Classifying and Deconstructing Tensor Correlators in 4D CFTs

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Introduction and Motivation

CFTs play a fundamental role in theoretical physics

They are the starting and ending points of renormalization group flows in QFTs

They describe second order phase transitions in critical phenomena

They are an essential part of string theory and quantum gravity by means of the AdS/CFT correspondence

Recently there has been a revival of interest in CFTs after 0807.004 (Rattazzi, Rychkov, Tonni and Vichi) have resurrected the so called bootstrap approach (Ferrara, Grillo, Gatto '73; Polyakov, '74)

Basic idea is to make some assumption about the structure of some CFT and check if it is consistent with fundamental principles such as unitarity and crossing symmetry. If it is not, that CFT is ruled out.

The CFT does **not** need to have a Lagrangian description. In fact, in bootstrap analysis a CFT is axiomatically defined by the so called CFT data:

- Spectrum of primary operators (scaling dimensions Δ_i and spins l_i)
- Their three point function coefficients λ_{ijk}

Starting point is typically a four-point correlation function. For 4 identical scalars we have

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}}$$

$$x_{ij}^2 = (x_i - x_j)_\mu (x_i - x_j)^\mu \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

The dynamical information of the CFT is encoded in the function $g(u,v)$. Relation to the CFT data is made using the OPE

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} W_{\Delta,l}(u,v)$$

Sum over all symmetric traceless primary operators in the OPE of the two scalars.

$\lambda_{\phi\phi\mathcal{O}}$ is the coefficient of the $\langle\phi\phi\mathcal{O}\rangle$ three-point function

$W_{\Delta,l}(u,v)$ are so called Conformal Partial Waves (CPW)

$$W_{\Delta,l}(u,v) = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} g_{\Delta,l}(u,v)$$

The functions $g_{\Delta,l}(u,v)$, contrary to $g(u,v)$, are kinematically determined

For each primary of dimension Δ and spin l, $g_{\Delta,l}(u,v)$ encodes the contribution of all its descendants in the exchange.

These functions are called conformal blocks and are the key players in this talk

For scalar operators, modulo a kinematical factor, CPW and conformal blocks are identical. This will no longer be the case for tensor operators.

The conformal blocks depend on the quantum numbers of the external and the exchanged operators, as well as the space-time dimension d.

