Two-dimensional massless light-front fields and solvable models

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ABSTRACT: Quantum field theory formulated in terms of light front (LF) variables has a few unusual features which have been sometimes interpreted as indicating its problematic nature. One of the apparent problems has been a lack of description of two-dimensional massless fields, in particular fermions, since one half of degrees of freedom seemed to vanish. The attempts to obtain a LF solution of e.g. the Schwinger model by quantizing at two characteristic surfaces did not produce acceptable results. Here we show a simple and natural way how the 2D massless fields (scalar, fermion) in the front form of the theory can be recovered as limits of massive fields and also consistently quantized without any loss of physical information. Bosonization of the fermion field then follows in a straightforward manner. This simple solution of the old problem opens also a road to a genuine LF understanding of solvable models. Here we present an operator solution and non-perturbative correlation functions of the LF Thirring and Thirring-Wess models. A few remarks concerning the LF Schwinger model and the LF conformal symmetry are also included.

I. INTRODUCTION

The light front (LF) form of field theory has been praised for its potential for decades

Distinguished features:

- minimal number (3) of dynamical Poincaré generators
- status of the vacuum state: Fock vacuum is (almost) the true ground state (lowest-energy eigenstate of the FULL Hamiltonian) – due to positivity and conservation of the LF momentum p^+
- consistent Fock expansion of the bound states, amplitudes with direct probabilistic interpretaion á la QM

- reduction of the number dynamical field variables, constrained components (this also technically complicates the theory)

Some doubts still present: how the LF scheme can cope with the vacuum structure, condensates, symmetry breaking...

Our approach: try to understand these aspects at the level of solvable models

Previous attempts did noy yield satisfactory results: Schwinger model – McCartor (1991, 1994) and McCartor, Pinsky and Robertson (1995) tried to introduce independent fermion variables quantized at $x^- = 0$

Theta vacuum constructed from a degenerate set of vacuum states built from the negative-momentum modes

Even this extension of the canonical LF quantization scheme did not reproduce the expected properties of the model (fermion condensate, massive Schwinger boson...)

Our motivation also: how are the two formulations (traditional - space-like (SL) and the LF) related

can the LF theory with its drastically simplified vacuum structure generate the same predictions as the SL form?

Solvable models: simple relativistic field theories in two-dimenional space-time in which explicit and non-approximate solutions of the field equations can be found on the quantum level

massless and massive model with derivative coupling, Rothe-Stamatescu, Thirring, Federbush, Thirring-Wess, Schwinger

Disadvantages: not realistic, "toy" models, mostly not gauge theories, 2-dimensional - extrapolation of the results to higher dimensions not guaranteed

Advantages: explicit solutions of the Heisenberg field equations, complete information on the NP dynamics, insights into the structure of QFT, vacuum properties under control

in the SL form, these models have been studied over decades using all kind of techniques

SURPRISINGLY, there are still some overlooked, unnoticed, neglected... aspects

- vacuum aspect: the Fock vacuum (no-particle state) often taken as the true ground state of the full (free + interacting) Hamiltonian, generally not correct

- choice of the right field variables - solutions of the field equations are composite operators given in terms of free fields $\Rightarrow$ the Hamiltonian (Lagrangian) has to be reexpressed in terms of the free fields also, similar to a treatment of a constraint

for a few solvable models, this step removed discrepancies between the SL and LF forms (the structure of Hamiltonians, e.g.)

- a consistent definition of quantum currents as local limits of nonlocal (point-split) *hermitean* products of composite fields (representing the interacting solutions)

the hermitian version of the point-split current automatically removes the divergent vacuum terms (cancellation), no subtractions done by hand needed

II. MASSLESS LIGHT FRONT FIELDS AND SOLVABLE MODELS

LF notation: $x^\mu = (x^+, x^-) = (x^0 + x^1, x^0 - x^1)$

the momentum k^μ

$$k^\mu = (k^+, k^-), \quad \partial_\pm = \frac{\partial}{\partial x^\pm}, \quad \hat{k} \cdot x = \frac{1}{2} k^+ x^- + \frac{1}{2} \hat{k}^- x^+, \quad k^2 = \mu^2 \Rightarrow \hat{k}^- = \frac{\mu^2}{k^+}. \tag{1}$$

$\hat{k}^-$ is the on-shell LF energy. No sign ambiguity analogous to $E(k^1) = \pm\sqrt{(k^1)^2 + \mu^2}$ of the conventional theory, both k^+, k^- can be taken positive. Quantization of the massless LF fields: start from the massive ones

II.1 Massless LF scalar field

The covariant Lagrangian density + the corresponding field equation

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2, \quad \left(\partial_\mu\partial^\mu + \mu^2\right)\phi(x) = 0, \tag{2}$$

takes in terms of the LF variables the form

$$\mathcal{L} = 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2, \quad \left(4\partial_+\partial_- + \mu^2\right)\phi(x) = 0. \tag{3}$$

The corresponding conjugate momentum and the field time derivative

$$\pi(x) = 2\partial_-\phi(x), \quad \partial_+\phi(x) = \frac{1}{4}\mu^2\partial_-^{-1}\phi(x). \tag{4}$$

∂_-^{-1} = inverse derivative. The quantum solution of the field equation (3)

$$\phi(x) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \left[a(k^+) e^{-\frac{i}{2}k^+ x^- - \frac{i}{2}\frac{\mu^2}{k^+}x^+} + a^\dagger(k^+) e^{\frac{i}{2}k^+ x^- + \frac{i}{2}\frac{\mu^2}{k^+}x^+}\right], \qquad (5)$$

with the Fock (creation and annihilation) operators satisfying

$$\left[a(k^+), a^\dagger(l^+)\right] = \delta(k^+ - l^+), \;\; a(k^+)|0\rangle = 0. \qquad (6)$$

REMARK: fields should be defined more carefully (smeared with test functions, more rigorous - work in progress (with P. Grange)).

