The pion renormalized light-cone wave function

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How to describe hadrons involved in high-energy interaction of scale λ that are solutions of QCD eigenvalue equation?

Hadron as a solution of QCD

Hadrons can be described in terms of constituents of any size.
Let us consider a meson as an example of a hadron.

$$
\begin{aligned}
|\text{Meson}\rangle = \; & \psi_{q\bar{q}}(\lambda)\,|q\bar{q};\lambda\rangle \\
& + \psi_{q\bar{q}g}(\lambda)\,|q\bar{q}\,g;\lambda\rangle \\
& + \psi_{q\bar{q}gg}(\lambda)\,|q\bar{q}\,gg;\lambda\rangle \\
& + \psi_{q\bar{q}q\bar{q}}(\lambda)\,|q\bar{q}\,q\bar{q};\lambda\rangle \\
& + \psi_{q\bar{q}q\bar{q}g}(\lambda)\,|q\bar{q}\,q\bar{q}\,g;\lambda\rangle \\
& + \dots
\end{aligned}
$$

In the front-form formulation of the theory, such a state satisfies the following eigenvalue equation:

$$H_{\text{QCD}}|\text{Meson}\rangle = [K + V]\,|\text{Meson}\rangle = M^2\,|\text{Meson}\rangle\,,$$

where K is a kinetic part of the full Hamiltonian H_{QCD} and V its interaction part.

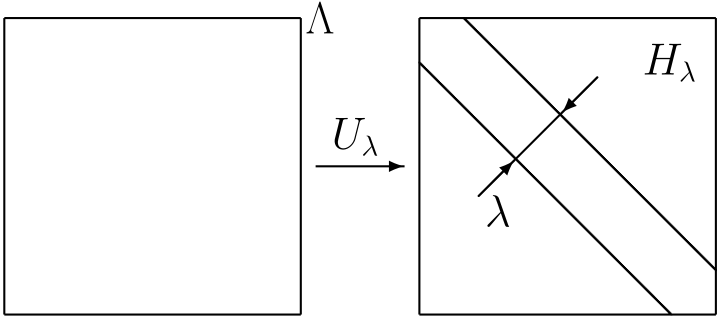
The QCD interaction

n	Sector	1 $q\bar{q}$	2 gg	3 $q\bar{q}\,g$	4 $q\bar{q}\,q\bar{q}$	5 $gg\,g$	6 $q\bar{q}\,gg$	7 $q\bar{q}\,q\bar{q}\,g$	8 $q\bar{q}\,q\bar{q}\,q\bar{q}$	9 $gg\,gg$	10 $q\bar{q}\,gg\,g$	11 $q\bar{q}\,q\bar{q}\,gg$	12 $q\bar{q}\,q\bar{q}\,q\bar{q}\,g$	13 $q\bar{q}\,q\bar{q}\,q\bar{q}\,q\bar{q}$
1	$q\bar{q}$					•		•	•	•	•	•	•	•
2	gg				•			•	•		•	•	•	•
3	$q\bar{q}\,g$								•	•		•	•	•
4	$q\bar{q}\,q\bar{q}$		•			•				•	•		•	•
5	$gg\,g$	•			•			•	•			•	•	•
6	$q\bar{q}\,gg$								•				•	•
7	$q\bar{q}\,q\bar{q}\,g$	•	•			•				•				•
8	$q\bar{q}\,q\bar{q}\,q\bar{q}$	•	•	•		•	•			•	•			
9	$gg\,gg$	•		•	•			•	•			•	•	•
10	$q\bar{q}\,gg\,g$	•	•		•				•				•	•
11	$q\bar{q}\,q\bar{q}\,gg$	•	•	•		•				•				•
12	$q\bar{q}\,q\bar{q}\,q\bar{q}\,g$	•	•	•	•	•	•			•	•			
13	$q\bar{q}\,q\bar{q}\,q\bar{q}\,q\bar{q}$	•	•	•	•	•	•	•		•	•	•		

Lepage, Brodsky, Huang, Mackenzie, Particles and fields: Proceedings (August 16-27, 1981) 83-141
Brodsky, Pauli, Pinsky, Phys.Rept. 301 (1998) 299-486

RGPEP

The similarity renormalization group procedure and its successors allow to change Hamiltonian from one λ to another. Especially, the renormalization group procedure for effective particles (RGPEP) diagonalizes Hamiltonian by a rotation parameterized by λ.