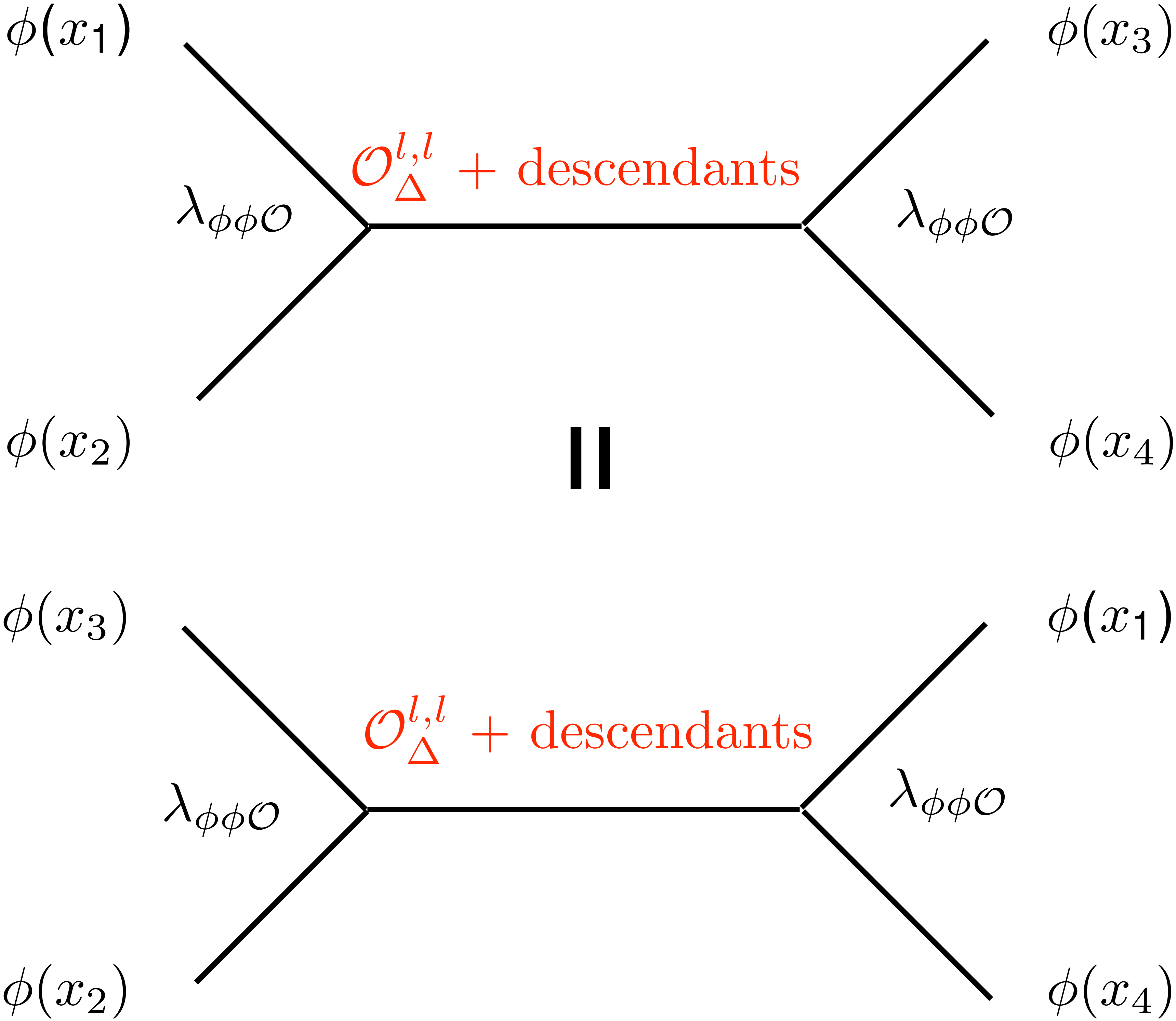
The conformal blocks for external scalars operators exchanging traceless symmetric operators of spin s is known for any s. For even space-time dimension d=2,4,6 they are known in a closed and compact form.

[Dolan,Osborn, hep-th/0011040, hep-th/0309180]

Demanding that the OPE in two different pairings (s and t channels) gives the same result, we get a crossing symmetry constraint

$$\sum_{\Delta,l} |\lambda_{\phi\phi\mathcal{O}}|^2 F_{d,\Delta,l}(u,v) = 1$$

$$F_{d,\Delta,l}(u,v) = \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}$$


$\phi(x_1)$
$\phi(x_3)$
$\mathcal{O}_\Delta^{l,l}$ + descendants
$\lambda_{\phi\phi\mathcal{O}}$
$\lambda_{\phi\phi\mathcal{O}}$
$\phi(x_2)$
$\phi(x_4)$
=
$\phi(x_3)$
$\phi(x_1)$
$\mathcal{O}_\Delta^{l,l}$ + descendants
$\lambda_{\phi\phi\mathcal{O}}$
$\lambda_{\phi\phi\mathcal{O}}$
$\phi(x_2)$
$\phi(x_4)$

Since the paper by Rattazzi et al. many results have been obtained in different space-time dimensions

All these results (with one exception in d=3 dimensions) derive from studying scalar 4-point functions only

Bootstrapping tensor correlators will be a major step in the field

Most likely more constraints (in more theories) will be discovered

Among the most interesting and universal correlators are four point functions involving **energy momentum tensors and conserved currents**

Bootstrapping tensor correlators is more challenging for a variety of reasons:

Tensor 4-point functions are determined in terms of several $g_n(u, v)$, one for each independent tensor structure.

Contribution of a primary operator is parametrized by one conformal block for each independent tensor structure.

