Systems of differential equations for Feynman Integrals; Schouten identities and canonical bases.

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Bologna, 18 Novembre 2014

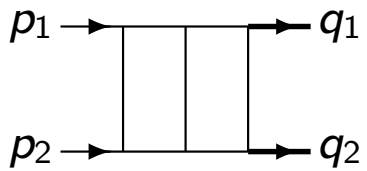
Based on collaboration with *Thomas Gehrmann*, *Andreas von Manteuffel*, *Pierpaolo Mastrolia*, *Ettore Remiddi*,...

Introduction

- Present understanding of the physical world through **Standard Model** of particle physics (SM).
- Most of the predictions that we can extract from the SM are based on **perturbative calculations** (as in most fields of physics!).
- Physical observables are computed perturbatively by **Expansion in Feynman Diagrams**.
- Computation of Feynman Diagrams is of *fundamental importance* for getting precise predictions from the **Standard Model**!

- Any Feynman Diagrams is (*after some tedious but elementary algebra!*) nothing but a collection of scalar **Feynman Integrals**

$\mathcal{I}(p_1, p_2, q_1) =$



with $\quad q_2 = p_1 + p_2 - q_1$

A (*possible*) representation in momentum space (*massless* case!)

$$\mathcal{I}(p_1, p_2, q_1) = \int \frac{\mathcal{D}^d\, k\, \mathcal{D}^d\, l}{k^2\, l^2\, (k-l)^2\, (k-p_1)^2\, (k-p_{12})^2\, (l-p_{12})^2\, (l-q_1)^2}$$

Typical **2-loop** Feynman Integral required for the computation of a $2 \to 2$ scattering process.

We will discuss how such integrals can be (*tentatively!*) computed **analytically**.

1. Integrals are ill-defined in $d = 4 \rightarrow$ need a **regularization procedure**!
2. Use of dimensional regularization to regulate **UV** and **IR divergences**.
3. Dimensional regularisation turned out to be much MORE than just a **regularization scheme**!

$$\Downarrow$$

Dimensionally regularized Feynman integrals always converge!

This allows to derive a large number of **unexpected relations...**

- Integration by Parts, Lorentz invariance identities, Schouten Identities,...
  $\rightarrow$ **differential equations!**

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This large set of identities makes it *simpler* to compute **Feynman integrals** in d continuous dimensions than in $d = 4$!

A general **scalar** Feynman Integral (l-loops) can be written as

$$\mathcal{I}(\sigma_1, ..., \sigma_s; \alpha_1, ..., \alpha_n) = \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{S_1^{\sigma_1} ... S_s^{\sigma_s}}{D_1^{\alpha_1} ... D_n^{\alpha_n}}$$

where

$$D_n = (q_n^2 + m_n^2)\,, \qquad \text{are the } \textbf{propagators}$$

$$S_n = k_i \cdot p_j\,, \quad \text{are } \textbf{scalar products} \text{ among internal and external momenta}$$

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1. **Integration By Parts Identities** (IBPs)

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^\mu}\, v_\mu\, \frac{S_1^{\sigma_1} \dots S_s^{\sigma_s}}{D_1^{\alpha_1} \dots D_n^{\alpha_n}} \right) = 0\,, \qquad v^\mu = k_j^\mu, p_k^\mu$$

2. They generate <u>huge systems of linear equations</u> which relates integrals with **different powers** of numerators and denominators.

3. The integrals always belong to the same **topology**, as defined above.

Inverting the IBPs we can express all Feynman Integrals in terms of a subset of basic integrals called **Master Integrals** (MIs).

- This was originally done with *generic "symbolic"* powers of denominators $\rightarrow$ becomes soon **prohibitive**.
- Can be automatised for *numerical* powers of denominators by the Laporta Algorithm (S. Laporta).

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In typical applications to 2-loop $2 \rightarrow 2$ processes one has $\approx$ **thousands** of scalar integrals to compute, which are reduced in this way to $\approx$ **tens** of MIs.

The are very powerful **public** codes to do this automatically:

Reduze2, FIRE5, AIR,...

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Important Remark: The choice of the BASIS of MIs (which scalar integrals are chosen to be MIs in the Laporta Algorithm!) is **ambiguous!**

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Choosing the **basis of MIs** in a "stupid" way, may make the intermediate expressions **blow up considerably**!

1. What does it mean choosing the basis properly?
2. How can we know a priori which basis is **good** and which one is **bad**?

