Light-front ϕ^4_{1+1} theory using a many-boson symmetric-polynomial basis

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Objectives

- compute the odd and even-parity massive eigenstate of ϕ^4_{1+1} theory.
  - eigenvalue problem solved in the form of a Fock-state expansion
  - truncated to obtain a finite set of equations
  - wave functions expanded in a new set of symmetric multivariate polynomials [PRE **90**, 063310 (2014)]
- study convergence with respect to Fock-space truncation
- compare results with those obtained with the light-front coupled-cluster method. [PRD **90**, 056003 (2014)]

ϕ^4 theory

The Lagrangian for two-dimensional ϕ^4 theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4,$$

where μ is the mass of the boson and λ is the coupling constant. The light-front Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4.$$

The mode expansion for the field at zero light-front time is

$$\phi = \int \frac{dp^+}{\sqrt{4\pi p^+}} \left\{ a(p^+) e^{-ip^+x^-/2} + a^\dagger(p^+) e^{ip^+x^-/2} \right\},$$

with the modes quantized such that

$$[a(p^+), a^\dagger(p'^+)] = \delta(p^+ - p'^+).$$

Hamiltonian

The light-front Hamiltonian is $\mathcal{P}^- = \mathcal{P}^-_{11} + \mathcal{P}^-_{13} + \mathcal{P}^-_{31} + \mathcal{P}^-_{22}$, with

$$\mathcal{P}^-_{11} = \int dp^+ \frac{\mu^2}{p^+} a^\dagger(p^+) a(p^+),$$

$$\mathcal{P}^-_{13} = \frac{\lambda}{6}\int \frac{dp_1^+ dp_2^+ dp_3^+}{4\pi\sqrt{p_1^+ p_2^+ p_3^+ (p_1^+ + p_2^+ + p_3^+)}} a^\dagger(p_1^+ + p_2^+ + p_3^+) a(p_1^+) a(p_2^+) a(p_3^+),$$

$$\mathcal{P}^-_{31} = \frac{\lambda}{6}\int \frac{dp_1^+ dp_2^+ dp_3^+}{4\pi\sqrt{p_1^+ p_2^+ p_3^+ (p_1^+ + p_2^+ + p_3^+)}} a^\dagger(p_1^+) a^\dagger(p_2^+) a^\dagger(p_3^+) a(p_1^+ + p_2^+ + p_3^+),$$

$$\mathcal{P}^-_{22} = \frac{\lambda}{4}\int \frac{dp_1^+ dp_2^+}{4\pi\sqrt{p_1^+ p_2^+}} \int \frac{dp_1'^+ dp_2'^+}{\sqrt{p_1'^+ p_2'^+}} \delta(p_1^+ + p_2^+ - p_1'^+ - p_2'^+) \times a^\dagger(p_1^+) a^\dagger(p_2^+) a(p_1'^+) a(p_2'^+).$$

The subscripts indicate the number of creation and annihilation operators in each term.

Fock-state expansion

The eigenstate with momentum P^+ is expanded as

$$|\psi(P^+)\rangle = \sum_m (P^+)^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta(1-\sum_i^m y_i)\psi_m(y_i)|y_i P^+; P^+, m\rangle,$$

with the individual Fock states defined by

$$|y_i P^+; P, m\rangle = \frac{1}{\sqrt{m!}} \prod_{i=1}^m a^\dagger(y_i P^+)|0\rangle.$$

The sum over m is restricted to odd or even numbers.
The Hamiltonian does not mix the two cases, and we solve for the lowest eigenstate in each case.

Coupled equations

The light-front Hamiltonian eigenvalue problem

$$\mathcal{P}^-|\psi(P)\rangle = \frac{M^2}{P}|\psi(P)\rangle$$

reduces to a coupled set of integral equations for the Fock-state wave functions:

$$\frac{m}{y_1}\psi_m(y_i) + \frac{g}{4}\frac{m(m-1)}{\sqrt{y_1 y_2}}\int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}}\delta(y_1+y_2-x_1-x_2)\psi_m(x_1,x_2,y_3,\ldots,y_m)$$
$$+\tfrac{g}{6}m\sqrt{(m+2)(m+1)}\int \tfrac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}}\delta(y_1-x_1-x_2-x_3)\psi_{m+2}(x_1,x_2,x_3,y_2,\ldots,y_m)$$
$$+\tfrac{g}{6}\tfrac{(m-2)\sqrt{m(m-1)}}{\sqrt{y_1 y_2 y_3(y_1+y_2+y_3)}}\psi_{m-2}(y_1+y_2+y_3,y_4,\ldots,y_m) = \tfrac{M^2}{\mu^2}\psi_m(y_i).$$

with a dimensionless coupling

$$g \equiv \frac{\lambda}{4\pi\mu^2}$$

Basis function expansion

Solve this coupled system by truncating at some maximum number of constituents and expanding each wave function in a basis of symmetric multivariate polynomials $P_{ni}^{(m)}$

$$\psi_m(y_i) = \sqrt{\prod_i y_i} \sum_{ni} c_{ni}^{(m)} P_{ni}^{(m)}(y_1, \ldots, y_m)$$

