{\mathbf{p}}) + i\tilde{f}_i(\tilde{\mathbf{p}})$$

$$U(\tilde{f}_r, \tilde{f}_i) := e^{i\phi(\tilde{f})}$$

$$U(\tilde{f}_r, \tilde{f}_i)U(\tilde{g}_r, \tilde{g}_i) = U(\tilde{f}_r + \tilde{g}_r, \tilde{f}_i + \tilde{g}_i)e^{-\frac{1}{2}((\tilde{f}_r,\tilde{g}_i)-(\tilde{f}_i,\tilde{g}_r))}$$

Same algebraic structure (Weyl algebra) as canonical equal time algebra.

$$U(f_1, f_2) = e^{i\phi(f_1,t=0)+i\pi(f_2,t=0)}$$

- **Light-front Fock algebra is irreducible**

- **Light-front Fock algebra is kinematically invariant but not Poincaré invariant**

- **Light-front Fock algebra does not contain all local observables.**

- **The light-front Fock algebra and light-front vacuum do not determine the dynamics (mass).**

Algebraic normal ordering

$$\phi_+(\tilde{\mathbf{p}}) := \theta(p^+)\phi(\tilde{\mathbf{p}}) = \frac{\theta(p^+)}{\sqrt{p^+}} a(\tilde{\mathbf{p}})$$

$$\phi_-(\tilde{\mathbf{p}}) := \theta(-p^+)\phi(\tilde{\mathbf{p}}) = \frac{\theta(-p^+)}{\sqrt{-p^+}} a^\dagger(-\tilde{\mathbf{p}})$$

$$\phi(\tilde{\mathbf{p}}) = \phi_+(\tilde{\mathbf{p}}) + \phi_-(\tilde{\mathbf{p}})$$

$$: e^{i\phi(\tilde{f})} := e^{i\phi_-(\tilde{f})} e^{i\phi_+(\tilde{f})}$$

$$\boxed{e^{i\phi(\tilde{f})} =: e^{i\phi(\tilde{f})} : e^{\frac{1}{2}(\tilde{f},\tilde{f})}}$$

Implications

- **if** $f(\tilde{\mathbf{p}}) = 0$ **for** $p^+ = 0$

$$\langle 0| : e^{i\phi(\tilde{f})} : |0\rangle = 1 \quad \langle 0| e^{i\phi(\tilde{f})} |0\rangle = e^{\frac{1}{2}(\tilde{f},\tilde{f})}$$

  - **Light-front vacuum fixed!**

- **if** $f(\tilde{\mathbf{p}}) \neq 0$ **for** $p^+ = 0$ **then**

  - $(\tilde{f}, \tilde{f})$ **is divergent and** $\langle 0| : e^{i\phi(\tilde{f})} : |0\rangle \neq 1$

  - **All contributions to** $\langle 0| : e^{i\phi(\tilde{f})} : |0\rangle$ **have** $p^+ = 0$ **(zero modes).**

Zero modes

$$\langle 0| : e^{i\phi(\tilde{f})} : |0\rangle = \sum \frac{i^n}{n!} \int z_n(\mathbf{p}_{1\perp}, \cdots, \mathbf{p}_{n\perp}) \times$$

$$\prod_{i=1}^{n} \delta(p_i^+) \tilde{f}(\tilde{\mathbf{p}}_1) \cdots \tilde{f}(\tilde{\mathbf{p}}_n) d\tilde{\mathbf{p}}_1 \cdots d\tilde{\mathbf{p}}_n$$

- **Regulation of inner product breaks kinematic scale invariance; zero modes are needed to restore the full kinematic symmetry, positivity, $\cdots$.**

Extension to local algebra

$$\phi(x) = \frac{1}{(2\pi)^3} \int \frac{d\tilde{\mathbf{p}} d\tilde{\mathbf{y}}}{2} e^{-i\frac{p_\perp^2+m^2}{p^+}x^+ + i\tilde{\mathbf{p}}\cdot(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})}\phi(\tilde{\mathbf{y}})$$

structure of mapping

$$\phi(x) = \int F_m(x, \tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}})d\tilde{\mathbf{y}}$$

- **Extension still utilizes light-front creation operators and light-front Fock vacuum.**
- **Mass dependence (dynamics) is in the extension. Inner products with different mass extensions are not preserved in the extension to the local algebra.**

Mapping local test functions to light-front test functions

$$f(x) = f_+(x^+)\tilde{f}(\tilde{\mathbf{x}})$$

$$\phi(f) = \int f(x) F_m(x, \tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}}) d\tilde{\mathbf{y}} d^4x =$$

$$\int \tilde{f}_+\left(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}\right)\tilde{f}(\tilde{\mathbf{p}})\phi(\tilde{\mathbf{p}}) d\tilde{\mathbf{p}} = \int \tilde{g}(\tilde{\mathbf{p}})\phi(\tilde{\mathbf{p}}) d\tilde{\mathbf{p}}$$

- $\tilde{f}_+(\frac{\mathbf{p}_\perp^2 + m^2}{p^+})$ **vanishes faster than any power of** p^+ **near** $p^+ = 0$ **for** $f_+(x^+)$ **a Schwartz function.**

- $F_m(\cdot)$ **maps local algebra to a sub algebra of the light-front algebra.**

