Asymptotic freedom of gluons in Hamiltonian dynamics

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in collaboration with
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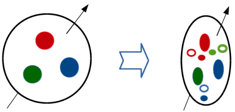
Motivation

- Connection between QCD & Structure of hadrons
  $\Rightarrow$ Hadron wave functions

$$|\text{Hadron}\rangle = |qqq\rangle + |qqq\,g\rangle + |qqq\,gg\rangle + |qqq\,q\bar{q}\rangle + |qqq\,q\bar{q}\,g\rangle + \ldots$$

  $\Rightarrow$ Relativistic properties of hadrons in QCD
  (Poincaré inv., confinement, asymptotic freedom, ...)



- Lagrangian $\rightarrow$ Hamiltonian $\rightarrow$ solve the bound state problem

$$\mathcal{L}_{QCD} \quad \rightarrow \quad \mathcal{H}_{QCD} \quad \rightarrow \quad H_{QCD}|\psi\rangle = E|\psi\rangle$$

- However, H_{QCD} is divergent! $\rightarrow$ Renormalization

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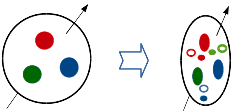

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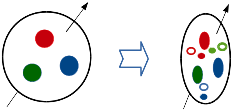

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The method of calculation

RGPEP

Consider only gluon fields $\mathcal{L}_{YM} = \frac{1}{2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu}$

1. **Canonical Hamiltonian** Use front-form dynamics:
   - $\mathcal{L}_{YM} \to \mathcal{T}^{\mu\nu}_{YM} \to H_{YM} = \int_{\mathrm{x}^+=0} \mathcal{H}_{YM}(\mathrm{x})d\mathrm{x}, \quad A^+ = 0$

     $k^+ = k^0 + k^3, \quad k^- = k^0 - k^3, \quad \vec{k}^\perp = (k^1, k^2); \quad x = k^+/P^+$
2. **Regularization**
   - UV and small-x cutoffs: $\int dx d\kappa^\perp \to \int dk^+ d\kappa^\perp r_\delta(x) r_\Delta(\kappa^\perp)$

     $\lim_{\delta\to 0} r_\delta(x) = 1, \ \lim_{\Delta\to\infty} r_\Delta(x) = 1$

     $r_\delta(x) = x^\delta \theta(x-\epsilon), \ r_\Delta(x) = \exp(-\kappa^{\perp 2}/\Delta^2)$
3. **Renormalization**

   $a_0^\dagger|0\rangle = |\mathrm{g}\rangle \quad \to \quad a_t^\dagger|0\rangle = |\mathrm{g}_t\rangle \qquad \mathcal{H}_0(a_0) = \mathcal{H}_t(a_t)$

   Size parameters for gluons introduced by RGPEP equation

   $$\frac{d}{dt}\mathcal{H}_t = [\mathcal{G}_t, \mathcal{H}_t] \ , \qquad a_t = \mathcal{U}_t a_0 \mathcal{U}_t^\dagger \ , \quad t = s^4 = 1/\lambda^4$$

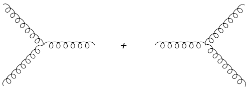
The initial Hamiltonian
RGPEP

$$H_{YM} = H_{A^2} + H_{A^3} + H_{A^4} + H_{[A\partial A]^2}$$

$$\begin{aligned} H_{A^2} &= \sum_{\sigma c} \int [k] \frac{k^{\perp\, 2}}{k^+} a^\dagger_{k\sigma c} a_{k\sigma c} \ , \\ H_{A^3} &= \sum_{123} \int [123]\, \tilde{\delta}(p^\dagger - p)\, \tilde{r}_{\Delta\delta}(3,1) \left[g\, Y_{123}\, a_1^\dagger a_2^\dagger a_3 + g\, Y^*_{123}\, a_3^\dagger a_2 a_1 \right] \ , \end{aligned}$$

where

$$Y_{123} = i f^{c_1 c_2 c_3} \left[\varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3 \kappa - \varepsilon_1^* \varepsilon_3 \cdot \varepsilon_2^* \kappa \frac{1}{x_{2/3}} - \varepsilon_2^* \varepsilon_3 \cdot \varepsilon_1^* \kappa \frac{1}{x_{1/3}} \right],$$