$P_{ni}^{(m)}$ are of order n and fully symmetric with respect to interchange of momenta.
Subscript i differentiates the various possibilities at a given order n.
For $m = 2$ constituents there is only one possibility at each order, but for $m > 2$ there can be more than one.

For example, $P_{61}^{(3)} = (y_1 y_2 y_3)^2$ and $P_{62}^{(3)} = (y_1 y_2 + y_1 y_3 + y_2 y_3)^3$

Momentum constraint

Number of linearly independent polynomials of a given order restricted by the momentum-conservation constraint

$$\sum_i y_i = 1$$

For example,

$$P_2^{(3)} = y_1 y_2 + y_1 y_3 + y_2 y_3$$

is equivalent to $y_1^2 + y_2^2 + y_3^2$, up to a constant, when y_3 is replaced by $1 - y_1 - y_2$.

Symmetric polynomials

[ssc and jrh, PRE **90**, 063310 (2014).]

- can be written as a product of powers of simpler polynomials, in the form
$$P_{ni}^{(N)} = C_2^{n_2} C_3^{n_3} \cdots C_N^{n_N},$$
with the powers restricted by $n = \sum_j j n_j$.
- each different way of decomposing n into a sum of integers greater than 1 yields a different polynomial.
- the C_m are sums of simple monomials $\prod_j^N y_j^{m_j}$
  - m_j is 0 or 1 and $\sum_j^N m_j = m$
  - the sum ranges over all possible choices for the m_j, making each C_m fully symmetric.

Sample monomials

- given N momentum variables:
  - C_2 is $\sum_j^N \left(y_j \sum_{k>j}^N y_k \right)$
  - C_{N-1} is $\sum_j^N \prod_{k \neq j} y_k$
  - C_N is $y_1 y_2 \cdots y_N$
- in particular, for N=3
  - $C_2 = y_1 y_2 + y_1 y_3 + y_2 y_3$
  - $C_3 = y_1 y_2 y_3$
- the first-order polynomial $C_1 = \sum_j y_j$ does not appear because the momentum constraint reduces it to a constant.

Matrix equations

$$\sum_{n'i'}\left[T^{(m)}_{ni,n'i'}+gV^{(m,m)}_{ni,n'i'}\right]c^{(m)}_{n'i'}+g\sum_{n'i'}V^{(m,m+2)}_{ni,n'i'}c^{(m+1)}_{n'i'}$$
$$+g\sum_{n'i'}V^{(m,m-2)}_{ni,n'i'}c^{(m-1)}_{n'i'}=\frac{M^2}{\mu^2}\sum_{n'i'}B^{(m)}_{ni,n'i'}c^{(m)}_{n'i'},$$

kinetic-energy matrix:

$$T^{(m)}_{ni,n'i'}=m\int\left(\prod_j dy_j\right)\delta(1-\sum_j y_j)\left(\prod_{j=2}^{m}y_j\right)P^{(m)}_{ni}(y_j)P^{(m)}_{n'i'}(y_j),$$

basis-function overlap matrix:

$$B^{(m)}_{ni,n'i'}=\int\left(\prod_j dy_j\right)\delta(1-\sum_j y_j)\left(\prod_j^{m}y_j\right)P^{(m)}_{ni}(y_j)P^{(m)}_{n'i'}(y_j).$$

Potential-energy matrices

$$V_{ni,n'i'}^{(m,m)} = \frac{g}{4} m(m-1) \int \left(\prod_j dy_j\right) \delta(1-\sum_j y_j)$$
$$\times \int dx_1 dx_2 \delta(y_1+y_2-x_1-x_2) \left(\prod_{j=3}^m y_j\right) P_{ni}^{(m)}(y_j) P_{n'i'}^{(m)}(x_1,x_2,y_3,\ldots,y_m),$$

$$V_{ni,n'i'}^{(m,m+2)} = \frac{g}{6} m\sqrt{(m+2)(m+1)} \int \left(\prod_j dy_j\right) \delta(1-\sum_j y_j)$$
$$\times \int dx_1 dx_2 dx_3 \delta(y_1-x_1-x_2-x_3) \left(\prod_{j=2}^m y_j\right) P_{ni}^{(m)}(y_j) P_{n'i'}^{(m+2)}(x_1,x_2,x_3,y_2,\ldots,y_m),$$