Physical cuts $\rightarrow$ NO spurious non-physical singularities in the game!

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Dimensionally regularised Feynman Integrals fulfil differential equations!

[Kotikov, Remiddi, Gehrmann,...]

Let us take a Feynman integral which depends on two external invariants

$$p^2\,,\ m^2 \quad \to x = \frac{p^2}{m^2}\,.$$

$$\mathcal{I}(p^2, m^2) = \int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d}\, \frac{1}{D_1 \ldots D_n}\,, \qquad \text{(with scalar products generalisation trivial)}$$

Consider now the integral with generic powers of the denominators

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Suppose that, by exploiting the IBPs identities above, one can prove that **any integral** can be expressed as a linear combination of say N MIs.

$$m_1(p^2; d)\,, \quad m_2(p^2; d)\,, \; \ldots\,, \quad m_N(p^2; d)\,.$$

Such that **any integral** can be written

$$\mathcal{I}(p^2, m^2; \alpha_1\,, \ldots, \alpha_n) = \sum_{j=1}^{N} C_j(p^2; d)\, m_j(p^2; d)\,.$$

1. $C_j(p^2; d)$ are rational functions which dependent only on p^2 and d .

This information can be exploited to derived **differential equations** for the m_j.

1. **Differentiation** w.r.t to an external invariant $\rightarrow$ differentiation w.r.t to **external momenta**

$$p^2 = p_\mu \, p^\mu \qquad \rightarrow \qquad \frac{\partial}{\partial \, p^2} = \frac{1}{2 \, p^2} \left(p^\mu \frac{\partial}{\partial p^\mu} \right)$$

2. Applying differential operator on Feynman integral (say on m_1) we get a linear combination of new integrals with the **same propagators** but raised to **different powers**.

3. These new integrals can be in turn reduced to the same MIs!

$$\frac{\partial}{\partial \, p^2} m_1(p^2; d) = \sum_{j=1}^{N} c_{1j}(p^2; d) \, m_j(p^2; d) \, .$$

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This is a **differential equation** for $m_1(p^2; d)$!!!

Repeating the procedure for **all N MIs** one get a **system of linear differential equations** in the external invariant p^2

$$\frac{\partial}{\partial\, p^2}\begin{pmatrix} m_1 \\ \dots \\ m_N \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1N} \\ \dots & \dots & \dots \\ c_{N1} & \dots & c_{NN} \end{pmatrix}\begin{pmatrix} m_1 \\ \dots \\ m_N \end{pmatrix}$$

where the $c_{ij}(p^2; d)$ are only **rational functions** of p^2 and of d.

- Supplemented with N **boundary conditions**, these equations allow, in principle, to compute all master integrals!

Is this really easier than attempting a **direct integration**?

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Makes the **expansion** in $d \to 4$ (or any other integer dimension) **trivial**...

What if **N = 1**? (*There is only 1 MI!*)

If there is only 1 master integral the situation is in principle trivial:

$$\frac{\partial}{\partial\, p^2} m(p^2; d) = c(p^2; d)\, m(p^2; d)$$

Expanding in $d \to 4$ (we can rescale to make everything finite!)

$$m(p^2; d) = m^{(0)}(p^2) + (d-4)\, m^{(1)}(p^2) + \ldots$$
$$c(p^2; d) = c^{(0)}(p^2) + (d-4)\, c^{(1)}(p^2) + \ldots$$

$$\frac{\partial}{\partial\, p^2} m^{(0)}(p^2) = c^{(0)}(p^2)\, m^{(0)}(p^2)\,,$$
$$\frac{\partial}{\partial\, p^2} m^{(1)}(p^2) = c^{(0)}(p^2)\, m^{(1)}(p^2) + c^{(1)}(p^2)\, m^{(0)}(p^2)\,,$$

Chained system of differential equations which can be solved by quadrature order-by-order in $(d-4)$.

What if **N** $>$ **1**? (*Life is not that easy anymore!*)

If the system is coupled, it corresponds to a N-**th order** differential equation for any of the MIs. No general strategy for a solution is known.

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Observations

1. We are **free** of choosing our **basis of MIs**!
2. We are interested in the expansion for $d \to 4$!

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Changing the basis can **simplify** the structure of the differential equations!

At least in the limit for $d \to 4$!

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At least in the limit for $d \to 4$!