Equivalently, the field commutation relation at equal LF time ($z^- = x^- -$

$y^-)$

$$\left[\phi(x^+,x^-),\phi(x^+,y^-)\right] = \int\limits_0^\infty \frac{dk^+}{4\pi k^+}\left[e^{-\frac{i}{2}k^+(z^- - i\epsilon^-)} - e^{\frac{i}{2}k^+(z^- + i\epsilon^-)}\right] \equiv -\frac{i}{4}\epsilon(z^-), \tag{7}$$

$\epsilon(z^-)$ is the sign function. From (5) we directly find

$$\theta(x) \equiv 2\partial_+\phi(x) = -i\int\limits_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}}\frac{\mu^2}{k^+}\left[a(k^+)e^{-i\hat{k}\cdot x} - a^\dagger(k^+)e^{i\hat{k}\cdot x}\right],$$

$$\pi(x) = -i\int\limits_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}}k^+\left[a(k^+)e^{i\hat{k}\cdot x} - a^\dagger(k^+)e^{i\hat{k}\cdot x}\right]. \tag{8}$$

Various two-point correlation functions from the three field operators above:

$$D_0^{(+)}(z) = \langle 0|\phi(x)\phi(y)|0\rangle, \tag{9}$$

$$D_1^{(+)}(z) = \langle 0|\phi(x)\pi(y)|0\rangle, \tag{10}$$

$$D_2^{(+)}(z) = \langle 0|\phi(x)\theta(y)|0\rangle, \tag{11}$$

$$D_i^{(+)}(z) = i\int\limits_0^\infty \frac{dk^+}{4\pi} f_i(k^+) e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(z^+ - i\epsilon^+)},$$

$$f_0(k^+) = -\frac{i}{k^+},\ f_1(k^+) = 1,\ f_2(k^+) = \frac{\mu^2}{k^{+2}}. \tag{12}$$

The small imaginary parts in time and space coordinates (implicitly present also in (5) and 8)) introduced in order that the integrals exist. The resultant exponential damping factors replace the role of test functions.

The integrals explicitly evaluated in terms of the (modified) Bessel functions $J_\nu(z), N_\nu(z), K_\nu(z), \nu = 0, 1$. As in the conventional SL theory, the first one logarithmically diverges for vanishing mass μ. The second is given by

$$
\begin{aligned}
D_1^{(+)}(z) = \quad & - \quad \theta\big(z^2\big)\frac{\mu}{4}\sqrt{\frac{z^+}{z^-}}i\Big[J_1\big(\mu\sqrt{z^2}\big) - i\ \mathrm{sgn}(z^+)N_1\big(\mu\sqrt{z^2}\big)\Big] + \\
& - \quad \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{\mu}{4\pi}\sqrt{-\frac{z^+}{z^-}}K_1\big(\mu\sqrt{-z^2}\big). \qquad (13)
\end{aligned}
$$

$D_2^{(+)}$ obtained from $D_1^{(+)}$ by the interchange $x^+ \leftrightarrow x^-$. The important observation: both $D_1^{(+)}$ and $D_2^{(+)}$ have a non-vanishing massless limit:

$$D_1^{(+)}(x-y;\mu^2=0)=\frac{1}{2\pi}\frac{1}{(x^- - y^- - i\epsilon^-)},$$
$$D_2^{(+)}(x-y;\mu^2=0)=\frac{1}{2\pi}\frac{1}{(x^+ - y^+ - i\epsilon^+)}. \qquad (14)$$

Technically, this is due to the behaviour of the Bessel function $K_1(z)\sim\frac{1}{z}$ for small value of z (the same is true for $N_1(z)$ in the timelike region), so that

$$\frac{\mu}{2\pi}\sqrt{-\frac{z^\mp}{z^\pm}}K_1\big(-\mu\sqrt{z^+z^-}\big) \qquad (15)$$

has the finite massless limit (Bergknoff 1977).

The results (14) suggest that there must exist massless analogs of the fields $\phi(x),\pi(x),\theta(x)$, whose correlation functions reproduce the above massless limits of the massive correlators.