Głazek, Wilson, Phys.Rev. D48 (1993) 5863-5872
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Wilson, Walhout, Harindranath, Zhang, Perry, Głazek, Phys.Rev. D49 (1994) 6720-6766
Głazek, Acta Phys.Polon. B42 (2011) 1933-2010

Frequently Asked Questions

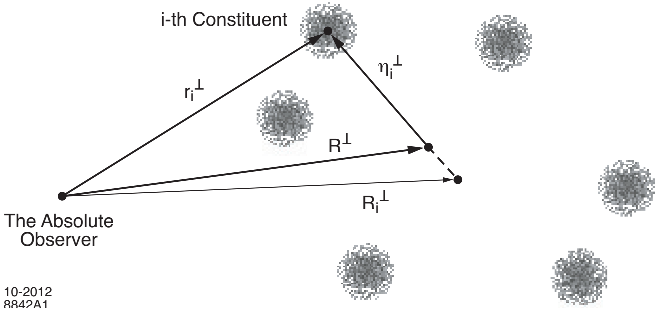
Does H depend on λ?
• Hamiltonian is written in terms of creation and annihilation operators that depend on λ. It can be written using operators at any λ but it is always **the same operator**.

What is the meaning of λ?
• An interaction term written for certain λ only allows changes of energy that are not greater than λ.

Who fixes λ?
• The choice of λ depends on us; we select λ to describe physics in a simplest way possible.
It is easier to solve an eigenvalue equation for a small λ than for the large one. Unfortunately, the eigenvector found for small λ is difficult to use for description of processes that involve large changes of energy.

Ehrenfest approach

This approach yields for any λ the eigenvalue equation that is similar to the eigenvalue equation for small λ. In n-th Fock sector, the active constituent is labeled by i and we average over others, and over all sectors.



$$M^2 = \left\langle \int_{k,x} \psi_{k,x}^{\dagger(n)} \frac{(k_i^\perp)^2 + (m_i^{(n)})^2}{x_i(1-x_i)} \psi_{k,x}^{(n)} \right\rangle + \left\langle \text{connected interactions} \right\rangle .$$

Głazek, Trawiński, Phys.Rev. D88 (2013) 10, 105025

Averging over spectators

$$\left\langle \int_{k,x} \left(m_i^{(n)}\right)^2 \left|\psi_{k,x}^{(n)}\right|^2 \right\rangle = m_{\text{active}}^2$$

$$\left\langle \int_{k,x} \left(\mathcal{M}_i^{(n)}\right)^2 \left|\psi_{k,x}^{(n)}\right|^2 \right\rangle = m_{\text{core}}^2$$

$$U_{\text{eff}} = \left\langle \text{connected interactions} \right\rangle$$

The averaged quantities are used to write the expectation value in term of one function $\psi(k^\perp, x)$

$$\int_{k,x} \psi(k^\perp, x)^\dagger \left[K_{\text{eff}} + U_{\text{eff}}\right] \psi(k^\perp, x) = M^2 \, .$$

Variation of the expectation value with respect to $\psi(k^\perp, x)$ yields

$$\left[K_{\text{eff}} + U_{\text{eff}}\right] \psi(k^\perp, x) = M^2 \, \psi(k^\perp, x) \, .$$

Głazek, Trawiński, Phys.Rev. D88 (2013) 10, 105025

Brodsky – de Téramond holography

The Light-Front (LF) Holography approach uses exacly the same equation as in Ehrenfest approach, and successfully predicts:

- hadron spectroscopy,
- hadron dynamics – hadron form factors,
- the QCD running coupling at small Q^2,
- ρ electroproduction,
- distribution amplitudes,
- LFWFs,

– all with a harmonic potential $U_{\text{eff}} = -\frac{1}{4}\varkappa^4 r_\perp^2$, where $r_\perp$ is a canonical conjugated variable to $p^\perp = k^\perp / \sqrt{x(1-x)}$ and $\varkappa = 550$ MeV.
Moreover U_{eff} does not depend on λ.

