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Nakanishi representation for two fermions bound state in Minkowski space.

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Outline

I. Nakanishi representation for two-fermions
II. Uniqueness of Nakanishi representation
III. Numerical method and preliminary results
IV. A method for Wick-Cutkosky model
V. Conclusions and perspectives

Nakanishi representation

- Nakanishi representation: “Parametric representation for any Feynmann diagram for interacting bosons, with a denominator carrying the overall analytical behavior in Minkowski space” (1962)

Bethe-Salpeter amplitude

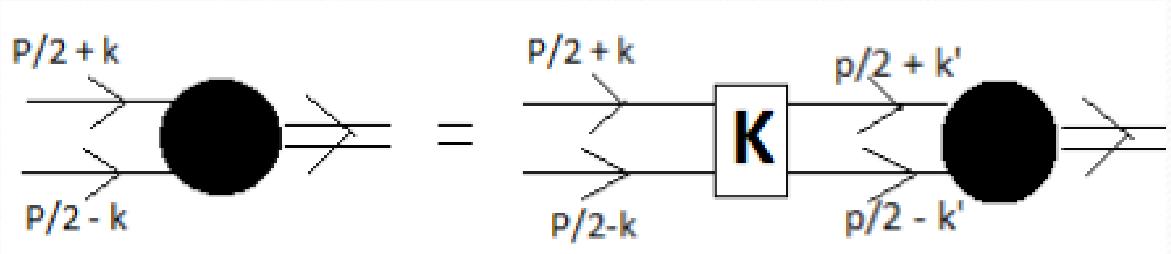
$$\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g(\gamma',z')}{(\gamma' + \kappa^2 - k^2 - p.kz' - i\epsilon)^3}$$

- Solution of the Bethe-Salpeter equation in Minkowski space with Nakanishi representation for bosons:
- Kusaka and Williams, PRD 51 7026 (1995); Karmanov and Carbonell, EPJA 27 1 (2006), EPJA27 11 (2006), EPJA27 11 (2010); Frederico, Salmè and Viviani PRD 85 036009(2012), PRD 89, 016010 (2014), EPJC(2015).
- For Fermions: Karmanov and Carbonell, EPJA 46 387 (2010)

Nakanishi representation for two-fermions

Carbonell and Karmanov (2010)

- Bethe-Salpeter equation



Ladder approximation: one scalar boson exchange

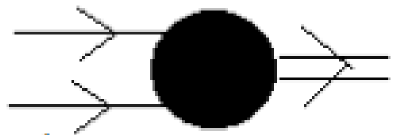
$$\Phi(k,p) = \frac{i\left(m + \not{p}/2 + \not{k}\right)}{\left((p/2+k)^2 - m^2 + i\epsilon\right)} \int \frac{d^4k'}{(2\pi)^4} \Phi(k',p) \frac{(-ig)^2 F^2(k-k')}{(k-k')^2 - \mu^2 + i\epsilon} \frac{i\left(m - \not{p}/2 + \not{k}\right)}{\left((p/2-k)^2 - m^2 + i\epsilon\right)}$$

Vertex Form-Factor $F(q) = \dfrac{\mu^2 - \Lambda^2}{q^2 - \Lambda^2 + i\epsilon}$

Nakanishi representation for two-fermions

Carbonell and Karmanov (2010)

Bethe-Salpeter amplitude: fermion-fermion 0^+ state



$$\Phi(k,p) = S_1\phi_1 + S_2\phi_2 + S_3\phi_3 + S_4\phi_4$$

$$S_1 = \gamma_5 \quad S_2 = \frac{1}{M}\not{p}\gamma_5 \quad S_3 = \frac{k\cdot p}{M^3}\not{p}\gamma_5 - \frac{1}{M}\not{k}\gamma_5 \quad S_4 = \frac{i}{M^2}\sigma_{\mu\nu}p^\mu k^\nu\gamma_5$$

Multiplying BSE by S_i and taking the trace

$$\phi_i(k,p) = \frac{i}{\left((p/2+k)^2 - m^2 + i\epsilon\right)}\frac{i}{\left((p/2-k)^2 - m^2 + i\epsilon\right)}$$

$$\times \int \frac{d^4k'}{(2\pi)^4}\frac{(-ig)^2F^2(k-k')}{(k-k')^2 - \mu^2 + i\epsilon}\sum_{j=1}^{4} c_{ij}(k,k',p)\phi_j(k',p)$$

Nakanishi representation for two-fermions

$$\phi_i(k,p) = \int_{-1}^{+1} dz' \int_0^{\infty} d\gamma' \frac{g_i(\gamma', z')}{(k^2 + p\cdot k\ z' + M^2/4 - m^2 - \gamma' + i\epsilon)^3}$$