From the LF massless Klein-Gordon equation

$$\partial_+ \partial_- \tilde{\phi}(x) = 0 : \tag{16}$$

one should expect a general solution of the form

$$\tilde{\phi}(x) = \tilde{\phi}(x^+) + \tilde{\phi}(x^-). \tag{17}$$

The mass dependence resides only in the plane-wave factor[1], the massless

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[1] The measure in the LF momentum integrals is, contrary to the space-like form of the theory, mass-independent (Leutwyler, Klauder and Streit 1972).

limit of the massive solution (5) yields (IR cutoff needed) $\tilde{\phi}(x^-)$:

$$\tilde{\phi}(x^-) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \left[\tilde{a}(k^+) e^{-\frac{i}{2}k^+ x^-} + \tilde{a}^\dagger(k^+) e^{\frac{i}{2}k^+ x^-}\right]. \tag{18}$$

But where is the second piece $\tilde{\phi}(x^+)$? Also contained in (5)! To show this, change the variables as (more correctly at the classical level) $k^+ = \frac{\mu^2}{k^-}$. One obtains

$$\phi(x) = \int_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} \left[\frac{\mu}{k^-} a\left(\frac{\mu^2}{k^-}\right) e^{-\frac{i}{2}\frac{\mu^2}{k^-}x^- - \frac{i}{2}k^- x^+} + \frac{\mu}{k^-} a^\dagger\left(\frac{\mu^2}{k^-}\right) e^{\frac{i}{2}\frac{\mu^2}{k^-}x^- + \frac{i}{2}k^- x^+}\right]. \tag{19}$$

The Fock commutators in terms of the new variables:

$$\Big[\frac{\mu}{k^-}a\big(\frac{\mu^2}{k^-}\big), \frac{\mu}{l^-}a^\dagger\big(\frac{\mu^2}{l^-}\big)\Big] = \frac{\mu^2}{k^-l^-}\delta\big(\frac{\mu^2}{k^-} - \frac{\mu^2}{l^-}\big) = \delta(k^- - l^-). \qquad (20)$$

Since the rhs does not depend on mass and hence survives the massless limit, this should be true for the lhs as well. Thus (classical level first)

$$\lim_{\mu \to 0} \frac{\mu}{k^-}a\big(\frac{\mu^2}{k^-}\big) \equiv \tilde{a}(k^-) \qquad (21)$$

is non-vanishing, with the properties

$$\big[\tilde{a}(k^-), \tilde{a}^\dagger(l^-)\big] = \delta(k^- - l^-), \quad \big[\tilde{a}(k^+), \tilde{a}^\dagger(l^-)\big] = 0. \qquad (22)$$

The massless limit in (19) and in $\pi(x), \theta(x)$ (8):

$$\tilde{\phi}(x^+) = \int\limits_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} \left[\tilde{a}(k^-)e^{-\frac{i}{2}k^- x^+} + \tilde{a}^\dagger(k^-)e^{\frac{i}{2}k^- x^+}\right], \tag{23}$$

$$\tilde{\theta}(x) = -i \int\limits_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} k^- \left[\tilde{a}(k^-)e^{-\frac{i}{2}k^- x^+} - \tilde{a}^\dagger(k^-)e^{\frac{i}{2}k^- x^+}\right], \tag{24}$$

$$\tilde{\pi}(x) = -i \int\limits_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} k^+ \left[\tilde{a}(k^+)e^{-\frac{i}{2}k^+ x^-} - \tilde{a}^\dagger(k^+)e^{\frac{i}{2}k^+ x^-}\right]. \tag{25}$$

The change of variables (19) performed for $\partial_+\phi(x)$ to make it compatible with the equation of motion (16) - to depend only on x^+ $(\partial_-\tilde{\theta} = 0)$.

The basic commutators, following from (6) and (22):

$$\left[\tilde{\phi}(x^-), \tilde{\phi}(y^-)\right] = -\frac{i}{4}\epsilon(x^- - y^-), \quad \left[\tilde{\phi}(x^+), \tilde{\phi}(y^+)\right] = -\frac{i}{4}\epsilon(x^+ - y^+). \tag{26}$$

Thus, the second half of the solution of the wave equation has been recovered from the massive solution.

The variables k^+ and k^- in fact coincide - analogous to the SL case where $k^0 = |k^1|$. In the LF case, we have directly $k^- = k^+$ since both are positive-definite.

Consistency check: the two-point functions calculated from the massless fields should coincide with the massless limit of the massive functions. Indeed the case for $D_1^{(+)}(z)$ and $D_2^{(+)}(z)$:

$$\tilde{D}_1^{(+)}(z) = \langle 0|\tilde{\phi}(x^-)\tilde{\pi}(y^-)|0\rangle = \frac{1}{2\pi}\frac{1}{(x^- - y^- - i\epsilon^-)}, \tag{27}$$

since the $\tilde{\phi}(x^+)$ term does not contribute (see Eq.(22)). The massive $D_0^{(+)}(z)$ can be evaluated for small mass μ, it diverges as lnμ.

Note: $D_0^{(+)}(z)$ calculated from the massless solution is ill defined (infinite) since μ has been already set to zero. Upon introducing the infrared cutoffs $\lambda^+ = \lambda^- \equiv \lambda$ in the corresponding integrals,

$$\tilde{D}_0^{(+)}(z) = \int\limits_{\lambda}^{\infty} \frac{dk^-}{4\pi k^-} e^{-\frac{i}{2}k^-(z^+ - i\epsilon^+)} + \int\limits_{\lambda}^{\infty} \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-)}, \qquad (28)$$

the same (regularized) $\ln \lambda$ divergent behaviour found (see below).

II.2 Massless light front fermion field.