- **Vacuum is the light-front Fock vacuum.**

Interacting fields

Irreducibility of the asymptotic fields

(Haag expansion)

$$\Downarrow$$

$$\phi(x) = \sum \int L(x; x_1, \cdots, x_n) : \phi_{in}(x_1) \cdots \phi_{in}(x_n) : d^4x_1 \cdots d^4x^n$$

$$L(f, x_1 \cdots x_n) = \int f(x) L(x; x_1, \cdots, x_n) d^4x$$

$$f(x) \in \mathcal{S}(\mathbb{R}^4) \to L(f, x_1, \cdots, x_n) \in \mathcal{S}(\mathbb{R}^{4n})$$

$$\phi_{in}(x) = \int zF_m(x, \tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}})d\tilde{\mathbf{y}}$$

$$\phi(x) = \sum \int \mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n) : \phi_0(\tilde{\mathbf{y}}_1) \cdots \phi_0(\tilde{\mathbf{y}}_n) : d\tilde{\mathbf{y}}_1 \cdots d\tilde{\mathbf{y}}_n$$

$$\mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n) = \int L(x; x_1, \cdots, x_n) \prod z_i F_m(x_i, \tilde{\mathbf{y}}) d^4x_1 \cdots d^4x_n$$

expect

$$\int f(x)\tilde{\mathcal{L}}(x; \tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) d^4x \to 0 \quad \text{any} \quad p_i^+ \to 0$$

Light-front Haag expansion

$$\phi(x) = \sum \int \mathcal{L}(x; \tilde{\mathbf{x}}_1, \cdots, \tilde{\mathbf{x}}_n) : \phi_0(\tilde{\mathbf{x}}_1) \cdots \phi_0(\tilde{\mathbf{x}}_n) : d\tilde{\mathbf{x}}_1 \cdots d\tilde{\mathbf{x}}_n$$

- **Vacuum is light-front Fock vacuum**

- **Test functions on the local algebra are mapped into functions on the light front with Fourier transforms that vanish at $p^+ = 0$.**

- $\mathcal{L}$ **maps the local algebra on to a sub-algebra of light-front algebra with no zero-mode contributions.**

- **Locality and Poincaré symmetry recovered by extension.**

Origin of zero modes (local operator products)

$$\langle 0|\phi(x)\phi(y)|0\rangle =$$

$$\int F_m(x^+;\tilde{\mathbf{x}}-\tilde{\mathbf{z}})F_m(y^+;\tilde{\mathbf{y}}-\tilde{\mathbf{w}})\langle 0|\phi(0,\tilde{\mathbf{z}})\phi(0,\tilde{\mathbf{w}})|0\rangle =$$

$$\frac{1}{(2\pi)^3}\int \frac{d\tilde{\mathbf{p}}}{2p^+} e^{-i\frac{\mathbf{p}_\perp^2+m^2}{p^+}(x^+-y^+)+i\tilde{\mathbf{p}}(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})} =$$

$$\int \theta(p^+)\delta(p^2+m^2)e^{ip\cdot(x-y)}d^4p$$

Gives the two-point function on the local algebra in terms of the two-point function on light-front algebra.
$(x^+ - y^+) \neq 0$ **regularizes two-point function**

$$\int_0^a \frac{e^{ic/p^+}}{p^+}dp^+ = \int_{1/a}^\infty \frac{e^{iu}}{u}du = \frac{\pi}{2} - (Ci(1/a) + iSi(1/a))$$

- **Local operator products have $x^+ - y^+ = 0$.**

$$\Downarrow$$

- **Turns off the term that regulates the $p^+ = 0$ singularity.**

$$\Downarrow$$

- **Light front scalar product becomes divergent.**

$$\Downarrow$$

- **Regularization at $p^+ = 0$ needed; breaks kinematic symmetry.**

$$\Downarrow$$

Regularization

$$(f, g) \to \int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{p^+}(\tilde{f}(-\tilde{\mathbf{p}})\tilde{g}(\tilde{\mathbf{p}}) - \tilde{f}(0, -\mathbf{p}_\perp)\tilde{g}(0, \mathbf{p}_\perp)e^{-p^+/b})$$

Removes log divergence.

Breaks longitudinal boost invariance.

Positivity of scalar product?

Additional terms $\sim \delta^{(n)}(p^+)$ **(zero modes) needed.**

- **Zero modes needed to recover broken kinematic symmetry, full rotational covariance, and positivity of the Hilbert space norm of the extension.**

- **Zero modes play a role in the proper definition of local operator products, but they do not play a role in the Haag expansion of the Heisenberg field.**

- **They will appear in the local products of Heisenberg fields since the leading term in the Haag expansion is**

$$\phi(x) \sim Z \int F_m(x, \tilde{\mathbf{y}}) \phi_0(\tilde{\mathbf{y}}) d\tilde{\mathbf{y}} + \cdots$$

- **Generators: Algebraic normal ordering replaced by zero-mode normal ordering.**

Conclusions

- **A vacuum is an invariant linear functional on an algebra; the definition of the vacuum depends on the algebra.**
- **The relevant algebra is the algebra of local observables.**
- **For both free and interacting fields there are dynamical maps from the local algebra to a sub algebra of the light-front Fock algebra.**
- **This leads to a formulation of full Poincaré invariance and locality on a sub algebra of the light front Fock algebra.**

- **Models with different maps lead to inequivalent representations of the local algebra on the light-front algebra.**

- **Zero modes play no role in these mappings.**

- **The mappings carry the dynamics. Zero modes can play a role in the explicit construction of these mappings.**

- **There is a unitary map, $U_0(R)U(R)^\dagger$, that relates theories with different light fronts - $p^+ = 0$ singularities $\leftrightarrow$ ultraviolet singularities. $\therefore$ the-zero mode problem in $3+1$ dim. is more complicated than $1+1$ dim..**

Thanks to
Organizers
INFN
US DOE