In this approach m_{active} and m_{core} is set 0.

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de Téramond, Brodsky, Phys.Rev.Lett. 94 (2005) 201601
de Téramond, Brodsky, Phys.Rev.Lett. 102 (2009) 081601
Brodsky, de Téramond, Phys.Rev. D77 (2008) 056007
Brodsky, de Téramond, Deur, Phys.Rev. D81 (2010) 096010
Forshaw, Sandapen, Phys.Rev.Lett. 109 (2012) 081601

Beyond LF-Holography

- The light-front holographic confining potential,

$$U_{\text{eff}} \sim \varkappa^4 r_\perp^2 \,.$$

- It is natural to replace $\varkappa^4 r_\perp^2$ in U_{eff} by $\varkappa^4(r_\perp^2 + r_3^2)$ in the case of massive quarks, where for $m = m_{\text{active}} = m_{\text{core}}$

$$p_3 = \frac{2m}{\sqrt{x(1-x)}}\left(x - \frac{1}{2}\right)$$

- Then U_{eff} becomes a 3-dimensional harmonic oscillator.
- Thus, excitations in the transverse plane are paired with excitations in the 3rd-direction, and 3-dimensional rotational symmetry is restored in the massive case, $m \neq 0$.
- The quadratic potential in the FF agrees with a linear potential in the IF ($\sqrt{\sigma} = \varkappa/\sqrt{2}$).

Trawiński, Głazek, Brodsky, Téramond, Dosch, Phys.Rev. D90 (2014) 7,074017

So, how to describe hadrons involved in high-energy interaction of scale λ that are solutions of QCD eigenvalue equation?

The Ehrenfest equation

$$\left[\frac{(k^{\perp})^2 + m^2_{\text{active}}}{x} + \frac{(k^{\perp})^2 + m^2_{\text{core}}}{1-x} + U_{\text{eff}}\right]\psi(k^{\perp}, x) = M^2\psi(k^{\perp}, x)$$

The QCD equation for λ_c

$$\left[\frac{(k^{\perp})^2 + m_q(\lambda_c)^2}{x} + \frac{(k^{\perp})^2 + m_{\bar{q}}(\lambda_c)^2}{1-x} + V_{\text{QCD}}(\lambda_c)\right]\psi_{q\bar{q}}(k^{\perp}, x; \lambda_c) \approx M^2\psi_{q\bar{q}}(k^{\perp}, x; \lambda_c)$$

How to describe hadrons involved in high-energy interaction of scale λ that are solutions of QCD eigenvalue equation?

1. Substitute $V_{\text{QCD}}(\lambda_c)$ by U_{eff}.
2. Solve equation for lowest Fock sector.
3. Find $\psi_{q\bar{q}}(\lambda_c)$.
4. Using RGPEP find $\psi_{q\bar{q}}(\lambda)$, $\psi_{q\bar{q}\,g}(\lambda)$, ...

Głazek, Acta Phys.Polon. B42 (2011) 1933-2010

Pion as an example of a hadron

Solving an eigenvalue equation with U_{eff} gives

$$\psi_{q\bar{q}/\pi}(x, k^{\perp}; \lambda_c) \propto \exp\left[-\frac{m_c^2 + (l^{\perp})^2}{2\,x(1-x)\,\varkappa^2}\right] ,$$

including spin factor, finally we get

$$\psi_{q\bar{q}/\pi}(x, k^{\perp}; \lambda_c) = \sqrt{\mathcal{N}}\,\bar{u}\gamma^5 v\,\exp\left[-\frac{m_c^2 + (k^{\perp})^2}{2\,x(1-x)\,\varkappa^2}\right] ,$$

where $m_c = m(\lambda_c)$, and

$$\bar{u}\gamma^5 v = \begin{cases} \pm\dfrac{m_c}{\sqrt{x(1-x)}} & \text{for } \begin{smallmatrix}(\uparrow\downarrow)\\(\downarrow\uparrow)\end{smallmatrix} \\ \dfrac{k_1 \pm ik_2}{\sqrt{x(1-x)}} & \text{for } \begin{smallmatrix}(\uparrow\uparrow)\\(\downarrow\downarrow)\end{smallmatrix} \end{cases} .$$