The situation with fermions simpler since no infrared divergencies present. The *massive* field equation (the two-dimensional version of the Dirac equation):

$$i\gamma^{\mu}\partial_{\mu}\psi(x) = m\psi(x). \tag{29}$$

In the LF variables decomposes into a dynamical and a constraint equation:

$$2i\partial_{+}\psi_2(x) = m\psi_1(x), \quad 2i\partial_{-}\psi_1(x) = m\psi_2(x) \Rightarrow \psi_1(x) = \frac{m}{2i}\partial_{-}^{-1}\psi_2(x). \tag{30}$$

Chiral representation for the Dirac matrices: $\gamma^{\pm} = \gamma^0 \pm \gamma^1, \gamma^0 = \sigma^1, \gamma^1 = i\sigma^2, \gamma^5 = \gamma^0\gamma^1$, where σ^1, σ^2 are Pauli matrices. In the massless case, it follows from Eqs.(30) that

$$\psi_2(x) = \psi_2(x^-), \quad \psi_1(x) = \psi_1(x^+). \tag{31}$$

So ψ_1 is a zero mode (x^-−independent quantity) that seemingly needs to be quantized independently on the surface $x^- = 0$ (McCartor 1992,1995). This assumption however does not generate a consistent theoretical framework (two evolution parameters, negative momenta of new modes).

INSTEAD: Start again from the massive fields in the momentum representation and study its massless limit

The equations (30) solved by

$$\psi_2(x) = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} \mathrm{d}p^+ \left[b(p^+) e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} + d^\dagger(p^+) e^{\frac{i}{2}p^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+} \right], \tag{32}$$

$$\psi_1(x) = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} \mathrm{d}p^+ \frac{m}{p^+} \left[b(p^+) e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} - d^\dagger(p^+) e^{\frac{i}{2}p^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+} \right], \tag{33}$$

with the Fock operators obeying

$$\{b(p^+), b^\dagger(q^+)\} = \{d(p^+), d^\dagger(q^+)\} = \delta(p^+ - q^+). \tag{34}$$

NOTE: the massless limit of ψ_2 well defined, that of ψ_1 seems to vanish (?!)

The two-point functions $S^{(+)}_{\alpha\beta}(x-y) = \langle 0|\psi_\alpha(x)\psi^\dagger_\beta(y)|0\rangle, \alpha, \beta = 1, 2$, are expressed in terms of the scalar-field functions as

$$S^{(+)}_{22}(z) = -iD^{(+)}_1(z), \;\; S^{(+)}_{11}(z) = -iD^{(+)}_2(z), \;\; S^{(+)}_{12}(z) = mD^{(+)}_0(z) \tag{35}$$

with $\mu \to m$.

Due to the behaviour of the $K_1(z)$ and $N_1(z)$ functions for small z analogous to (15), the massless limit of the functions $S^{(+)}_{22}$ and $S^{(+)}_{11}$ is finite:

$$S_{22}^{(+)}(x-y;m=0)=\frac{1}{2i\pi}\frac{1}{(x^- - y^- - i\epsilon^-)}, \tag{36}$$

$$S_{11}^{(+)}(x-y;m=0)=\frac{1}{2i\pi}\frac{1}{(x^+ - y^+ - i\epsilon^+)}. \tag{37}$$

The mixed two-point function $S_{12}^{(+)}$ vanishes for $m=0$.

The result (36) easily obtained directly from the massless $\psi_2(x^-)$ (well defined simply by setting $m=0$ in the plane wave factors of Eq.(32)).

ALSO: Eq.(37) clearly indicates that there should exist a massless fermion-field component $\psi_1(x^+)$ whose 2-point function is given by (37). The only difference: the massless limit of the massive $\psi_1(x)$ (33) is due to its constrained nature more tricky than the massless limit of $\psi_2(x)$. Change of variables $p^- = \frac{m^2}{p^+}$ again required first to be able to perform the limit.

The anticommutators in terms of the new variables:

$$\Big\{\frac{m}{p^-}b\big(\frac{m^2}{p^-}\big),\frac{m}{q^-}b^\dagger\big(\frac{m^2}{q^-}\big)\Big\}=\frac{m^2}{p^-q^-}\delta\big(\frac{m^2}{p^-}-\frac{m^2}{q^-}\big)=\delta(p^- - q^-). \quad (38)$$

By the same reasoning as for the scalar field, we conclude that

$$\lim_{m\to 0}\frac{m}{p^-}b\big(\frac{m^2}{p^-}\big)\equiv\tilde{b}(p^-),\ \ \lim_{m\to 0}\frac{m}{p^-}d\big(\frac{m^2}{p^-}\big)\equiv\tilde{d}(p^-) \quad (39)$$

are non-vanishing.

Thus we have found

$$\tilde{\psi}_2(x^-)=\frac{1}{\sqrt{4\pi}}\int_0^{+\infty}dp^+\big[\tilde{b}(p^+)e^{-\frac{i}{2}p^+x^-}+\tilde{d}^\dagger(p^+)e^{\frac{i}{2}p^+x^-}\big], \quad (40)$$

$$\tilde{\psi}_1(x^+) = \frac{1}{\sqrt{4\pi}} \int\limits_0^{+\infty} dp^- \left[\tilde{b}(p^-) e^{-\frac{i}{2}p^- x^+} - \tilde{d}^\dagger(p^-) e^{\frac{i}{2}p^- x^+}\right], \quad (41)$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^-)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^-)\} = \delta(p^- - q^-), \quad (42)$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^+)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^+)\} = 0. \quad (43)$$

The expected form of the field anticommutators obtained:

$$\left\{\tilde{\psi_1}(x^+), \tilde{\psi_1}^\dagger(y^+)\right\} = \delta(x^+ - y^+), \quad \left\{\tilde{\psi_2}(x^-), \tilde{\psi_2}^\dagger(y^-)\right\} = \delta(x^- - y^-). \quad (44)$$

A SIMPLE AND CONSISTENT FRAMEWORK ESTABLISHED

No new variables have to be intoduced, the necessary information contained in the original massive solution.