We can very often find a basis of MIs where the equations become

triangular as $d \to 4$.

$$\frac{\partial}{\partial p^2}\begin{pmatrix} m_1 \\ \dots \\ m_N \end{pmatrix} = \begin{pmatrix} c_{11}^{(0)} & c_{12}^{(0)} & \dots & c_{1N}^{(0)} \\ 0 & c_{22}^{(0)} & \dots & c_{2N}^{(0)} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_{NN}^{(0)} \end{pmatrix}\begin{pmatrix} m_1 \\ \dots \\ m_N \end{pmatrix} + \mathcal{O}(d-4)$$

This reduces, *in principle*, the problem to the case $N = 1$

$$\Downarrow$$

In order to obtain expansion in $(d-4)$ we must perform **many repeated integrations by quadrature**!

We can do better $\rightarrow$ **Canonical Form by J. Henn**

Suppose we are able to find a **basis of Master Intergrals** such that the system of differential equations takes the following form:

$$\frac{\partial}{\partial p^2}\begin{pmatrix} m_1 \\ \dots \\ m_N \end{pmatrix} = (d-4)\begin{pmatrix} c_{11}(p^2) & \dots & c_{1N}(p^2) \\ c_{21}(p^2) & \dots & c_{2N}(p^2) \\ \dots & \dots & \dots \\ c_{N1}(p^2) & \dots & c_{NN}(p^2) \end{pmatrix}\begin{pmatrix} m_1 \\ \dots \\ m_N \end{pmatrix}$$

So that <u>the dependence from the kinematics is **factorised** from d!</u>

Plus, the functions $c_{jk}(p^2)$ must be in d-log form, i.e.

$$\int^{p^2} dt\, c_{jk}(t) \propto \ln\left(f(p^2)\right)$$

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If the differential equations are in **canonical form**

$$\partial_x f(d,x) = (d-4)\, A(x)\, f(d,x)\,, \quad \text{with} \quad A(x) \ \text{in} \ d\,log \ \text{form}$$

$$\rightarrow d\, f(d,x) = (d-4)\, d\tilde{A}(x)\, f(d,x)$$

- We can trivially integrate the equations **order-by-order in** $(d-4)$
- Order-by-order the result is expressed as multiple polylogarithms only
  $\longrightarrow$ **NO rational factors!**
- Order-by-order the result has **uniform degree of transcendentality**

$$f(d,x) = 1 + (d-4)\ln(x) + (d-4)^2\left(\frac{\pi^2}{6} - Li_2(x) + \frac{1}{2}\ln^2(x)\right) + \ldots$$

Formal solution as **path-ordered exponential**

$$f = P\, \mathrm{e}^{(d-4)\int_C d\tilde{A}} f_0 \quad \rightarrow \quad f_0 \text{ boundary condition}$$

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This goes back to original question: how do we define a **good basis** of MIs?

1. It was observed that if differential equations are in **canonical form**, then MIs have only **physical cuts**! (no *proof* here but *common sense*!)

2. So the **canonical basis** really seems to be the right thing to do!

   - Makes **straightforward** the solution of the **differential equations**!
   - Makes **more compact** the expression of any integral in terms of MIs!

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   - They do not introduce any **spurious singularities**!

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Guiding criteria to choose a **canonical basis** by **J.Henn**

- Look at the **Maximal Cut**
- Check the **Leading singularity**
- **Direct inspection** through representation in Feynman parameters.

Magnus Expansion [Argeri, Vita, Mastrolia, Mirabella, Schlenk, Schubert, L.T.]

- Consider a **linear system of differential equations**

$$\partial_x f(d,x) = A(d,x)\, f(d,x)\,, \quad \rightarrow \quad \text{f(d,x) is the vector of MIs}$$

- Assume that $A(d,x)$ is **linear** in d. The system can be therefore written as

$$\partial_x f(d,x) = A_0(x)\, f(d,x) + (d-4)\, A_1(x)\, f(d,x)\,,$$

- **Change** then **basis of MIs** as

$$f(d,x) = B_0(x)\, g(d,x)\,, \quad \text{such that} \quad B_0(x) = e^{\Omega[A_0](x)}$$

where $\Omega[A_0](x)$ is defined through the **Magnus expansion**:

$$\Omega[A_0](x) = \sum_{n=1}^{\infty} \Omega_n[A_0](x)$$

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The different terms can be obtained from $A_0(x)$ as:

1. $$\Omega_1[A_0](x) = \int_{x_0}^{x} d\tau_1\, A_0(\tau_1)\,,$$

2. $$\Omega_2[A_0](x) = \frac{1}{2}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2\, [A_0(\tau_1), A_0(\tau_2)]\,,$$