After using the Nakanishi representation, we project on the light-front obtaining

$$\int_0^{\infty} \frac{d\gamma'}{(\gamma + \gamma' + m^2 z^2 + (1-z^2)\kappa^2)^2} g_i(\gamma', z) = \int_0^{\infty} d\gamma' \int_{-1}^{1} dz' \sum_j V_{ij}(\gamma, z, \gamma', z') g_j(\gamma', z')$$

$$V_{ij}(\gamma, z, \gamma', z') = W_{ij}(\gamma, z, \gamma', z'), z' \leq z$$
$$V_{ij}(\gamma, z, \gamma', z') = \sigma_{ij} W_{ij}(\gamma, -z, \gamma', -z'), z' > z$$

$$\sigma = \begin{pmatrix} +1 & +1 & -1 & +1 \\ +1 & +1 & -1 & +1 \\ -1 & -1 & +1 & -1 \\ +1 & +1 & -1 & +1 \end{pmatrix}$$

$$\Lambda \to \infty$$

$$W_{ij}(\gamma, z, \gamma', z') = \frac{\alpha m^2}{2\pi} \frac{1}{\gamma + m^2 z^2 + (1-z^2)\kappa^2} \frac{(1-z)^2}{(1-z')^2} \int_0^1 \frac{dv}{v^2} \frac{C_{ij}(\gamma, z; v)}{D(\gamma, z)^2}$$

$$D(\gamma, z) = \gamma + m^2 z^2 + \kappa^2(1-z^2) + \Gamma.$$
$$\Gamma = \frac{1-z}{1-z'} \left(\frac{1-v}{v} \left[z'^2 \frac{M^2}{4} + \kappa^2 \right] + \frac{\mu^2}{(1-v)} + \frac{\gamma'}{v} \right)$$

Uniqueness for Nakanishi representation

The goal is to reorganize the RHS of the BSE in order to obtain the same denominator of the LHS

Using Feynman parametrization we have

$$W_{ij}(\gamma,z,\gamma',z') = \frac{\alpha m^2}{\pi} \frac{(1-z)^2}{(1-z')^2} \int_0^1 \frac{dv}{v^2} \int_0^1 d\zeta \, \zeta \, \frac{C_{ij}(\gamma,z;v)}{[\gamma + m^2 z^2 + (1-z^2)\kappa^2 + \zeta\Gamma]^3}$$

$$\mathcal{D}(\gamma,\zeta\Gamma) = \gamma + \zeta\Gamma + m^2 z^2 + (1-z^2)\kappa^2$$

$$C_{ij}(\gamma,z;v) = [\mathcal{D}(\gamma,\zeta\Gamma)]^3 A^3_{ij}(z,v) + [\mathcal{D}(\gamma,\zeta\Gamma)]^2 B^2_{ij}(\zeta\Gamma,z,v) + \mathcal{D}(\gamma,\zeta\Gamma) B^1_{ij}(\zeta\Gamma,z,v) + B^0_{ij}(\zeta\Gamma,z,v)$$

For example:

$$\frac{B^2_{ij}(\zeta\Gamma,z,v)}{\mathcal{D}(\gamma,\zeta\Gamma)} = \int_0^\infty d\gamma'' \, \theta(\gamma''-\zeta\Gamma) \, \frac{B^2_{ij}(\zeta\Gamma,z,v)}{[\mathcal{D}(\gamma,\gamma'')]^2}$$

Uniqueness for Nakanishi representation

$$\frac{B^0_{ij}(\zeta\Gamma,z,v)}{[\mathcal{D}(\gamma,\zeta\Gamma)]^3} = \frac{B^0_{ij}(\zeta\Gamma,z,v)}{2}\ \frac{\partial^2}{\partial(\zeta\Gamma)^2}\frac{1}{\mathcal{D}(\gamma,\zeta\Gamma)} =$$

$$= \frac{B^0_{ij}(\zeta\Gamma,z,v)}{2}\ \frac{\partial^2}{\partial(\zeta\Gamma)^2}\int_0^\infty d\gamma''\ \theta(\gamma''-\zeta\Gamma)\frac{1}{[\mathcal{D}(\gamma,\gamma'')]^2} =$$

$$= \frac{1}{2}\int_0^\infty d\gamma''\ \frac{B^0_{ij}(\zeta\Gamma,z,v)}{[\mathcal{D}(\gamma,\gamma'')]^2}\ \frac{\partial}{\partial\gamma''}\delta(\gamma''-\zeta\Gamma)$$