The two-point function calculated from the massless $\tilde{\psi}_1(x^+)$ coincides with the massless limit (37) of the massive 2-point function.

The massless vector current $j^\mu = \overline{\psi}\gamma^\mu\psi$ from the massless fields (40,41):

$$j^+(x^-) = \lim_{\epsilon^- \to 0} \left[\tilde{\psi_2}^\dagger(x^- + \frac{\epsilon^-}{2})\tilde{\psi}_2(x^- - \frac{\epsilon^-}{2}) + H.c.\right] = 2 : \tilde{\psi}_2^\dagger(x^-)\tilde{\psi}_2(x^-) : \quad (45)$$

$$j^-(x^+) = \lim_{\epsilon^+ \to 0} \left[\tilde{\psi_1}^\dagger(x^+ + \frac{\epsilon^+}{2})\tilde{\psi}_1(x^+ - \frac{\epsilon^+}{2}) + H.c.\right] = 2 : \tilde{\psi}_1^\dagger(x^+)\tilde{\psi}_1(x^+) : \quad (46)$$

Solvable models are based on free Heisenberg fields $\Rightarrow$ the above derivation of the two-dimensional massless LF fermion fields opens the avenue for the genuine light-front solution of this class of models

II.3 LF bosonization.

A remarkable property of two-dimensional field theory: fermion fields can be represented in terms of boson variables (Coleman, Mandelstam,..).

Our quantization of the massless LF scalar and fermion fields: formulate the bosonization property in a genuine LF form. Since the massless $\phi(x)$ and $\psi(x)$ fields decompose as

$$\phi(x) = \phi(x^+) + \phi(x^-), \, \psi^T(x) = (\psi_1(x^+), \psi_2(x^-))$$

(we omit tilde for the massless fields henceforth), the demonstration of bosonization is very simple.

Start with $\psi_2(x^-)$. Assume that it can be represented as

$$\varphi_2(x^-) = C : e^{i\alpha\phi(x^-)} : \; = C e^{i\alpha\phi^{(-)}(x^-)} e^{i\alpha\phi^{(+)}(x^-)}. \tag{47}$$

Adjust the constants C and α plus use the properties of $D_0^{(+)}$. Adjust in such a way, that two φ_2 with different arguments anticommute and $\varphi_2(x^-)$, $\varphi_2^\dagger(y^-)$ satisfy the anticommutation relation (44). The first condition fixes α to the value $\alpha = 2\sqrt{\pi}$. Form the product $\varphi_2(x^-)\varphi_2(y^-)$ and performs the necessary commutations to obtain the opposite order of the operators. This generates the expression

$$\varphi_2(x^-)\varphi_2(y^-) = e^{-\alpha^2\left(D_0^{(+)}(x^- - y^-) - D_0^{(+)}(y^- - x^-)\right)}\varphi_2(y^-)\varphi_2(x^-). \qquad (48)$$

The two commutator functions $D_0^{(+)}(\pm(x^- - y^-))$, where

$$D_0^{(+)}(x^- - y^-) = \left[\phi^{(+)}(x^-), \phi^{(-)}(y^-)\right] = \int_0^\infty \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(x^- - y^- - i\epsilon^-)}, \qquad (49)$$

individually diverge, but upon introducing the infrared cutoff λ (cf. Eq.(28))

the divergent parts cancel in (48) producing the sign function $\epsilon(x^- - y^-)$. With $\alpha = 2\sqrt{\pi}$, the net result is $e^{i\pi\epsilon(x^- - y^-)} = -1$ for all x^-, y^-.
This is the required anticommutativity.

To prove the the second property, form the anticommutator

$$A(x^-, y^-) \equiv \varphi_2(x^-)\varphi_2^\dagger(y^-) + \varphi_2^\dagger(y^-)\varphi_2(x^-) =$$

$$C^2\left[e^{4\pi D_0^{(+)}(x^- - y^-)} : \varphi_2(x^-)\varphi_2^\dagger(y^-) : +e^{4\pi D_0^{(+)}(y^- - x^-)} : \varphi_2^\dagger(y^-)\varphi_2(x^-) : \right] \quad (50)$$

Taking into account the explicit form of the infrared-regularized $D_0^{(+)}$ function

$$D_0^{(+)}(z^-) = -\frac{1}{4\pi}\ln\left[\frac{\lambda}{2}e^{\gamma_E}(iz^- + \epsilon^-)\right], \quad z^- = x^- - y^-, \quad (51)$$

and the fact that two normal-ordered expressions in (50) actually coincide, we find

$$A(x^-, y^-) = \frac{2}{i\lambda e^{\gamma_E}}\Big[\frac{1}{z^- - i\epsilon^-} - \frac{1}{z^- + i\epsilon^-}\Big] : \varphi_2(x^-)\varphi_2^\dagger(y^-) := \frac{4\pi}{\lambda e^{\gamma_E}}\delta(x^- - y^-). \tag{52}$$