3. $$\Omega_3[A_0](x) = \frac{1}{6}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \times \left([A_0(\tau_1), [A_0(\tau_2), A_0(\tau_3)]] + [A_0(\tau_3), [A_0(\tau_2), A_0(\tau_1)]]\right)\,,$$

By definition one finds

$$\partial_x\, B_0(x) = A_0(x)\, B_0(x)\,, \quad \text{and} \quad \partial_x g(d,x) = (d-4)\hat{A}_1(x) g(d,x)$$

with

$$\hat{A}_1(x) = B_0^{-1}(x)\, A_0(x)\, B_0(x)\,.$$

If the Magnus expansion **converges** in a finite number of steps we **might** end up with a canonical form!

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By definition one finds

$$\partial_x\, B_0(x) = A_0(x)\, B_0(x)\,, \quad \text{and} \quad \partial_x g(d,x) = (d-4)\hat{A}_1(x) g(d,x)$$

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This is to me *somewhat* **unsatisfactory**

1. Can we **always** obtain such linear form in d? Is it enough?
2. If not, how do we distinguish case when **we can** and when **we cannot**?
3. How do we know if, applying any of these tricks, we will get anywhere?

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What has become clear (*at least in my opinion*) is that

1. if we have a basis where the DE are in triangular form $\rightarrow$ **canonical basis** is obtainable with limited effort [T. Gehrmann, A. Manteuffel, L.T., E. Weihs].

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Seems to be a property connected to the expansion for $d \to 4$ which is equivalent to the expansion for $d \to 2\,n$, any **(even) integer** number of **dimensions** (Tarasov-Lee, see later on...)!

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Schouten Identities in $d = n$ with $n \in \mathbb{N}$ [E.Remiddi, L.T.]

1. $d = 1$ **dimension**:
   a^μ, b^μ cannot be independent $\to a^\mu\, b^\nu\, \epsilon_{\mu\nu} = 0$.

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The idea is

- It has been observed that if we start from a basis where the equations are in **triangular form**, then getting to a canonical form is **quite easy**.
- Use the information coming from the **Schouten identities** in order to put the equations in triangular form! $\rightarrow$ to decouple them!
- Finally "clean up" things in order to put them in canonical form.

Let us see how this can be used to decouple the equations in an explicit example, *easy but non-trivial*.

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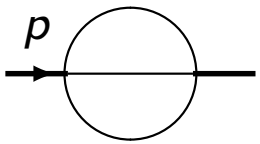

$$= \mathcal{I}(p^2, m^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

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1. Using **IBPs** we can reduce all these integrals to only **2 Master Integrals**. Say we randomly make a "reasonable choice"

$$I_1 = \mathcal{I}(p^2, m^2; 1, 1, 1, 0, 0)\,, \qquad I_2 = \mathcal{I}(p^2, m^2; 1, 1, 2, 0, 0)\,.$$

2. Derive now DE for these two integrals, we find:

$$\frac{d}{d\,p^2} I_1 = \frac{(d-3)}{p^2} I_1 - \frac{m^2}{p^2} I_2$$

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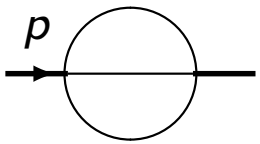

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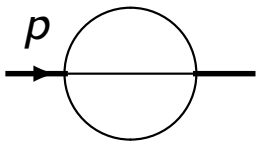

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How can we make these equations look better when expanded in $d \to 2\,n$?

1. Schouten Identities say that in $d = 2$ dimensions there can be only 2 vectors that are **linearly independent**.
2. The **two-loop sunrise** depends on 3 vectors $\to k, l, p$!
3. In $d = 2 \to \epsilon(k, l, p) = k^\mu\, l^\nu\, p^\rho\, \epsilon_{\mu\nu\rho} = 0\,.$
4. Now **square it** and **contract** the epsilon tensors in $d = 3$

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What do I do with this polynomial that is zero in $d = 2$?

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We know *UV* and *IR* properties of these polynomials $\to$ they can *partly* cure **IR divergences**!

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Integrals in **sunrise topology** have at most a squared pole $1/(d-2)^2$ as $d \to 2$.

Take for example these two new master integrals

$$Z_1 = (d-1)(d-2)\, I_1$$

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What do I mean?