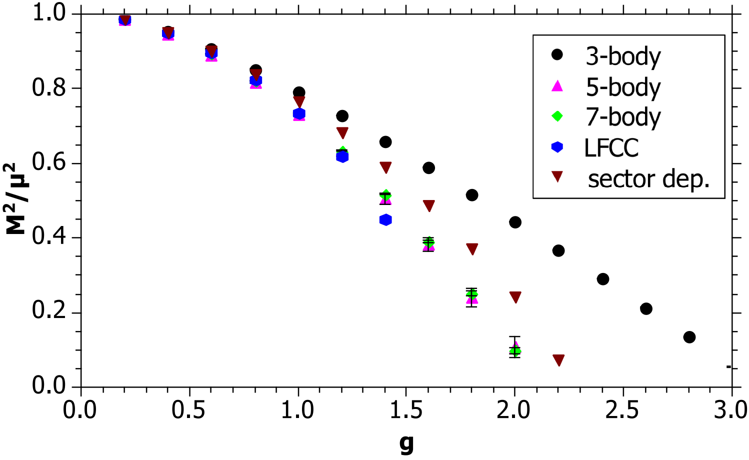
$$V_{ni,n'i'}^{(m,m-2)} = \frac{g}{6} (m-2)\sqrt{m(m-1)} \int \left(\prod_j dy_j\right) \delta(1-\sum_j y_j)$$
$$\times \left(\prod_{j=4}^m y_j\right) P_{ni}^{(m)}(y_j) P_{n'i'}^{(m-2)}(y_1+y_2+y_3,y_4,\ldots,y_m),$$

All of the integrals can be done analytically in terms of a generalized beta function.

Generalized eigenvalue problem

- need to solve problem of the form $H\vec{c} = (M^2/\mu^2)B\vec{c}$.
- standard approach: factorize B and convert to ordinary eigenvalue problem.
- factorization can fail in practice due to round-off errors
  - implicit orthogonalization of the basis
- reliable factorization is a singular-value decomposition (SVD) $B = UDU^T$
  - columns of the matrix U are the eigenvectors of B
  - D is a diagonal matrix of the eigenvalues of B
- solve $H'\vec{c}\,' = (M^2/\mu^2)\vec{c}\,'$
  - $H' = D^{-1/2}U^T HUD^{-1/2}$ and $\vec{c}\,' = D^{1/2}U^T\vec{c}$

Odd number of constituents



LFCC method: Phys Lett B **711**, 417 (2012)

To solve the light-front eigenvalue problem

$$\mathcal{P}^-|\psi\rangle = \frac{M^2+P_\perp^2}{P^+}|\psi\rangle$$

without making a Fock-space truncation,
build eigenstate as

$$|\psi\rangle = \sqrt{Z}e^T|\phi\rangle$$

from valence state $|\phi\rangle$
and operator T that increases particle number:

$$e^{-T}\mathcal{P}^-e^T|\phi\rangle = e^{-T}\frac{M^2+P_\perp^2}{P^+}e^T|\phi\rangle,$$

$\longrightarrow$ new effective Hamiltonian $\overline{\mathcal{P}^-} = e^{-T}\mathcal{P}^-e^T$.

LFCC equations

Eigenvalue problem becomes $\overline{\mathcal{P}^-}|\phi\rangle = \frac{M^2+P_\perp^2}{P^+}|\phi\rangle$

Project onto the valence and orthogonal sectors

$$P_v\overline{\mathcal{P}^-}|\phi\rangle = \frac{M^2+P_\perp^2}{P^+}|\phi\rangle, \qquad (1-P_v)\overline{\mathcal{P}^-}|\phi\rangle = 0.$$

Second (auxiliary) equation determines T.

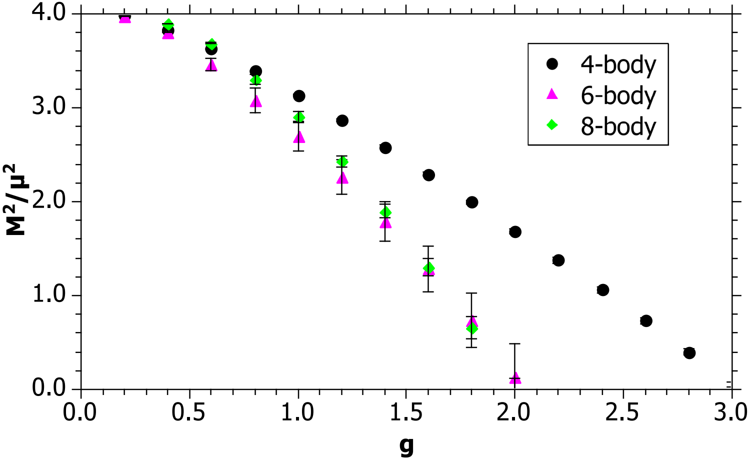
Construct effective Hamiltonian from Baker–Hausdorff expansion

$$\overline{\mathcal{P}^-} = \mathcal{P}^- + [\mathcal{P}^-, T] + \frac{1}{2}[[\mathcal{P}^-, T], T] + \dots$$