Used: the term in the square bracket is equal to $2i\pi\delta(z^-)$. The operator part on the rhs has reduced to unity due to the presence of this delta-function. It follows that the rescaled operator

$$\hat{\varphi}_2(x^-) = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}} e^{i2\sqrt{\pi}\phi^{(-)}(x^-)} e^{i2\sqrt{\pi}\phi^{(+)}(x^-)} \tag{53}$$

obeys the correct anticommutation relation (44) and represents the bosonized form of the fermion field $\psi_2(x^-)$. The construction of the second component $\varphi_1(x^+)$ is completely paralel, with $x^- \to x^+$, etc. Thus

$$\hat{\varphi}_1(x^+) = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}} e^{i2\sqrt{\pi}\phi^{(-)}(x^+)} e^{i2\sqrt{\pi}\phi^{(+)}(x^+)}. \tag{54}$$

The vector current in the bosonic form: Inserting the bosonic form (53) into (45), one gets a product of four exponential operators, which is NOT in the normal order. Commute the two middle terms to normal-order the expression, one generates a term $e^{\alpha^2 D_0^{(+)}(\epsilon^-)}$, which according to (51) behaves as $1/\epsilon^-$. This singularity is canceled by the terms from the exponential, linear in ϵ^-. No vacuum subtractions are needed, since the second (conjugate) term in (45) cancels them automatically. The net result

$$j^+(x^-) = \frac{2}{\sqrt{\pi}} \partial_- \phi(x^-), \quad j^-(x^+) = \frac{2}{\sqrt{\pi}} \partial_+ \phi(x^+). \tag{55}$$

The second current component was obtained in a completely analogous way. The boson representation correctly reproduces the Schwinger term in both current-current commutators:

$$\left[j^+(x^-), j^+(y^-)\right] = \frac{i}{\pi}\partial_x \delta(x^- - y^-), \quad \left[j^-(x^+), j^-(y^+)\right] = \frac{i}{\pi}\partial_x \delta(x^+ - y^+). \tag{56}$$

Similarly, for the scalar densities (no singularities - no splitting)

$$\overline{\psi}(x)\psi(x) = \psi_1^\dagger(x)\psi_2(x) + \psi_2^\dagger(x)\psi_1(x) = \tilde{\psi_1}^\dagger(x^+)\psi_2(x^-) + H.c., \tag{57}$$

$$\overline{\psi}(x)\gamma^5\psi(x) = \psi_1^\dagger(x)\psi_2(x) - \psi_2^\dagger(x)\psi_1(x) = \tilde{\psi_1}^\dagger(x^+)\psi_2(x^-) - H.c., \tag{58}$$

one obtains ($\phi(x) = \phi(x^+) + \phi(x^-)$)

$$\overline{\psi}(x)\psi(x) = \frac{\lambda e^{\gamma_E}}{4\pi}\cos\left(2\sqrt{\pi}\phi(x)\right), \quad \overline{\psi}(x)\gamma^5\psi(x) = i\frac{\lambda e^{\gamma_E}}{4\pi}\sin\left(2\sqrt{\pi}\phi(x)\right). \tag{59}$$

CONCLUSION: the LF version of bosonization yields the results known from the SL theory, but in a simpler and more transparent form.

On the other hand, **certain care** required: the current has to be defined in a mathematically correct manner. Otherwise, the relations (55) would not follow since the bosonized current would be manifestly ill-defined (singular).

The above construction enables one to study the bosonized LF Thirring model as well as the LF version of the sine-Gordon - massive Thirring model correspondence (Coleman 1976, Mandelstam 1976).

III. THE THIRRING MODEL

no truly LF solution available, a hybrid formulation by Frishman, Del' Antonio and Zwanziger (1972)

HERE: a genuine LF solution sketched, treatment paralel to the SL one (bosonization of the free current, explicit solution of the field equation), simplifications (no Bogoliubov transformation, physical vacuum = Fock vacuum)

Classical Lagrangian density of the model and the corresponding Euler-Lagrange equation are

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overleftrightarrow{\partial}_{\mu} \Psi + \frac{1}{2} g J_{\mu} J^{\mu}, \quad i\gamma^{\mu}\partial_{\mu}\Psi = -g\gamma_{\mu}J^{\mu}, \quad J^{\mu}(x) = \overline{\Psi}(x)\gamma^{\mu}\Psi(x). \tag{60}$$

The LF field equation decomposes into two equations:

$$2i\partial_+\Psi_2 = -gJ^-\Psi_2,\;\; 2i\partial_-\Psi_1 = -gJ^+\Psi_1, \quad J^+ = 2\Psi_2^\dagger\Psi_2,\;\; J^- = 2\Psi_1^\dagger\Psi_1. \tag{61}$$

Introducing the potentials $J(x)$ and $J(x^\pm)$ (specified in detail below), where $J(x) = -(J(x^+) + J(x^-))$, by $J^+(x^-) = -\frac{2}{\sqrt{\pi}}\partial_- J(x)$, $J^-(x^+) = -\frac{2}{\sqrt{\pi}}\partial_+ J(x)$, the Eqs.(61) are solved by

$$\Psi_1(x) = e^{i\frac{g}{\sqrt{\pi}}J(x)}\psi_1(x^+),\;\; \Psi_2(x) = e^{i\frac{g}{\sqrt{\pi}}J(x)}\psi_2(x^-). \tag{62}$$