Finally

$$W_{ij}(\gamma,z,\gamma',z') = \frac{\alpha m^2}{\pi}\frac{(1-z)^2}{(1-z')^2}\int_0^1\frac{dv}{v^2}\int_0^1 d\zeta\ \zeta\int_0^\infty d\gamma''\ \frac{1}{[\mathcal{D}(\gamma,\gamma'')]^2}$$
$$\times\Big[3A^3_{ij}(z,v)\ (\gamma''-\zeta\Gamma)\theta(\gamma''-\zeta\Gamma) + B^2_{ij}(\zeta\Gamma,z,v)\theta(\gamma''-\zeta\Gamma)\ + B^1_{ij}(\zeta\Gamma,z,v)\delta(\gamma''-\zeta\Gamma)$$
$$+\frac{1}{2}B^0_{ij}(\zeta\Gamma,z,v)\frac{\partial}{\partial\gamma''}\delta(\gamma''-\zeta\Gamma)\Big]$$

Uniqueness for Nakanishi representation

$$\int_0^\infty \frac{d\gamma''}{[\mathcal{D}(\gamma,\gamma'')]^2} g_i(\gamma'',z) = \int_0^\infty d\gamma'' \frac{1}{[\mathcal{D}(\gamma,\gamma'')]^2} \frac{\alpha m^2}{\pi} \sum_{j=1}^{4} \int_0^\infty d\gamma' \int_{-1}^{1} dz' K_{ij}\left(\gamma'',z,\gamma',z'\right) g_j(\gamma',z')$$

$$g_i(\gamma'',z) = \frac{\alpha m^2}{\pi} \sum_{j=1}^{4} \int_0^\infty d\gamma' \int_{-1}^{1} dz' K_{ij}\left(\gamma'',z,\gamma',z'\right) g_j(\gamma',z')$$

$$K_{ij}(\gamma'',z;\gamma',z') = \theta(z-z')K'_{ij}(\gamma'',z;\gamma',z') + \sigma_{ij}\theta(z'-z)K'_{ij}(\gamma'',-z;\gamma',-z')$$
$$K'_{ij}(\gamma'',z;\gamma',z') = \frac{(1-z)^2}{(1-z')^2} \int_0^1 \frac{dv}{v^2} \int_0^1 d\zeta\, \zeta\, \mathcal{M}_{ij}(\gamma'',z;\gamma',z';v,\zeta)$$

$$\begin{aligned}\mathcal{M}_{ij}(\gamma,z,\gamma',z') = {} & 3A^3_{ij}(z,v)\ (\gamma''-\zeta\Gamma)\theta(\gamma''-\zeta\Gamma) + B^2_{ij}(\zeta\Gamma,z,v)\theta(\gamma''-\zeta\Gamma) \\ & + B^1_{ij}(\zeta\Gamma,z,v)\delta(\gamma''-\zeta\Gamma) + \frac{1}{2}B^0_{ij}(\zeta\Gamma,z,v)\frac{\partial}{\partial\gamma''}\delta(\gamma''-\zeta\Gamma)\end{aligned}$$

Numerical Methods and Preliminary Results

Basis expansion:

$$g_i(\gamma'', z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A^i_{mn} G_m(z) \mathcal{L}_n(\gamma'')$$

$$G_m(z) = 4(1-z^2)\Gamma(5/2)\sqrt{\frac{(m+5/2)(m)!}{\pi\Gamma(m+5)}} C_m^{5/2}(z)$$

$$\mathcal{L}_n(\gamma) = \sqrt{a} L_n(a\gamma) e^{-a\gamma/2}$$

Applying basis expansion to the uniqueness equation eliminates the delta and derivative of delta.

Numerical Methods and Preliminary Results

Checking the basis expansion method for the two-fermions Bethe-Salpeter coupled equations (without uniqueness):

Coupling constant for the coupled equations in terms of the binding energies.

Results obtained for $\Lambda = 2$, $m = 1$ and $\mu = 0.5$

B	$g1 + g2$	$g1 + g2 + g3$	$Carbonell/Karmanov$
0.01	25.28	25.29	25.23
0.02	29.32	29.35	29.49
0.05	38.86	39.00	39.19
0.10	52.34	52.41	52.82
0.20	77.73	77.84	78.25
0.50	159.87	157.97	157.4

We are implementing g_4.

In the basis is used 6 Laguerre and 6 Gegenbauer polynomials

Method for Wick-Cutkosky limit: two bosons

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{(\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2)^2} = \frac{\alpha m^2}{2\pi}\frac{1}{d_0(\gamma,z)}\int_{-1}^{1} dz' \left[\omega(z,z')\theta(z-z')+\omega(z,z')\theta(z'-z)\right]$$

$$\omega(z,z') = \int_0^\infty d\gamma' \frac{g(\gamma',z)}{(\gamma'+a(z'))(\gamma'+c(z,z',\gamma))}$$

$$d_0(\gamma,z) = \gamma + z^2m^2 + (1-z^2)\kappa^2 \quad a(z') = z'^2m^2 + (1-z'^2)\kappa^2 \quad c(z,z',\gamma) = \frac{1-z'}{1-z}d_0(\gamma,z)$$