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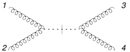
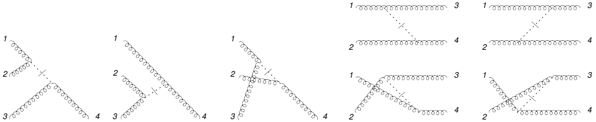
$$\begin{aligned} H_{A^4} &= \sum_{1234} \int [1234]\, \tilde{\delta}(p^\dagger - p)\, \frac{g^2}{4} \Big[\Xi_{A^4\,1234} a_1^\dagger a_2^\dagger a_3^\dagger a_4 + X_{A^4\,1234} a_1^\dagger a_2^\dagger a_3 a_4 \\ &\quad + \Xi^*_{A^4\,1234} a_4^\dagger a_3 a_2 a_1 \Big] \end{aligned}$$

$$\begin{aligned} H_{[\partial A A]^2} &= \sum_{1234} \int [1234]\, \tilde{\delta}(p^\dagger - p)\, g^2 \Big[\Big(\Xi_{[\partial A A]^2\,1234} a_1^\dagger a_2^\dagger a_3^\dagger a_4 + h.c. \Big) \\ &\quad + X_{[\partial A A]^2\,1234} a_1^\dagger a_2^\dagger a_3 a_4 \Big] \end{aligned}$$





In front form: $k^+ > 0$

$$\Rightarrow \cancel{a_1^\dagger a_2^\dagger}, \quad \cancel{a_1 a_2}, \quad \cancel{a_1^\dagger a_2^\dagger a_3^\dagger}, \quad \cancel{a_1 a_2 a_3}, \quad \cancel{a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger}, \quad \cancel{a_1 a_2 a_3 a_4},$$

no particle production out of the vacuum.

Renormalization group procedure

→ Solve the RGPEP equation

$$\mathcal{H}_t(a_t) \quad = \quad \mathcal{H}_0(a_0)$$

$$\frac{d}{dt}\mathcal{H}_t \quad = \quad [\mathcal{G}_t, \mathcal{H}_t] \qquad a_t = \mathcal{U}_s a_0 \mathcal{U}_t^\dagger$$

$$\mathcal{G}_t \quad = \quad [\mathcal{H}_f, \mathcal{H}_{Pt}]$$

→ Initial condition: $\mathcal{H}_{t=0} = H_{A^2} + H_{A^3} + H_{A^4} + H_{[A\partial A]^2} + \text{counterterms}$.
Counterterms are such that the UV-cutoff Δ is removed entirely.

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Difficult to solve nonperturbatively...

→ Solve perturbatively up to order g^3 and extract the Hamiltonian running coupling

$$H_t = E + g^2\hat{\mu}_t^2 + gY_{21t} + gY_{12t} + g^2X_{22t} + g^2\Xi_{31t} + g^2\Xi_{13t} + g^3Y_{h21t} + g^3Y_{h12t}$$

$$g^3 \sim g \times g \times g; \quad g^3 \sim g \times g^2$$

S. D. Głazek, Acta Phys. Polon. B **43**, 1843 (2012).

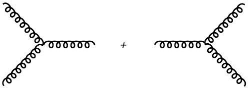
Solve perturbatively up to order g^3

Extract the Hamiltonian running coupling

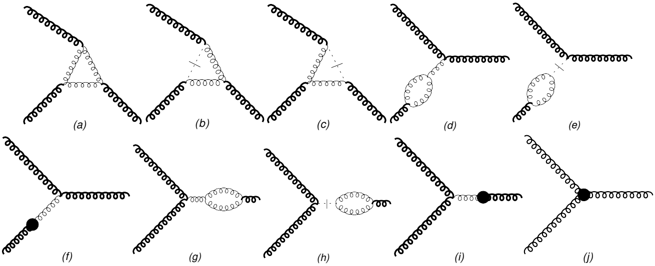
→ Solve order by order up to g^3

$$H_t = E + g^2 \hat{\mu}_t^2 + \underline{gY_{21t} + gY_{12t}} + g^2 X_{22t} + g^2 \Xi_{31t} + g^2 \Xi_{13t} + \underline{g^3 Y_{h21t} + g^3 Y_{h12t}}$$

1st order terms $\sim g$



3rd-order terms $\sim g^3$



The Hamiltonian running coupling

→ From three-gluon interaction term in the effective Hamiltonian

$$H_{tA^3} = \sum_{123} \int [123] \, \tilde{\delta}(p^\dagger - p) \; f_{12t} \; g Y_t(x_1, \kappa_{12}^\perp, \sigma) \; a_{t1}^\dagger a_{t2}^\dagger a_{t3} + H.c.$$

→ Extract the running coupling from the three-gluon vertex in $\lim_{\kappa_{12} \to \infty} Y_t$

$$Y_t = g Y_{1t} + g^3 Y_{3t} + \ldots \,,$$

$$Y_t = \sum_{123} \int [123] \; \tilde{\delta}(1 + 2 - 3) \; f_{12t} \; \tilde{Y}_t(x_1, \kappa_{12}^\perp, \sigma) \; a_1^\dagger a_2^\dagger a_3$$

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$$\Rightarrow g_\lambda = g_0 - \frac{g_0^3}{48\pi^2} N_c\, 11\, \ln \frac{\lambda}{\lambda_0} , \qquad \lambda = 1/t^4$$

$$\lambda \frac{d}{d\lambda} g_\lambda = \beta_0 g_\lambda^3 , \qquad \beta_0 = -\frac{11 N_c}{48\pi^2} .$$

Asymptotic freedom result

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Asymptotic freedom result identical to
D. J. Gross and F. Wilczek Phys. Rev. Lett. **30**, 1343 (1973)
H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).