In $D > 3$ the exchanged operator is not necessarily traceless symmetric, but is in an arbitrary representation of $SO(D)$

Bootstrapping tensor correlators require to compute all the conformal blocks entering in the various channels

General Tensor 4-Point Functions

Consider 4-point function of four primary tensor operators

$$\langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_3^{I_3}(x_3)\mathcal{O}_4^{I_4}(x_4)\rangle = \mathcal{K}_4 \sum_{n=1}^{N_4} g_n(u,v)\mathcal{T}_n^{I_1 I_2 I_3 I_4}(x_i)$$

$$\mathcal{K}_4 = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\tau_1-\tau_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\tau_3-\tau_4}{2}} \frac{1}{x_{12}^{\tau_1+\tau_2} x_{34}^{\tau_3+\tau_4}}, \qquad \tau \equiv \Delta_i + \frac{l_i+\bar{l}_i}{2},$$

Similarly, 3-point functions have different tensor structures:

$$\langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_r^{I_r}(x)\rangle = \sum_{n=1}^{N_3} \lambda^n_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} \langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_r^{I_r}(x)\rangle_n$$

$$\langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_r^{I_r}(x)\rangle_n = \mathcal{K}_3\, \mathcal{I}_n^{I_1 I_2 I_r}(x_i)$$

$$\mathcal{K}_3 = x_{12}^{\tau_3-\tau_1-\tau_2} x_{13}^{\tau_2-\tau_1-\tau_3} x_{23}^{\tau_1-\tau_2-\tau_3}$$

$\mathcal{I}_n^{I_1 I_2 I_3}(x_i)$ and $\mathcal{I}_n^{I_1 I_2 I_3 I_4}(x_i)$ are tensor structures that can be determined

Non-trivial information of 4-point function is in the N_4 functions $g_n(u,v)$

Using OPE in (12-34) s-channel

$$\langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_3^{I_3}(x_3)\mathcal{O}_4^{I_4}(x_4)\rangle = \sum_r \sum_{p=1}^{N_{3r}^{12}} \sum_{q=1}^{N_{3\bar{r}}^{34}} \sum_{\mathcal{O}_r} \lambda^p_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_r} \lambda^q_{\bar{\mathcal{O}}_{\bar{r}}\mathcal{O}_3\mathcal{O}_4} W^{(p,q)I_1I_2I_3I_4}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r}(x_i)$$

Number of allowed tensor structures in three and four-point functions is related:

$$N_4 = \sum_r N_{3r}^{12} N_{3\bar{r}}^{34}$$

We now have N_4 CPW

$$W^{(p,q)I_1I_2I_3I_4}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r}(x_i) = \mathcal{K}_4 \sum_{n=1}^{N_4} \mathcal{G}^{(p,q)}_{\mathcal{O}_r,n}(u,v)\mathcal{T}_n^{I_1I_2I_3I_4}(x_i)$$

$\mathcal{G}^{(p,q)}_{\mathcal{O}_r,n}(u,v)$ is a set of N_4^2 conformal blocks

The bootstrap equations derive, as usual, by taking OPE in different channels

Two tasks are needed for that:

1) Computation of the CPW

2) Classification of 4-point functions

Computing CPW directly is possible, but technically not straightforward.

It is however possible to relate CPW associated to different 4-point functions, where the **same operator** is exchanged.

If one finds a relation between three-point functions of the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_r(x)\rangle_p = \mathcal{D}_{pp'}(x_1,x_2)\langle \mathcal{O}'_1(x_1)\mathcal{O}'_2(x_2)\mathcal{O}_r(x)\rangle_{p'}$$

$$\Rightarrow \quad W^{(p,q)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r}(x_i) = \mathcal{D}^{12}_{pp'}\mathcal{D}^{34}_{qq'}W^{(p',q')}_{\mathcal{O}'_1\mathcal{O}'_2\mathcal{O}'_3\mathcal{O}'_4,\mathcal{O}_r}(x_i)$$

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Focus on this → Two tasks are needed for that:

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It has been shown (using a vector embedding formalism), how to reduce CPW of arbitrary traceless symmetric operators, **exchanging** a traceless symmetric operator, to the known CPW of scalar correlators

[Costa, Penedones, Poland, Rychkov, 1109.6321]

Our aim is to generalize this reduction (using a twistor embedding formalism) of CPW for arbitrary **external and exchanged** operators.

Twistor formalism allows to consider mixed tensor operators, but its technical implementation (contrary to the vector embedding) is d-dependent.