Quantum solution:

$$\Psi_1(x) = Z^{-1/2}(\epsilon)e^{i\frac{g}{\sqrt{\pi}}J^{(-)}(x+\frac{\epsilon}{2})+i\frac{g}{\sqrt{\pi}}J^{(+)}(x-\frac{\epsilon}{2})}\psi_1(x^+) = \tag{63}$$

$$= e^{i\frac{g}{\sqrt{\pi}}J^{(-)}(x+\frac{\epsilon}{2})}e^{i\frac{g}{\sqrt{\pi}}J^{(+)}(x-\frac{\epsilon}{2})}\psi_1(x^+), \tag{64}$$

$$\Psi_2(x) = Z^{-1/2}(\epsilon) e^{i\frac{g}{\sqrt{\pi}} J^{(-)}(x+\frac{\epsilon}{2}) + i\frac{g}{\sqrt{\pi}} J^{(+)}(x-\frac{\epsilon}{2})} \psi_2(x^+) = \quad (65)$$

$$= e^{i\frac{g}{\sqrt{\pi}} J^{(-)}(x+\frac{\epsilon}{2})} e^{i\frac{g}{\sqrt{\pi}} J^{(+)}(x-\frac{\epsilon}{2})} \psi_2(x^+), \quad (66)$$

where the renormalization constant is

$$Z(\epsilon) = [J^{(-)}(x+\frac{\epsilon}{2}), J^{(+)}(x-\frac{\epsilon}{2})] = -G^2(g)\big(D_0^{(+)}(-\epsilon^+) + D_0^{(+)}(-\epsilon^-)\big),$$

$$D_0^{(+)}(z^\pm) = \int\limits_{\lambda}^{\infty} \frac{dk^\mp}{4\pi k^\mp} e^{-\frac{i}{2}k^\mp(z^\pm - i\delta^\mp)} = -\frac{1}{4\pi} \ln\big[\frac{\lambda}{2} e^{\gamma_E}(iz^\pm + \delta^\pm)\big]. \quad (67)$$

The quantum mode expansion of the free massless fermion field is (the tilde for $\tilde{b}, \tilde{d}$ ommited)

$$\psi_2(x^-) = \int_0^\infty \frac{dp^+}{\sqrt{4\pi}} \left[b(p^+) e^{-\frac{i}{2}p^+x^-} + d^\dagger(p^+) e^{\frac{i}{2}p^+x^-} \right],$$

$$\{b(p^+), b^\dagger(q^+)\} = \{b \to d\} = \delta(p^+ - q^+),$$

$$\psi_1(x^+) = \int_0^\infty \frac{dp^-}{\sqrt{4\pi}} \left[b(p^-) e^{-\frac{i}{2}p^-x^+} - d^\dagger(p^-) e^{\frac{i}{2}p^-x^+} \right],$$

$$\{b(p^-), b^\dagger(q^-)\} = \{b \to d\} = \delta(p^- - q^-). \tag{68}$$

The interacting currents calculated from the solutions (62) by means of the (hermitean) point-splitting. Using

$$\psi_2^\dagger(x^- + \epsilon^-/2)\psi_2(x^- - \epsilon^-/2) =: \psi_2^\dagger(x^-)\psi_2(x^-) : +V(\epsilon^-),$$

$$\psi_1^\dagger(x^+ + \epsilon^+/2)\psi_1(x^+ - \epsilon^+/2) =: \psi_1^\dagger(x^+)\psi_1(x^+) : +V(\epsilon^+),$$

$$V(\epsilon^\pm) = \frac{1}{4\pi}\int_0^\infty \mathrm{d}p^- e^{-\frac{i}{2}p^\mp(\epsilon^\pm - i\eta)} = -\frac{i}{2\pi}\frac{1}{\epsilon^\pm - i\eta}, \tag{69}$$

one obtains

$$J^+(x) = G(g)j^+(x^-), \;\; J^-(x) = G(g)j^-(x^+), \;\; G(g) = \left(1 - \frac{g}{2\pi}\right)^{-1}, \tag{70}$$

i.e. the interacting vector current is a ”renormalized” free current.

Convenient to bosonize the currents by a Fourier transformation:

$$j^+(x^-) = -\frac{i}{\sqrt{\pi}}\int_0^\infty \frac{dk^+\; k^+}{\sqrt{4\pi k^+}}\left[c(k^+)e^{-\frac{i}{2}k^+x^-} - H.c.\right],$$

$$c(k^+) = \frac{i\hat{c}(k^+)}{2\sqrt{k^+}}, \ \left[c(k^+), c^\dagger(l^+)\right] = \delta(k^+ - l^+), \tag{71}$$

$$\hat{c}(k^+) = \int\limits_0^\infty ds^+ \left[b^\dagger(s^+)b(s^+ + k^+) - (b \to d) + d(p^+)b(k^+ - p^+)\theta(k^+ - s^+)\right].$$