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Identical result obtained using other generator $\mathcal{G}_t$ in
S. D. Głazek, Phys. Rev. D **63**, 116006 (2001)

Summary and Conclusions

- We derived the 3rd-order effective Hamiltonian for gluons from solving the RGPEP equations perturbatively.
- We have extracted the Hamiltonian running coupling $\rightarrow$ asymptotic freedom.
- Universality:
  - Hamiltonian g_λ resembles the one obtained from other formalism or renormalization schemes.
  - Hamiltonian g_λ identical to the one obtained using a different RGPEP generator $\mathcal{G}_t$.
- 4th-order terms are necessary to pose any physical eigenvalue problem (e.g. glueball) that includes the running of the coupling.

Additional slides

RGPEP details

$$\mathcal{H}_t(a_t) \quad = \quad \mathcal{H}_0(a_0)$$

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$$\mathcal{G}_t \quad = \quad [\mathcal{H}_f, \mathcal{H}_{Pt}]$$

$$\mathcal{H}_t(a_0) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \ldots, i_n} c_t(i_1, \ldots, i_n)\ a_{0i_1}^\dagger \cdots a_{0i_n}$$

$$\mathcal{H}_{Pt}(a_0) \quad = \quad \sum_{n=2}^{\infty} \sum_{i_1, i_2, \ldots, i_n} c_t(i_1, \ldots, i_n) \left(\frac{1}{2} \sum_{k=1}^{n} p_{i_k}^+ \right)^2 \ a_{0i_1}^\dagger \cdots a_{0i_n}$$

$$\mathcal{U}_t \quad = \quad T \exp\left(- \int_0^t d\tau\, \mathcal{G}_\tau \right)$$

Initial condition: $\mathcal{H}_{t=0} = H_{A^2} + H_{A^3} + H_{A^4} + H_{[A\partial A]^2} +$ counterterms.
Counterterms are such that the UV-cutoff Δ is removed entirely.

The Hamiltonian running coupling

Other regularizations?

- → counterterms removed cutoff-Δ dependence entirely
- → r_δ removes small-x divergences but a finite regularization dependence remains

$$g_\lambda = g_0 - \frac{g_0^3}{48\pi^2} N_c \, [11 + h(x_0)] \, \ln \frac{\lambda}{\lambda_0} \, ,$$

$$\text{a) } r_\delta(x) = x/(x+\delta), \quad b) \; r_\delta(x) = \theta(x-\delta), \quad c) \; r_\delta(x) = x^\delta \, \theta(x-\epsilon),$$

$$\text{a)} \qquad h(x_0) = 12 \left[3 + \frac{1 - x_0 - x_0^2}{(1-x_0)(1-2x_0)} \ln x_0 + \frac{(1-x_0)^2 - x_0}{x_0(1-2x_0)} \ln(1-x_0) \right] \, ,$$

$$\text{b)} \qquad h(x_0) = 12 \ln \min(x_0, 1-x_0) \, ,$$

$$\text{c)} \qquad h(x_0) = 0.$$

Identical result obtained in S. D. Głazek, Phys. Rev. D **63**, 116006 (2001).with other generator $\mathcal{G}_t$.
The $\mathcal{G}_t$ used in the present work leads to simpler equations.

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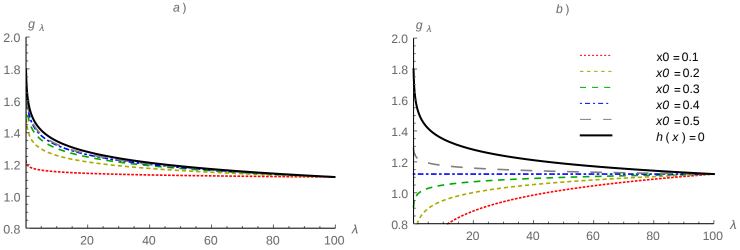

a) $r_\delta(x) = x/(x+\delta)$, b) $r_\delta(x) = \theta(x-\delta)$, c) $r_\delta(x) = x^\delta \, \theta(x-\epsilon)$,

$g_0 = 1.1$, $\lambda_0 = 100$ GeV