We will focus in the following on the d=4 case.

d=4 is the minimal space-time dimensionality where non-trivial mixed tensor operators can appear.

The starting point is the recent general classification and computation of three-point functions in 4d CFT's

[Elkhidir, Karateev, MS, 1412.1796]

We will not enter into any detail of the derivation and report only the final results

We have found a basis of differential operators that allows to construct all the operators $\mathcal{D}_{pp'}(x_1, x_2)$ needed to relate any tensor (bosonic or fermionic) three-point function to any other

In particular we focused on the problem of simplifying the computation of CPW appearing in four-point functions of symmetric traceless operators.

The Lorentz quantum numbers of a 4D operator are labelled by two integers $(l, \bar{l})$.

(1,0) = Chiral fermion, (1,1) = Vector, (2,0) = Self-dual antisymmetric tensor, ...

Two traceless symmetric operators $\mathcal{O}_1^{(l_1,l_1)}$ and $\mathcal{O}_2^{(l_2,l_2)}$ can exchange operators of the form $\mathcal{O}^{(l+\delta,l)}$, where

$|\delta| = 0, 1, ..., 2(l_1 + l_2)$ $\qquad$ l any integer

The number of conformal blocks to compute is large!
For instance, in a four point function with 4 traceless symmetric field with spin l, it goes like

$$N_4(l) \times N_4(l) \propto l^{14} \qquad \text{at large } l$$

Key question: what are the 4-point functions with the simplest CPW that allows us to reconstruct the more complicated CPW entering traceless symmetric 4-point correlators?

Clearly those with the **lowest** number of tensor structures where the operators of the form $\mathcal{O}^{(\delta+l,l)}$ can be exchanged in some of the OPE channels. These are

$$\langle\phi(x_i)F^{(2\delta,0)}(x_2)\phi(x_3)F^{(0,2\delta)}(x_4)\rangle$$

The number of tensor structures is just $2\delta+1$

ϕ and $F^{(2\delta,0)}$ can exchange mixed tensor operators $\mathcal{O}^{(l+p,l)}$, with $|p|=0,1,\ldots,\delta$.

Just one CPW for each exchange!

$$W^{n,m}_{O^{l+2\delta,l}}(l_1,l_2,l_3,l_4)=\mathcal{D}^{(n)}_{(12)}\mathcal{D}^{(m)}_{(34)}W_{O^{l+2\delta,l}}(\delta,0,\delta,0)$$

$$|\delta|=2\min(l_1+l_2,l_3+l_4)$$

Example: correlators with four spin 2 operators

	$O_{l,l}$	$O_{l+2,l}$	$O_{l+4,l}$	$O_{l+6,l}$	$O_{l+8,l}$
N_O^{12}	19	16	10	4	1

Number of tensor structures in the 3-point function $\langle T_1 T_2 O^{l,\bar{l}} \rangle$.
For conjugate fields we have $N^{12}_{O(l,l+\delta)} = N^{12}_{O(l+\delta,l)}$.

$$N_4 = 19^2 + 2\times 16^2 + 2\times 10^2 + 2\times 4^2 + 2\times 1 = 1107$$

Total number of conformal blocks to compute is more than a milion!
Using the differential basis and the minimal seed CPW, one has to determine 5 CPW for a total of 3+5+7+9 = 24 conformal blocks.
Moreover, the problem decouples.

For each 4-point function of the form

$$\langle \phi(x_i) F^{(2\delta,0)}(x_2) \phi(x_3) F^{(0,2\delta)}(x_4) \rangle$$

we need to determine just 1 CPW, so at most we have a system of 9 conformal blocks (for $\delta = 4$)

Conclusions

Using the 6D Embedding Space Formalism in Twistor Space, we have

1) Introduced a Differential Basis to Deconstruct 4-Point Functions. Explicitly constructed for correlators of traceless symmetric operators

2) Proposed a minimal set of “seed” CPW to dramatically simplify the computation of conformal blocks in tensor correlators.

For the Future

Two main tasks:

1) Explicit Computation of the Seed CPW

[work in progress]

2) General Classification of 4-Point Functions

Although significant work has yet to be done, bootstrapping tensor correlators in 4D is closer to the horizon now!

Thank You