Let us have a closer look at Z_2. It can be written in terms of the two previous masters:

$$Z_2 = \frac{d-1}{12}\left\{[(d-2)p^2 + (d-3)m^2]\, I_1 + m^2(p^2 - m^2)\, I_2\right\}$$

- We can **expand everything in** $(d-2)$, knowing that I get <u>at most</u> $1/(d-2)^2$.

$$I_j = \frac{1}{(d-2)^2}\, I_j^{(-2)} + \frac{1}{(d-2)}\, I_j^{(-1)} + \ldots$$

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- The first "non-zero" order is $1/(d-2)^2$ which gives:

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- We can **expand everything in** $(d-2)$, knowing that I get <u>at most</u> $1/(d-2)^2$.

$$I_j = \frac{1}{(d-2)^2}\, I_j^{(-2)} + \frac{1}{(d-2)}\, I_j^{(-1)} + \ldots$$

$$Z_2 = \frac{1}{(d-2)}\, Z_2^{(-1)} + \ldots \quad \rightarrow \quad \text{at least one pole less!}$$

- The first "non-zero" order is $1/(d-2)^2$ which gives:

$$0 = m^2\left(I_1^{(-2)} - (p^2 - m^2) I_2^{(-2)}\right) \quad \rightarrow \quad \text{First poles } \textbf{not independent}!$$

This means that **one of the two integrals** can be written, at first order, in terms of **the other one**!

- **Rescaling everything** by $(d-2)^2$ this means:

$$I_2 = A(p^2, m^2)\, I_1 + \mathcal{O}((d-2))$$

- Now if the two differential equations are **coupled**

$$\frac{d}{d\,p^2} I_1 = a_{11} I_1 + a_{12}\, I_2$$
$$\frac{d}{d\,p^2} I_2 = a_{21} I_1 + a_{22} I_2$$

- The the recipe is

$$\text{choose as new master} \quad m_2 = I_2 - A(p^2, m^2)\, I_1 = \mathcal{O}((d-2))$$

- Then you find:

$$\frac{d}{dp^2} m_2 = \mathcal{O}((d-2)) \quad \rightarrow \quad \text{The equation } \textbf{decouples!}$$

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This is all true and for $d \to 2$... but we are interested in $d \to 4$!

One more property of **dimensionally regularised Feynman Integrals**

- Every Feynman Integral in d dimensions can be related to the same Feynman integral in $d \pm 2$ dimensions [Tarasov, Lee]

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- Once we know a basis which is canonical in $d \approx 2$, we can shift it **back and forth** to $d \approx 4$ or any $d \approx 2\,n$!

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If $M(d)$ fulfils $\partial_x M(d) = (d-2)\,A(x)\,M(d)$

then of course $M(d-2)$ and $M(d+2)$ will fulfil

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Can we expect that such procedure will always work?

We know at least one example where this does not work: **The massive two-loop sunrise**

$$= \mathcal{I}(p^2, m_1^2, m_2^2, m_3^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$= \int \mathcal{D}^d k \mathcal{D}^d l \frac{(k \cdot p)^{\alpha_4} (l \cdot p)^{\alpha_5}}{(k^2 - m_1^2)^{\alpha_1} (l^2 - m_2^2)^{\alpha_2} ((k - l + p)^2 - m_3^2)^{\alpha_3}}$$

1. One finds **4** MIs, which respect **coupled differential equations** in $d \approx 2$
2. Using Schouten identities one can **prove** that **2** MIs are <u>not independent</u> in $d = 2$
3. The Schoutens give automatically two **new MIs** whose differential equations decouple from the other two! $\rightarrow$ The other two **stay coupled!!!!**
4. No canonical basis (in the **"canonical sense"!**) $\rightarrow$ Elliptic integrals ?????

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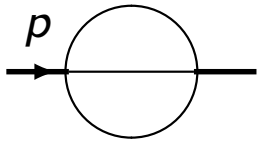

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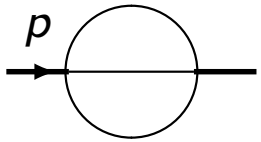

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Conclusions

1. The differential equations method is one of the most effective methods for solving **multi-loop** and **multi-scale** Feynman Integrals.

2. Recently fast development with the introduction of the idea of a **Canonical Basis** by J.Henn.

3. Many hints on how to find it, but the problem of **how to do** it in general, and **when this is possible**, remains open.

4. I showed with an easy example how the **Schouten Identities** could be used in order to systematically find such basis.

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Thanks!