Let´s define the sequence $G^{(n)}(\gamma',z)$

$$G^{(0)}(\gamma',z) = g(\gamma',z) \qquad G^{(n+1)}(\gamma',z) = -\int_{\gamma'}^{\infty} d\gamma''\, G^{(n)}(\gamma'',z)$$

$$G^{(n)}(\gamma',z) = \frac{\partial}{\partial\gamma'}G^{(n+1)}(\gamma',z) \qquad b_n(z) = G^{(n)}(0,z)$$

Method for Wick-Cutkosky limit: two bosons

The LHS

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{(\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2)^2} = \int_0^\infty d\gamma' \frac{g(\gamma',z)}{(\gamma'+d_0(\gamma,z))^2}$$

$$= \int_0^\infty d\gamma' \left(\frac{\partial}{\partial\gamma'} G^{(1)}(\gamma',z)\right) \frac{1}{(\gamma'+d_0(\gamma,z))^2}$$

$$= \frac{b_1(z)}{d_0(\gamma,z)^2} - \int_0^\infty d\gamma' \; G^1(\gamma',z) \frac{\partial}{\partial\gamma'} \left(\frac{1}{(\gamma'+d_0(\gamma,z))^2}\right)$$

$$d_0(\gamma,z) = \gamma + z^2m^2 + (1-z^2)\kappa^2$$

After perfoming n integrations and taking the limit to infinity

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{(\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2)^2} = \sum_{i=1}^{\infty} i! \frac{b_i(z)}{d_0(\gamma,z)^{i+1}}$$

Method for Wick-Cutkosky limit: two bosons

The RHS $$\int_0^\infty d\gamma' \frac{g(\gamma',z')}{(\gamma'+a(z'))(\gamma'+c(z,z',\gamma))} = \sum_{i=1}^{\infty}\left(\frac{1-z}{(1-z')d_0(\gamma,z)}\right)^i \sum_{j=i}^{\infty}\frac{(j-1)!b_j(z')}{a(z')^{j-i+1}}$$

In both LHS and RHS we obtain a series on d_0. Therefore

$$b_i(z) = \frac{\alpha m^2}{2\pi}\int_{-1}^{1} dz' \frac{1}{i!}\Big[\left(\frac{1-z}{1-z'}\right)^i \theta(z-z') + \left(\frac{1+z}{1+z'}\right)^i \theta(z'-z)\Big]\sum_{j=i}^{\infty}\frac{(j-1)!b_j(z')}{a(z')^{j-i+1}}$$

To solve the system of coupled equations we set $b_i(z)=0$ for $i> i_0$ for some i_0

The matrix is triangular by blocks, therefore the eigenvalues are determined by $i=i_0$

$$b_{i_0}(z) = \frac{\alpha m^2}{2\pi}\int_{-1}^{1} dz' \frac{1}{i_0}\Big[\left(\frac{1-z}{1-z'}\right)^{i_0} \theta(z-z') + \left(\frac{1+z}{1+z'}\right)^{i_0} \theta(z'-z)\Big]\frac{b_{i_0}(z')}{a(z')}$$

Nakanishi weigh function is then given by a series of derivative of delta

$$g(\gamma',z') = b_1(z')\delta(\gamma') + b_2(z')\delta'(\gamma') + b_3(z')\delta''(\gamma') + \ldots + b_i(z')\delta^{(i-1)}(\gamma')$$

Wick-Cutkosky limit: Numerical results

Eingenvalues for fundamental and excited states for different binding energies.

Series Expansion

$\frac{B}{m}$	α_1	α_2	α_3	α_4
0.1	1.12	2.93	5.37	13.29
0.2	1.79	4.85	9.15	17.20
0.3	2.35	6.53	12.49	20.21
0.4	2.84	8.05	15.53	22.72
0.5	3.30	9.42	18.34	24.88

Legendre Polynomials

Wick-Cutkosky

$\frac{B}{m}$	α_1	α_2
0.1	1.11	2.90
0.2	1.78	4.90
0.3	2.34	6.58
0.4	2.84	8.09
0.5	3.29	9.44

Laguerre Polynomials – in γ

Legendre Polynomials – in z

Conclusions and Perspectives

I. We analyzed the BSE for two-fermions bound state using the Nakanishi representation and light-front projection.

II. We derived the integral equation for the Nakanishi weight function using the conjecture of uniqueness.

III. We applied the basis expansion method to obtain the numerical solution for the bound state. The next step is to solve the integral equation after applying uniqueness.

IV. We presented a method for solving the Wick-Cutkosky model for bosons with Nakanishi representation and then we will apply for the Fermion case.

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