Test #1

The form-factor formula

$$F(q^2) = \int \frac{d^2k^\perp dx}{16\pi^3 x(1-x)}\ \psi_{q\bar{q}/\pi}\left(x, k^\perp + (1-x)q^\perp; \lambda_c\right) \\ \times \psi_{q\bar{q}/\pi}\left(x, k^\perp; \lambda_c\right)\ ,$$

thus reads

$$F(q^2) = 3\mathcal{N} \int \frac{d^2k^\perp dx}{16\pi^3 x(1-x)}\, 2\left[\frac{m_c^2 + (k^\perp)^2}{x(1-x)} - \frac{1-x}{x}\frac{q^2}{4}\right] \\ \times \exp\left[-\frac{m_c^2 + (k^\perp)^2}{x(1-x)\varkappa^2}\right] \exp\left[-\frac{1-x}{x}\,\frac{q^2}{4\varkappa^2}\right]\ ,$$

where 3 comes from summing over colors and 2 comes from summing over spins.

Test #1

The condition $F(0) = 1$ fixes the value of $\mathcal{N}$.

$$F(q^2) = 3\mathcal{N} \int \frac{d^2k^\perp dx}{16\pi^3 x(1-x)}\, 2 \left[\frac{m_c^2 + (k^\perp)^2}{x(1-x)} - \frac{1-x}{x}\frac{q^2}{4} \right] \times \exp\left[-\frac{m_c^2 + (k^\perp)^2}{x(1-x)\varkappa^2} \right] \exp\left[-\frac{1-x}{x}\frac{q^2}{4\varkappa^2} \right]$$

Then the pion radius is calculable,

$$r_\pi^2 = -6\, \frac{d}{dq^2} F(q^2)\Big|_{q^2=0}\,,$$

and can be compared with the PDG value $\langle r_\pi^2 \rangle = 0.45(1)\ \mathrm{fm}^2$.

For $m_c = 300$ MeV and $\varkappa = 550$ MeV one gets
$r_\pi^2 = 10.86\ \mathrm{GeV}^{-2} = 0.43\ \mathrm{fm}^2$.

Particle Data Group Collaboration, Chin.Phys. C38 (2014) 090001

Test #2

The pion decay constant f_π is defined from the pion to vacuum matrix element of the axial current,

$$\langle 0|\,\bar{\Psi}_u(0)\,\gamma^+\,\Psi_d(0)\,\left|\pi^-\right\rangle = if_\pi\,P_\pi^+\,.$$

Thus, we have

$$f_\pi = 3\sqrt{\mathcal{N}}\int \frac{d^2k^\perp dx}{16\pi^3 x(1-x)}\,4\,m_c\,\exp\left[-\frac{m^2+(k^\perp)^2}{2x(1-x)\varkappa^2}\right]\,,$$

where 3 comes from summing over colors.

And can be compared with the PDG value $\langle f_\pi\rangle = 130.4(2)$ MeV.

For $m_c = 300$ MeV and $\varkappa = 550$ MeV one gets $f_\pi = 134$ MeV.

Particle Data Group Collaboration, Chin.Phys. C38 (2014) 090001

What is left from our *To Do* list?

1. Substitute $V_{\mathrm{QCD}}(\lambda_c)$ by U_{eff}.
2. Solve equation for lowest Fock sector.
3. Find $\psi_{q\bar{q}}(\lambda_c)$.
4. Using RGPEP find $\psi_{q\bar{q}}(\lambda)$, $\psi_{q\bar{q}\,g}(\lambda)$, ...

Will be presented soon.

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Thank you for your attention.