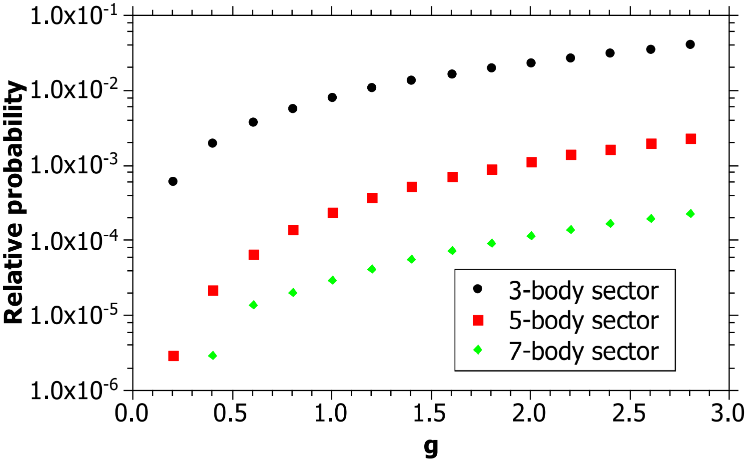
which can be terminated when the increase in particle number matches the truncation of the projection $1 - P_v$.

No spectator dependence and no uncanceled divergences!

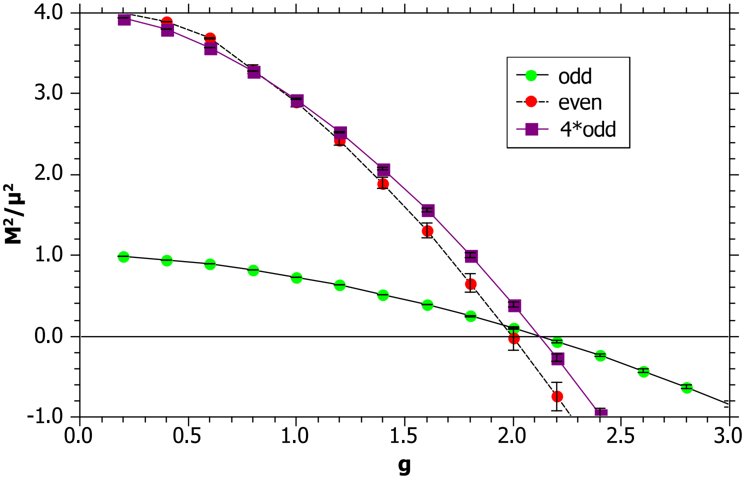
Even number of constituents



Odd-case Fock-sector probabilities



Estimate of critical coupling



Comparison of critical coupling values

From Rychkov and Vitale, PRD **91**, 085011 (2015), with $\bar{g} = \frac{\pi}{6} g$:

Method	$\bar{g}_c$	Reported by
LF Symmetric Polys	1.1	this talk
DLCQ	1.38	Harindranath & Vary
Quasi-sparse eigenvector	2.5	Lee & Salwen
Density matrix ren. gp	2.4954(4)	Sugihara
Lattice Monte Carlo	$2.70 \begin{cases} +0.025 \\ -0.013 \end{cases}$	Schaich & Loinaz
Uniform matrix product	2.766(5)	Milsted et al.
Ren. Hamiltonian truncation	2.97(14)	Rychkov & Vitale

Implies systematic difference between equal-time and light-front values.

Summary

- have developed a high-order method for 1+1 LF theories distinct from DLCQ
  - based on fully symmetric multivariate polynomials that respect the momentum conservation constraint
  - allows separate tuning of resolutions in each Fock sector
  - could be combined with transverse discretization or basis functions for 3+1 theories
- applied to ϕ^4_{1+1}
  - computed lowest mass eigenvalues
  - extracted estimate of critical coupling for $+\mu^2$
    - identified systematic difference with ET quantization
  - can immediately extend to $-\mu^2$
- compared lowest-order LFCC with high-order Fock space truncation
  - LFCC shows promise for rapid convergence

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