Similarly

$$j^-(x^+) = -\frac{i}{\sqrt{\pi}} \int\limits_0^\infty \frac{dk^- \ k^-}{\sqrt{4\pi k^-}} \left[a(k^-)e^{-\frac{i}{2}k^- x^+} - H.c.\right],$$

$$a(k^-) = \frac{i\hat{a}(k^-)}{2\sqrt{k^-}}, \ \left[a(k^-), a^\dagger(l^-)\right] = \delta(k^- - l^-), \tag{72}$$

$$\hat{a}(k^-) = \int_0^\infty ds^- \left[b^\dagger(s^-)b(s^- + k^-) - (b \to d) + d(p^-)b(k^- - p^-)\theta(k^- - s^-)\right].$$

With an implicit infrared regularization, the potentials (”integrated currents”) are represented as

$$j(x^-) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \left[c(k^+)e^{-\frac{i}{2}k^+x^-} + c^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\right],$$
$$j(x^+) = \int_0^\infty \frac{dk^-}{\sqrt{4\pi k^-}} \left[a(k^-)e^{-\frac{i}{2}k^-x^+} + H.c.\right], \tag{73}$$

i.e. they are simply two components of a massless LF scalar field $j(x)$, which is related to the interacting potential by $J(x) = G(g)j(x)$.

Express now the Lagrangian and Hamiltonian in terms of independent fields, using the solution (62):

$$\mathcal{L} = i\Psi_2 \overleftrightarrow{\partial_+} \Psi_2 + i\Psi_1 \overleftrightarrow{\partial_-} \Psi_1 + \frac{1}{2} g J^+ J^- \to \mathcal{L} = i\psi_2 \overleftrightarrow{\partial_+} \psi_2 + i\psi_1 \overleftrightarrow{\partial_-} \psi_1 - \frac{1}{2} g J^+ J^- . \tag{74}$$

The Hamiltonian derived from the above Lagrangian

$$P^- = \frac{1}{2} \int\limits_{-\infty}^{+\infty} dx^- T^{+-}(x) = \frac{1}{2} g \int\limits_{-\infty}^{+\infty} dx^- J^+(x^-) J^-(0) \tag{75}$$

acquires a very simple form

$$P^- = g G^2(g) \tilde{Q} Q = \frac{g}{(1 - \frac{g}{2\pi})^2} \int\limits_0^\infty \frac{dp^-}{4\pi} \left[\hat{a}(p^-) + \hat{a}^\dagger(p^-)\right] \hat{c}(0), \tag{76}$$

where

$$Q = \hat{c}(0) = \int_0^\infty \frac{\mathrm{d}p^+}{2\sqrt{2\pi}} \left[b^\dagger(p^+) b(p^+) - d^\dagger(p^+) d(p^+) \right],$$

$$\tilde{Q} = \frac{1}{4\pi} \int_0^\infty \mathrm{d}p^- \left[\hat{a}(p^-) + \hat{a}^\dagger(p^-) \right] \tag{77}$$

and $\hat{a}(p^-)$ is given by the Eq.(73). The form of the Hamiltonian seems to indicate absence of asymptotic states. This is in agreement with the conclusions of the important paper by K. Johnson.

NEXT: calculate n-point functions, their Fourier transform will give information about particle spectrum

Two-point function (a matrix) $\langle vac|\Psi(x)\overline{\Psi}(y)|vac\rangle$:

We can consider $\langle 0|\Psi_1(x)\overline{\Psi}(y)|0\rangle$i. After canceling the infinite (regularized) constant, one Fourier transform: $\frac{p^+}{\lambda}$ terms appear. No poles $\Rightarrow$ no asymptotic states

Higher correlation functions...

IV. THIRRING-WESS MODEL

The classical Lagrangian density

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - \frac{1}{4}\tilde{G}^{\mu\nu}\tilde{G}_{\mu\nu} + \frac{1}{2}\mu_0^2 \tilde{B}^\mu \tilde{B}_\mu - g J_\mu \tilde{B}^\mu \tag{78}$$

A system of coupled field equation (Dirac + Proca) can be solved exactly. The axial current has an anomaly. The vector current is equak to a free current plus a quantum correction (finite mass renormalization). Hamiltonian in terms of the independent (free) fields B^+, ψ. NP correlation function from the exact solution...

REMARKS:

- LF SCHwinger model: analogous solution possible in a covariant gauge, need to understand quantization of 2D LF gauge field
- conformal symmetry (CS): a tool to restrict form of correlation functions in the SL theory, LF CS works differently - initial study (P. Grange)

V. SUMMARY AND OUTLOOK

- solvable models – good laboratory for studying subtleties of QFT, the vacuum problem and comparison between the SL and LF forms of the relativistic theory

- importance of the correct choice of the field variables and form of the Hamiltonians

- massless light-front fields can be obtained as the limits of the massive fields, consistent quantization, full physical information available

- massless LF bosonization simple and transparent, one has to be careful about the correct mathematical treatment

- massless Thirring model quantized in terms of LF variables, operator solution at quantum level, Hamiltonian, vacuum structure trivial, correlation function non-perturbatively, no free asymptotic states, more complicated coupling-constant dependence than known (Klaiber's) solution

- analogous study of the Thirring-Wess and the Schwinger model underway

- THE GENERAL MESSAGE: LF theory is a healthy scheme, no failures, we need to be careful and creative